Miloš Zahradník

ℓ_1-partitions of unity on normed spaces

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If X is a uniform space, denote by $\mathcal{U} = \{U_\beta\}$ a uniform covering of X, and denote by $\mathcal{A} = \{f_\alpha\}_{\alpha \in I}$ a partition of unity on X. A partition of unity $\mathcal{A}$ is subordinated to $\mathcal{U}$ (we write $\mathcal{A} \prec \mathcal{U}$) if the supports of f_α form a covering which refines $\mathcal{U}$. The following notion will play the basic role:

<u>Definition</u>. A partition of unity $\{f_\alpha\}_{\alpha \in I}$ is ℓ_p continuous ($1 \leq p \leq \infty$) if the mapping

$$\{x \longrightarrow \{f_\alpha(x)\}_{\alpha \in I}: X \longrightarrow \ell_p(I)$$

is uniformly continuous.

The simplest assertion of the type studied here is the following (see [2]): For each uniform covering $\mathcal{U}$ there exists a partition of unity $\mathcal{A} = \{f_\alpha\}_{\alpha \in I}$, subordinated to $\mathcal{U}$ such that the family $\{f_\alpha\}_{\alpha \in I}$ is equiuniformly continuous. (In other words, $\mathcal{A}$ is ℓ_∞ continuous.)

We will show in Section 1 that the analogous result holds for ℓ_p continuity, if $1 < p < \infty$. This is quite elementary. The case $p = 1$ is the most important, and it seems to be non-trivial. Remark the assertion: a partition of unity $\{f_\alpha\}_{\alpha \in I}$ is ℓ_1 continuous if and only if $\{\sum_{I'} f_\alpha\}_{I' \subset I}$ is uniformly equicontinuous (see [1]). For $p = 1$, our preceding result (the existence of ℓ_1-continuous partition of unity subordinated to given $\mathcal{U}$) does not hold for any infinite-dimensional normed space, and this is the main result.

To obtain the main theorem 1.4 we have to make a detailed study of the partitions of unity in Euclidean spaces E_m.

We define a useful notion, namely integral partition of unity. This is done in Section 2, the main result is the theorem 2.9.

In Section 3, we refer to 2 and show, how "the module of continuity" of an arbitrary partition of unity in E_m depends on the dimension.

Finally, in Section 4, we use the preceding results (Corollary 3.9 and Lemma 3.10). This is immediate for Hilbert spaces. In the case of an arbitrary normed space, we use a theorem of Dvoretzky.

1.

In this section we deal with ℓ_p continuity for $p > 1$ and state the main theorem for $p = 1$.

1.1. Proposition. Let $\mathcal{U}$ be a uniform covering of a uniform space X. Then there exists an ℓ_∞ continuous partition of unity subordinated to $\mathcal{U}$.

Proof: see [2], p. 62.

1.2. Proposition. Let $\mathcal{U}$ be a uniform covering of X, let $1 < p < \infty$. Then there exists an ℓ_p continuous partition of unity subordinated to $\mathcal{U}$.

Proof: By 1.1, there exists a partition of unity $\mathcal{A} = \{f_\alpha\}_{\alpha \in I}$ such that the family $\{f_\alpha\}_{\alpha \in I}$ is uniformly equicontinuous. Define the functions

$$f_{\alpha,n} = \left(\left(f_\alpha \wedge \frac{1}{n} \right) - \frac{1}{n+1} \right) \vee 0 \qquad n = 1, 2, \ldots$$

It is clear that $\{ f_{\alpha,n} \}$ forms a partition of unity. Finally, define (for some $M(n)$)

$$g_{\alpha,n,i} = \frac{f_{\alpha,n}}{M(n)} \qquad i = 1, 2, \ldots, M(n) .$$

We will show that the family $\{ g_{\alpha,n,i} \}_{\alpha \in I, n \in N, i \leq M}$ is the desired partition of unity, if $M(n)$ is suitably chosen.

