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Inversion closed uniform spaces have the Daniell property

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Definition 1. A uniform space X is inversion closed iff for each positive uniformly continuous function f the function $\frac{1}{f}$ is uniformly continuous.

Definition 2. A uniform space X has the Daniell property iff each family $\{f_n\}$ of uniformly continuous functions such that $1 \geq f_n \searrow 0$ is uniformly equicontinuous.

It can be shown (see [1]) that the Daniell property of X implies that X is inversion closed. It is the aim of this note to prove the converse statement.

Proposition. Any inversion closed space X has the Daniell property.

Proof: Let $\{f_{n,n=1,2,\dots}\}$ be a family of uniformly continuous functions such that $1 \geq f_n \searrow 0$. We have to show that $\{f_n\}$ is uniformly equicontinuous. It is possible to assume that $1 > f_n(x) > f_{n+1}(x)$ for each $x \in X$ and $n = 1,2,\dots$. Define a function f_t for each $t \in \langle 0, \infty)$ as follows:

$f_t(x) = s . f_{n+1}(x) + (1 - s) f_n(x)$ where $s = t - [t]$, $n = [t]$ (we put $f_0 = 1$). Obviously f_t are uniformly continuous functions.

We shall show that the family $\{f_t\}$ is uniformly equicontinuous. For each δ positive, $\delta \leq 1$, consider the function

$$\Phi_\delta(x) = \inf_{f_t(x) \leq \delta} \{t\} .$$

Lemma. Each Φ_δ is a cozero function. Then the inversion closed property of X will imply that Φ_δ is uniformly continuous (see [1]).

Proof of Lemma. We have

i) $\Phi_{\delta}^{-1}(c,\infty) = \{x, f_c(x) > \delta\}$,

ii) $\Phi_{\delta}^{-1}\langle 0,c) = \{x, f_c(x) < \delta\}$.

Thus $\Phi_{\delta}^{-1}(\Omega)$ is a cozero set for each open $\Omega \subset \langle 0,\infty)$, q.e.d.

Proof of uniform equicontinuity of $\{f_t\}$:

Notice that for $n \leq t_1 \leq t_2 \leq n+1$

$$|f_{t_1}(x) - f_{t_2}(x)| \leq |t_2 - t_1|\, f_n(x) + |t_2 - t_1|\, f_{n+1}(x) .$$

Hence

(1) $\| f_{t_1} - f_{t_2} \| \leq 2\, |t_2 - t_1|$ for each $t_1, t_2 \in (0,\infty)$.

This implies

(2) $$\inf_{x \in X} | \Phi_{\delta_1}(x) - \Phi_{\delta_2}(x) | \geq \frac{|\delta_2 - \delta_1|}{2} .$$

Fix $\varepsilon > 0$. Choose a uniformly continuous pseudometric ϱ_ε on X such that

(3) $$\varrho_\varepsilon(x,y) < 1 \Longrightarrow |\Phi_{n\varepsilon}(x) - \Phi_{n\varepsilon}(y)| < \frac{\varepsilon}{2}$$

for each $n = 1,2,\dots [\frac{1}{\varepsilon}]$.

If $f_t(x) - f_t(y) \geq 2\varepsilon$, then $f_t(x) \leq n\varepsilon$ and $f_t(y) \geq (n+1)\varepsilon$ holds for some integer n . This means that $\Phi_{n\varepsilon}(x) \leq t$, $\Phi_{(n+1)\varepsilon}(y) \geq t$. Using (2) and (3) we get

$$\Phi_{n\varepsilon}(y) \geq t + \frac{\varepsilon}{2} \quad \text{and} \quad \varrho_\varepsilon(x,y) \geq 1 .$$

Thus $\varrho_\varepsilon(x,y) < 1$ implies $|f_t(x) - f_t(y)| < 2\varepsilon$ for each t , q.e.d.

Reference:

[1] Frolík Z.: Three uniformities associated with uniformly continuous functions. Proc.Conf.Algebras Continuous Functions, Rome, 1973. To appear.