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A LEMMA ON FINITE-DIMENSIONAL COVERS

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The aim of this note is to prove a lemma which enables every uniform cover with finite order to be refined by a uniform cover consisting of finite uniformly discrete subcollections. This lemma, in a slightly weaker form (for one cover of the all space) is well-known, the usual proof uses the technic of uniform complexes - see e.g. J.R. Isbell: Uniform spaces, IV. 25. The proof presented below uses elementary properties of pseudometrics and a more general assertion is obtained quite easily. Therefore the lemma is formulated for pseudometric spaces; its corollary is, in fact, the lemma in a form which seems more usable for uniform spaces.

Remember one theorem on uniform dimensions only, the proof of which uses essentially the lemma. Let X be a uniform space, $\delta d\, X < \infty$; then each finite-dimensional uniform cover of X can be refined by a uniform cover with the order less or equal to $\delta d\, X + 1$. Corollary: Finite $\Delta d\, X$ implies $\delta d\, X = \Delta d\, X$.

Let $(X, \mathcal{U})$ be a uniform space ($\mathcal{U}$ is the set of entourages), $U \in \mathcal{U}$, $Z \subset X$. A collection $\mathcal{G}$ of subsets of X is said to be a U -cover of Z , if for each point x of Z there is G in $\mathcal{G}$ such that $U[x] \cap Z \subset G$; $\mathcal{G}$ is U -discrete if $U[G] \cap \cap H = \emptyset$ for any G , H from $\mathcal{G}$, $G \neq H$. If ρ is a pseudometric on a set X, $\varepsilon > 0$ then $S_{\rho,\varepsilon} = \{(x,y) \in X \times X \mid \rho(x,y) < \varepsilon\}$. In a pseudometric space (X,ρ), $S_{\rho,\varepsilon}$-covers are called simply ε-covers, $S_{\rho,\varepsilon}$-discrete collections are called ε-discrete. Given a collection $\mathcal{G}$ of sets, the order of $\mathcal{G}$ is

$$\operatorname{ord} \mathcal{G} = \sup \{\operatorname{card} \mathcal{A} \mid \mathcal{A} \subset \mathcal{G},\ \bigcap \mathcal{A} \neq \emptyset\}.$$

Now, let us formulate and prove the lemma.

Lemma. Let (X,ρ) be a pseudometric space, m be a natural number. If a collection $\mathcal{H}$ of subsets of X is a 1-cover of a set $Z \subset X$ and $\operatorname{ord} \mathcal{H} \leq p$ for some natural $p \leq m$ then there exist $\frac{1}{3m}$-discrete collections $\mathcal{K}_1, \ldots, \mathcal{K}_p$ such that $\bigcup_{i=1}^{p} \mathcal{K}_i$ is a $\frac{1}{3m}$-cover of Z and refines $\mathcal{H}$.

Proof. We may suppose that $\operatorname{card} \mathcal{H} > p$ and $Z \setminus H \neq \emptyset$ for each H in $\mathcal{H}$, otherwise the assertion would be evident. Put, for any $\mathcal{A} \subset \mathcal{H}$ with $1 \leq \operatorname{card} \mathcal{A} \leq p$,

$$K_{\mathcal{A}} = \{x \in Z \mid G \in \mathcal{A},\ H \in \mathcal{H} \setminus \mathcal{A} \Longrightarrow$$

$$\Longrightarrow \operatorname{dist}(x, Z \setminus G) > \operatorname{dist}(x, Z \setminus H) + \frac{1}{3m}\}.$$

Let $\mathcal{K}_i$ be the collection of all $K_{\mathcal{A}}$ with $\operatorname{card} \mathcal{A} = i$ for $i = 1, \ldots, p$; we shall show that these $\mathcal{K}_i$ have the required properties.

If $\mathcal{A} \subset \mathcal{H}$, $\mathcal{B} \subset \mathcal{H}$, $\mathcal{A} \neq \mathcal{B}$, $\operatorname{card} \mathcal{A} = \operatorname{card} \mathcal{B} = i$, choose G in $\mathcal{A} \setminus \mathcal{B}$ and H in $\mathcal{B} \setminus \mathcal{A}$. Let $x \in K_{\mathcal{A}}$, $y \in K_{\mathcal{B}}$. Then we have

$$dist(x, Z \setminus G) > dist(x, Z \setminus H) + \frac{1}{3m},$$

$$dist(y, Z \setminus H) > dist(y, Z \setminus G) + \frac{1}{3m}.$$

Suppose $\varrho(x, y) < \dfrac{1}{3m}$; then

$$dist(y, Z \setminus H) \leqq dist(x, Z \setminus H) + \frac{1}{3m} < dist(x, Z \setminus G) \leqq$$

$$\leqq dist(y, Z \setminus G) + \frac{1}{3m} < dist(y, Z \setminus H)$$

which is a contradiction; thus $\varrho(x, y) \geqq \dfrac{1}{3m}$ and $\mathcal{K}_i$ are $\dfrac{1}{3m}$ -discrete.

Given a point x of Z, choose G in $\mathcal{H}$ such that $S_{\varrho,1}[x] \cap Z \subset G$. Hence $dist(x, Z \setminus G) \geqq 1$. The numbers $dist(x, Z \setminus H)$ are positive for at most p sets H from $\mathcal{H}$, one is at least 1 and at most $p - 1$ of them are less than 1, thus a gap with length $\dfrac{1}{p}$ must appear. Therefore there exists $\mathcal{a} \subset \mathcal{H}$ with $1 \leqq card\, \mathcal{a} \leqq p$ such that

$$min\{dist(x, Z \setminus H) \mid H \in \mathcal{a}\} \geqq max\{dist(x, Z \setminus H) \mid H \in \mathcal{H} \setminus \mathcal{a}\} + \frac{1}{p}.$$

Now, clearly, if $y \in Z$, $\varrho(x, y) < \dfrac{1}{3m}$ then $y \in K_{\mathcal{a}}$. Thus $\bigcup_{i=1}^{p} \mathcal{K}_i$ is a $\dfrac{1}{3m}$ -cover of Z and it refines $\mathcal{H}$ because $K_{\mathcal{a}} \subset H$ for any H in $\mathcal{a}$.

Corollary. Let $(X, \mathcal{U})$ be a uniform space. For each V in $\mathcal{U}$ and a natural number m there exists

W in $\mathcal{U}$ with the following property. If a collection $\mathcal{H}$ of subsets of X is a V-cover of a set $Z \subset X$ and $\operatorname{ord} \mathcal{H} \leq p$ for some natural $p \leq m$ then there exist W-discrete collections $\mathcal{H}_1, \ldots, \mathcal{H}_p$ such that $\bigcup_{i=1}^{p} \mathcal{H}_i$ is a W-cover of Z and refines $\mathcal{H}$.