For $x, y \in X$ put

$$\varphi(x,y) = \sup_I \{ | f_\alpha(x) - f_\alpha(y) | \} .$$

Since $\{ f_\alpha \}$ satisfies 1.1, φ is uniformly continuous. We have

$$\| g_{\alpha,n,i}(x) - g_{\alpha,n,i}(y) \|_{\ell_p} \stackrel{def}{=\!=}$$

$$\stackrel{def}{=\!=} \left(\sum_{\alpha,n,i} | g_{\alpha,n,i}(x) - g_{\alpha,n,i}(y) |^p \right)^{\frac{1}{p}}$$

$$= \left(\sum_n M^{1-p} \sum_\alpha | f_{\alpha,n}(x) - f_{\alpha,n}(y) |^p \right)^{\frac{1}{p}}$$

$$\leq \left(\sum_n M^{1-p} \, 2(n+1) \, \varphi(x,y)^p \right)^{\frac{1}{p}}$$

(because $f_{\alpha,n}(x) > 0$ for at most $n+1$ indices $\alpha \in I$). Choose $M(n)$ such that $2 M^{1-p}(n+1) < \frac{1}{2^n}$.

We get

$$\| \{ g_{\alpha,n,i}(x) - g_{\alpha,n,i}(y) \} \|_{\ell_p} \leq \varphi(x,y)$$

and the ℓ_p continuity of $\{ g_{\alpha,n,i} \}$ is proved, q.e.d.

Now we will investigate the case $p = 1$.

1.3. Definition. Let $\mathcal{A} = \{f_\alpha\}_{\alpha \in I}$ be a partition of unity on a metric space (X, ρ). Define a function ("module of continuity")

$$(1) \quad \mathcal{H}_{\mathcal{A}}(t) = \sup_{\rho(x,y) \leq t} \|\{f_\alpha(x) - f_\alpha(y)\}\|_{\ell_1} . \quad t \geq 0 .$$

Remark that $\mathcal{H}_{\mathcal{A}}$ is monotone, $\mathcal{H}_{\mathcal{A}} \leq 2$: Obviously $\mathcal{A}$ is ℓ_1 continuous if and only if $\mathcal{H}_{\mathcal{A}}$ is continuous at $t = 0$.

1.4. Theorem. Let $\mathcal{U}$ denote the covering consisting of all open balls with radius 1 on an infinite dimensional normed space X. Then there is no ℓ_1-continuous partition of unity subordinated to $\mathcal{U}$.

Remark. This theorem answers the problem in [1], p.107.

Proof is contained in Sections 2, 3, 4. Let us sketch the idea of proof. Suppose, for the simplicity, that X is a Hilbert space. Let X_n be an n-dimensional subspace of X, let $\mathcal{A}_n$ be the restriction to X_n of some partition of unity $\mathcal{A}$ in X.

It is obvious that

$$(2) \quad \mathcal{H}_{\mathcal{A}}(t) \geq \mathcal{H}_{\mathcal{A}_n}(t) \qquad \text{for each } t \geq 0 .$$

X_n is an Euclidean space. If $\mathcal{A} \prec \mathcal{U}$, we will show in Section 3 that

$$\lim_{n \to \infty} \mathcal{H}_{\mathcal{A}_n}(t) = 2 \qquad \text{for each } t > 0 .$$

Then we use (2) to show that $\mathcal{H}_{\mathcal{A}}$ is not continuous at $t = 0$.

2.

In this section we investigate partitions of unity on

Euclidean spaces.

2.1. Denote by $E = E_m$ an Euclidean space with the Euclidean norm $\| \ \|$. The group G of all isometries on E has the following properties:

(3) i) for each $x, y \in E$ there exists $T \in G$ such that $Tx = y$.

ii) for each $x, y, z \in E$ satisfying $\|y - z\| = \|x - z\|$ there exists $T \in G$ such that $Tz = z$, $Tx = y$.

G is a locally compact group in the topology, defined by all the pseudometrics of the type

$$\rho_F(T_1, T_2) = \sup_{x \in F} \| T_1 x - T_2 x \|$$

where F is a finite subset of E.

On G, there exists the Haar unimodular (left and right) measure m (see for example [4], (2, 7, 16), example 7). For any $x \in E$ consider the mapping

(4) $$\{ T \longrightarrow Tx \} : G \longrightarrow E$$

The preimage of an arbitrary ball $\{ z, \| z - y \| \le 1 \}$ equals to the set $\{ T, \| Tx - T_y x \| \le 1 \}$ where $T_y x = y$. This is a compact set in G. Thus the image of m with respect to $\{ T \longrightarrow Tx \}$ exists. Denote it by μ. Since m is right-invariant, μ does not depend on the choice of $x \in E$. Since m is left-invariant, μ is invariant with respect to G. Then μ is up to a constant the Lebesgue measure.

2.2. <u>Definition.</u> Let A be a function on $E \times E$. We say that A is an integral partition of unity if the following holds:

(5) i) $A \geq 0$

ii) $A(\cdot, y) \in \mathcal{L}^1(E, \mu)$ for almost each $y \in E$.

iii) $\int_E A(x, y)\, dx(\mu) = 1$ for almost each $y \in E$.

If further $\|x - y\| \geq 1$ implies $A(x, y) = 0$ we say that A is subordinated to $\mathcal{U}$ ($\mathcal{U}$ will always denote the covering of all open balls with radius 1), and write $A \prec \mathcal{U}$. We put analogously as in (1)

$$(1') \quad \mathcal{H}_A(t) = \sup_{\|x-y\| \leq t} \int_E |A(z, x) - A(z, y)|\, dz(\mu).$$

Our aim is to restrict ourselves to the partitions of the type (5), which are much more convenient for the following computations. This enables us the following lemma which is the main step in the proof of 1.4.

2.3. Lemma. Let $\mathcal{A} = \{f_\alpha\}_{\alpha \in I}$ be a partition of unity. Then for each $\varepsilon > 0$ there exists an integral partition of unity A such that

$$\mathcal{H}_A(t) \leq (1 + \varepsilon)\mathcal{H}_{\mathcal{A}}(t) \quad \text{for each } t \geq 0.$$

Further, if $\mathcal{A} \prec \mathcal{U}$, then $A \prec \mathcal{U}$.

Proof. First we introduce some preliminary definitions and constructions.

2.4. Definition. If $\mathcal{A} = \{f_\alpha\}_{\alpha \in I}$ is a partition of unity, $T \in G$, put

$$\mathcal{A}^T = \{f_\alpha \circ T\}_{\alpha \in I}$$

Analogously, for integral partitions of unity, put

$$A^T(x,y) = A(Tx, Ty) \ .$$

2.5. Put

$$G_{xy} = \{ T \in G, \ Tx = y \} \ .$$

G_{xy} equals to the preimage of y with respect to the mapping $\{T \longrightarrow Tx\}$. Using the desintegration theorem (see [3], VI, § 3,1) we obtain:

There exist Radon measures m_{xy} on G_{xy} such that $m_{xy} \geq 0$ and $\{m_{xy}, y \in E\}$ form a desintegration of m with respect to the mapping $\{T \longrightarrow Tx\}$: $: (G, m) \longrightarrow (E, \mu)$.

2.6. First we describe the construction of A in a simpler situation, namely S (instead of E) is a sphere S in E_{n+1}, and G is a compact group of all isometries on S. The measures m, μ, m_{xy} are defined analogously as before. We suppose $\|m\| = 1$ and put

$$A(x,y) = \sum_I \int_{G_{x_\alpha x}} f_\alpha(T^{-1}y) \, dT(m_{x_\alpha x})$$

where for each $\alpha \in I$ we choose $x_\alpha \in S$ such that

(6) $\quad \|y - x_\alpha\| > 1 \quad$ implies $\quad f_\alpha(y) = 0$.

2.7. In the case of E there is a little technical complication (m is not finite). Choose, for each $\alpha \in I$, $x_\alpha \in X$ satisfying (6). Let i be an integer. Put:

$K_i = \{ y \in E, \ \|y\| \leq i \}$,

$$I_i = \{\alpha \in I,\ X_\alpha \in K_i\} .$$

It is easy to show that

$$(7) \qquad \sum_{I_i} f_\alpha(x) = 1 \quad \text{for each} \quad x \in K_{i-1}$$

$$\sum_{I_i} f_\alpha(x) = 0 \quad \text{for each} \quad x \notin K_{i+1} .$$

Now we construct, analogously as in 2.6

$$A_i(x,y) = \sum_{I_i} \int_{G_{X_\alpha} X} f_\alpha(T^{-1}(y))\, d\, T(m_{X_\alpha X}) .$$

Notice that $\mathcal{A} \prec \mathcal{U}$ implies $A_i \prec \mathcal{U}$.

I_i is countable, $\int_{G_{X_\alpha(\cdot)}} f_\alpha(T^{-1}(y))\, d\, m_{X_\alpha(\cdot)}$ is μ integrable function (according to the desintegration theorem).

Then we can write

$$(8) \quad \int_E \sum_{I_i} f_\alpha(x)\, dx(\mu) = \int_G \sum_{I_i} f_\alpha(T^{-1}y)\, d\, T(m)$$

$$= \sum_{I_i} \int_G f_\alpha(T^{-1}y)\, d\, T(m) = \quad \text{(desintegration theorem)}$$

$$= \sum_{I_i} \int_E \Big(\int_{G_{X_\alpha} X} f_\alpha(T^{-1}y)\, d\, T(m_{X_\alpha X}) \Big)\, dx(\mu) =$$

$$= \int_E \Big(\sum_{I_i} \int_{G_{X_\alpha} X} f_\alpha(T^{-1}y)\, d\, T(m_{X_\alpha X}) \Big)\, dx(\mu) =$$

$$= \int_E A_i(x,y)\, dx(\mu) .$$

Putting $\int_E \sum_{I_i} f_\alpha(x)\, dx(\mu) = \lambda_i$ (notice that $\lambda_i \geq \mu(K_{i-1})$), we get:

$$(9) \qquad \frac{A_i}{\lambda_i} \quad \text{is an integral partition of unity.}$$

Now estimate $\mathcal{H}_{A_i}$. Write

(10) $$\int_E |A_i(z,x) - A_i(z,y)|\,dz(\mu) \leq$$

$$\leq \sum_{I_i} \int_E \Big(\int_{G_{\chi_\alpha Z}} |f_\alpha(T^{-1}x) - f_\alpha(T^{-1}y)|\,dT(m_{\chi_\alpha Z})\Big)\,dz(\mu)$$

$$= \sum_{I_i} \int_G |f_\alpha(T^{-1}x) - f_\alpha(T^{-1}y)|\,dT(m)$$

according to the desintegration theorem. Put

$$G_i = \{T \in G,\ T^{-1}x \in K_i\} .$$

Suppose $\|x - y\| \leq t \leq 2$. (7) implies

$$T^{-1}x \notin K_{i+3} \Rightarrow T^{-1}y \notin K_{i+1} \Rightarrow f_\alpha(T^{-1}x) = f_\alpha(T^{-1}y) = 0 .$$

Returning to (10), we obtain

(11) $$\int_E |A_i(z,x) - A_i(z,y)|\,dz(\mu) \leq$$

$$\leq \sum_{I_i} \int_{G_{i+3}} |f_\alpha(T^{-1}x) - f_\alpha(T^{-1}y)|\,dT(m)$$

$$\leq \int_{G_{i+3}} \mathcal{H}_{\mathcal{A}T^{-1}}(t)\,dT(m) = \mathcal{H}_{\mathcal{A}}(t)\,\mu(K_{i+3}) .$$

Combining (11) with (9), we conclude

$$\mathcal{H}_{\frac{A_i}{\lambda_i}}(t) \leq \frac{\mu(K_{i+3})}{\mu(K_{i-1})}\ \mathcal{H}_{\mathcal{A}}(t) .$$

Obviously $\lim\limits_{i\to\infty} \dfrac{\mu(K_{i+3})}{\mu(K_{i-1})} = 1$, thus for sufficiently large i $\dfrac{A_i}{\lambda_i}$ is the desired integral partition of unity, q.e.d.

<u>Remark</u>. It is possible to show that the family $\left\{\dfrac{A_i}{\lambda_i}\right\}$ is equicontinuous. Taking limits, we can prove the lemma also for $\varepsilon = 0$.

2.8. Lemma. A_i from the preceding lemma satisfies the property $A_i^T = A_i$ for each $T \in G$ (A_i is "symmetric").

Proof. Remind that

$$A_i^T(x,y) = \sum_{I_i} \int_{G_{X_\alpha Tx}} f_\alpha(U^{-1}Ty)\, dU(m_{X_\alpha Tx})$$

Consider the diagram

$$(G,m) \xrightarrow{\{U \to T \circ U\}} (G,m) \xrightarrow{\{U \to Ux_\alpha\}} (E,\mu) \xrightarrow{\{x \to T^{-1}x\}} (E,\mu)$$

(the mappings will be shortly denoted by $\varphi_1, \varphi_2, \varphi_3$). The following holds: $\{m_{X_\alpha x}\}$ is the desintegration of m with respect to $\varphi_3 \circ \varphi_2 \circ \varphi_1$, then $\{\varphi_1(m_{X_\alpha x})\}$ is the desintegration of m with respect to $\varphi_3 \circ \varphi_2$. But $\{m_{X_\alpha Tx}\}$ is also the desintegration of m with respect to $\varphi_3 \circ \varphi_2$. Then we have for almost all $x \in (E,\mu)$

$$m_{X_\alpha Tx} = \varphi_1(m_{X_\alpha x})$$

and

$$A_i^T(x,y) = \sum_{I_i} \int_{G_{X_\alpha Tx}} f_\alpha(U^{-1}Ty)\, dU(m_{X_\alpha Tx})$$

$$= \sum_{I_i} \int_{G_{X_\alpha Tx}} f_\alpha(U^{-1}y)\, dU(\varphi_1(m_{X_\alpha x})) =$$

$$= \sum_{I_i} \int_{G_{X_\alpha x}} f_\alpha(U^{-1}y)\, dU(m_{X_\alpha x}) = A_i(x,y), \quad q.e.d.$$

Summarizing Lemma 2.3 and Lemma 2.8, we obtain:

2.9. Theorem. For each $\varepsilon > 0$, and for each partition of unity $\mathcal{A}$ there exists some symmetric integral partition of unity A such that

$$\mathcal{H}_A(t) \leq (1+\varepsilon)\,\mathcal{H}_{\mathcal{A}}(t) \quad \text{for each } t \geq 0$$

and $\mathcal{A} \prec \mathcal{U}$ implies $A \prec \mathcal{U}$.

Remark. Using the remark 2.7, one can prove the theorem also for $\varepsilon = 0$.

3

Now we will study the symmetric integral partitions of unity. With the help of (3) we can say that A is symmetric if and only if there exists a locally integrable function $f : \langle 0, \infty) \longrightarrow E_1$ such that

(12) $A(x, y) = f(\| x - y \|)$ for almost each $x, y \in$ $\in (E \times E, \mu \times \mu)$.

3.1. Definition. Let A be a symmetric integral partition of unity. Let $1 \geq s > 0$. Put

$$A^s(x, y) = f\left(\frac{\| x - y \|}{s}\right) .$$

3.2. Lemma. The following hold

i) $\int_E A^s(x, y)\,dx(\mu) = s^n$

(then $\dfrac{A^s}{s^n}$ is an integral partition of unity)

ii) $\mathcal{H}_{\frac{A^s}{s^n}}(t) = \mathcal{H}_A\left(\frac{t}{s}\right)$.

The proof is easy (consider the mapping

$\{x \longrightarrow s \circ x\} : E \longrightarrow E$).

3.3. Definition. Let A be an integral symmetric partition of unity. Put

$$\hat{f}(t) = \int_0^1 f\left(\frac{t}{s}\right)\frac{1}{s^n}\,ds ,$$

$$\hat{A}(x,y) = \hat{f}(\|x-y\|) .$$

It is easy to show that $\hat{A}$ is a symmetric integral partition of unity.

3.4. Lemma. $\mathcal{H}_{\hat{A}}(t) \le \int_0^1 \mathcal{H}_A\left(\frac{t}{s}\right) ds$.

Proof. Put $\|x-y\| = t$.

$$\int_E |\hat{A}(z,x) - \hat{A}(z,y)|\,dz \le$$

$$\le \int_E \left(\int_0^1 \left|\frac{A^s}{s^n}(z,x) - \frac{A^s}{s^n}(z,y)\right| ds\right) dz = \int_0^1 \mathcal{H}_A\left(\frac{t}{s}\right) ds$$

according to 3.2, ii), q.e.d.

3.5. Lemma. $\hat{f}$ is a decreasing and differentiable function on $(0, \infty)$.

Proof. Let $t' \ge t$. Obviously

$$\hat{f}(t') = \int_0^1 f\left(\frac{t'}{s}\right)\frac{1}{s^n}\,ds = \left(\frac{t}{t'}\right)^{n-1} \int_0^{\frac{t}{t'}} f\left(\frac{t}{s}\right)\frac{1}{s^n}\,ds$$

$$< \int_0^1 f\left(\frac{t}{s}\right)\frac{1}{s^n}\,ds = \hat{f}(t) .$$

Analogously one can compute

$$\hat{f}'(t) = -\left(\frac{n-1}{t}\hat{f}(t) + \frac{f(t)}{t^{n+1}}\right) , \qquad \text{q.e.d.}$$

3.6. Definition. Define a function ϕ : $\phi(x,y) = 0$ for each x, y satisfying $\|x-y\| \geq 1$, $\phi(x,y) = 1$ for each x, y satisfying $\|x-y\| < 1$.

We suppose that the μ volume of the unit ball equals to 1. Then ϕ is an integral partition of unity.

3.7. Proposition. Let A be a symmetric integral partition of unity, $A \prec \mathcal{U}$. Then

i) $\hat{A}(x,y) = -\int_0^1 \phi^{1-s}(x,y)\, \hat{f}'(1-s)\,ds$

ii) $1 \quad = -\int_0^1 (1-s)^m \quad \hat{f}'(1-s)\,ds$

iii) $\mathcal{H}_{\hat{A}}(t) = -\int_0^1 \mathcal{H}_{\phi^{1-s}}(t)\, \hat{f}'(1-s)\,ds$

Proof.

i) write $\|x-y\| = t$.

Then $\hat{f}(t) = -\int_t^1 \hat{f}'(s)\,ds = -\int_0^{1-t} \hat{f}'(1-s)\,ds =$
$= -\int_0^1 \hat{f}'(1-s)\,\phi^{1-s}(x,y)\,ds$;

ii) follows easily from i).

iii): $\int_E |\hat{A}(z,x) - \hat{A}(z,y)|\,dz\,(\mu) =$

$= \int_E \left| \int_0^1 (\phi^{1-s}(z,x) - \phi^{1-s}(z,y))\, \hat{f}'(1-s)\,ds \right| dz$

$= -\int_E \left(\int_0^1 \hat{f}'(1-s)\, |\phi^{1-s}(z,x) - \phi^{1-s}(z,y)|\,ds \right) dz$

$= -\int_0^1 \hat{f}'(1-s)\, \mathcal{H}_{\phi^{1-s}}(t)\,ds$, q.e.d.

3.8. Combining ii) with iii) we obtain

$$\mathcal{H}_{\frac{\phi^s}{s^m}}(t) \leq \mathcal{H}_{\hat{A}}(t) \quad \text{for some } s < 1 .$$

But it is obvious that $\mathcal{H}_{\phi}(t) \leq \mathcal{H}_{\frac{\phi^s}{s^m}}(t)$.

Then $\mathcal{H}_\phi(t) \leq \mathcal{H}_{\hat{A}}(t)$ holds for each $t \geq 0$. Using Lemma 3.4, we obtain

(13) $$\mathcal{H}_\phi(t) \leq \int_0^1 \mathcal{H}_A\left(\frac{t}{s}\right) ds .$$

An application of Theorem 2.9 gives:

3.9. Corollary. Let $\mathcal{A}$ be a partition of unity, let $\mathcal{A} \prec \mathcal{U}$. Then

$$\mathcal{H}_\phi(t) \leq \int_0^1 \mathcal{H}_{\mathcal{A}}\left(\frac{t}{s}\right) ds$$

holds for each $t \geq 0$.

In order to emphasize the dimension of $E = E_n$ write

(14) $$\mathcal{H}_{\phi_n}(t) \leq \int_0^1 \mathcal{H}_{\mathcal{A}_n}\left(\frac{t}{s}\right) ds .$$

3.10. Lemma. $\lim\limits_{n\to\infty} \mathcal{H}_{\phi_n}(t) = 2$ for each $t > 0$.

Proof. Denote by

$$K_0^n = \{x \in E_n , \|x\| \leq 1\} ,$$

$$K_t^n = \{x \in E_n , \|x - (t, 0, \dots, 0)\| \leq 1\} ,$$

$$K_t^{n'} = \{x \in E_n , \left\|x - \left(\frac{t}{2}, 0, \dots, 0\right)\right\| \leq \sqrt{1 - \frac{t^2}{4}}\} .$$

It is easy to check $K_0^n \cap K_t^n \subset K_t^{n'}$. We obtain

(15) $$\mathcal{H}_{\phi_n}(t) = \mu(K_0^n \,\Delta\, K_t^n) = 2 - \mu(K_0^n \cap K_t^n)$$

$$> 2 - \mu(K_t^{n'}) = 2 - \left(1 - \frac{t^2}{4}\right)^{\frac{n}{2}} , \qquad \text{q.e.d.}$$

4.

In this section we prove 1.4, using the following theorem of Dvoretzky:

Let $(X, \| \ \|)$ be an infinite dimensional normed space, let n be an integer, let $\varepsilon > 0$. There exists some n-dimensional subspace X_n of X and an Euclidean norm $\| \ \|^E$ on X_n such that

(16) $\quad \| \ \|^E \leq \| \ \| \leq (1+\varepsilon) \| \ \|^E$ holds on X_n.

(see [5], p. 123).

Now we can finish the proof of 1.4. Let $\mathcal{A}$ be a partition of unity on X, let $\mathcal{A} \prec \mathcal{U}$. Denote by $\mathcal{A}_n$ (resp. $\mathcal{U}_n$) the restriction of $\mathcal{A}$ (resp. of $\mathcal{U}$) to X_n. Obviously $\mathcal{A}_n \prec \mathcal{U}_n$. Each element of $\mathcal{U}_n$ has the $\| \ \|$ diameter less or equal to 2. Using (16), we obtain that $\mathcal{U}_n$ refines the covering of all $\frac{1}{2} \| \ \|^E$ open balls with radius 1. We use (14) and (15) for X_n, provided with $\frac{1}{2} \| \ \|^E$ and obtain

$$2 - \left(1 - \frac{t^2}{4}\right)^{\frac{n}{2}} < \int_0^1 \mathcal{H}^E_{\mathcal{A}_n}\left(\frac{t}{s}\right) ds ,$$

where $\mathcal{H}^E$ denotes the module of continuity in $(X_n, \frac{1}{2} \| \ \|^E)$. (16) implies $\mathcal{H}^E(t) \leq \mathcal{H}(2(1+\varepsilon)t)$.

Thus

$$2 - \left(1 - \frac{t^2}{4}\right)^{\frac{n}{2}} < \int_0^1 \mathcal{H}_{\mathcal{A}_n}\left(\frac{2(1+\varepsilon)t}{s}\right) ds .$$

Using (2)

$$2-\left(1-\frac{t^2}{4}\right)^{\frac{n}{2}} < \int_0^1 \mathcal{H}_{\mathcal{R}}\left(\frac{2(1+\varepsilon)t}{s}\right) ds$$

for each n .

This implies $\int_0^1 \mathcal{H}_{\mathcal{R}}\left(\frac{2(1+\varepsilon)t}{s}\right) ds = 2$, then $\mathcal{H}_{\mathcal{R}}(t) = 2$

for each $t > 0$, q.e.d.

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