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A CONTRIBUTION TO GÖDEL'S AXIOMATIC SET THEORY, I

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Some questions are discussed concerning models, dependences and independences (between some axioms and some theorems) in Gödel's set theory. (See KURT GÖDEL, The Consistency of the Axiom of Choice and of the Generalized Continuum Hypothesis with the Axioms of Set Theory, Princeton 1940; quoted as [G].)

One of the main results of the present paper is the following statement:

The existence of Russell's predicative sets (being an element of itself) and of the class of impredicative sets is consistent with the axioms of [G] sub A, B, C, E completed by the Generalized Continuum Hypothesis, provided the axioms sub A, B, C are consistent.

The results of the paper have been communicated at the session of the Mathematical Society held in Prague on the 28th of May 1956.

1. Introduction. Some metamathematical notions

The present paper is closely related to Gödel's fundamental treatise [G]. Therefore — and for the sake of brevity — I accept the mathematical and the logical signs (with little typographical modifications) and termini of [G] and I do not, as a rule, rewrite the corresponding definitions but I only quote them in the original notation (by ordinary numerals). In order to distinguish theorems and definitions not due to [G], I denote them by latin numerals. The reader not interested in technical details may be satisfied by the informal versions of the main notions and theorems as well as by the related comments.

Basic notions of Boolean algebras and of the lower predicate calculus are assumed, though the full formalization is not performed but always obviously possible. Less usual needed notions of mathematical logic will be restated in the following part of this introductory §. In the sequel, they will often be applied without quotation. For further purposes, they are stated in a more general and more explicite (algebraic) formulation than would be necessary for the purpose of the present paper alone.

The notion of a theory. An *elementary theory*[1] $\Theta = (\Sigma, \Sigma^*)$ is an ordered pair of two sets Σ and Σ^* of propositional functions so that the following holds:

(I) Σ contains all the propositional functions inductively formed with the help of an at most countable given set of individual variables, primitive individual constants and primitive predicate constants.

(II) Σ^* is a nonvoid subset of Σ of s. c. *theorems* of Θ so that the following is true:

(III) If Φ_1 and Φ_2 belong to Σ^*, then the conjuction $\Phi_1 \, . \, \Phi_2$ belongs to Σ^* too.

(IV) If Φ_1 belongs to Σ^* and the implication $\Phi_1 \supset \Phi_2$ is identical[2]), then Φ_2 belongs to Σ^*. (As a consequence, all the identical propositional functions belong to Σ^*.)

(V) If $\Phi = \Phi(x)$ with an arbitrary free individual variable x belongs to Σ^*, then the "generalized" propositional function $(x)\, \Phi(x)$ belongs to Σ^* too.

By an *axiomatizable* (in the finitary sense — no other will be considered here) theory $\Theta = (\Sigma, \Sigma^*)$ we understand one such that in Σ^* there is a propositional function without free variables, say Φ_0, so that Φ belongs to Σ^* if, and only if, $\Phi_0 \supset \Phi$ is identical. If Φ_0 has the form $A_1 \, . \, A_2 \ldots A_n$ (no A_i

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[1]) The fundamental metamathematical notion of a theory is essentially due to TARSKI, [T]. "Elementary" means "without predicate variables", "theory" means "the formalized and mathematically idealized side of a real mathematical theory". The s. c. absolute or semantical side of a theory (i. e. its relationship to its real objects) and the s. c. pragmatical side of a theory (i. e. the behaviours and psychical contents related to it), though perhaps more important than the formal one, remain disregarded here.

[2]) In the obvious sense of the lower predicate calculus (without identity; for identity see later).

Note that in this § the symbols like

$$x, y, z, \ldots, \Phi, \Psi, \ldots, A_1, A_2, \ldots, \Phi(x), \Psi(y), \ldots$$

etc. are *metamathematical* ("syntactical") *signs* (variables), i. e. the *letters* $x, y, z, \ldots$ are not individual variables, but signs for arbitrary individual variables, the *symbol* $\exists x$ means not "there is an x so that ...", but it exhibits the fact that we have to do with an existence-quantifier followed by an individual variable, the *letters* $\Phi, \Psi, \ldots$ are not propositional functions or abbreviations of them, but signs to denote arbitrary propositional functions, etc.

Of course, since individual variables, propositional functions, etc. become mathematical objects (in the sense of metamathematics as a theory of finite configurations of signs of the given symbolized mathematical theory), the letters $x, y, \ldots \Phi, \Psi, \ldots$ etc. are also mathematical variables (and moreover, e. g. variables of arithmetics of integers in the case of arithmetized („Gödelized") metamathematics).

Writing $x = y$ or $w = \xi$ we mean that the *letters* x and y (or w and ξ) denote the same individual variable (or constant), whereas writing x_0 ,=' y, we speak *about an identity-sign* lying between the individual variables x and y (see later).

being a conjunction) then Θ is said to be axiomatized with the axioms $A_1, A_2, \ldots,$ $\ldots, A_n$. (In the case of an axiomatizable Θ, the item (V) follows from the remaining items by the predicate calculus.)

The algebraic formulation.[3]) Let $\mathbf{L}_\Sigma$ be the s. c. *Lindenbaum algebra* of classes $[\Phi]$, $[\Psi]$, ... of logically equivalent and identity-free prop. functions Φ, Ψ, ... from Σ, i. e. the following Boolean algebra: Φ_2 belongs to the class $[\Phi_1]$ if and only if both the implications $\Phi_1 \supset \Phi_2$ and $\Phi_2 \supset \Phi_1$ are identical prop. functions (in the sense of the lower predicate calculus without identity),
$[\Phi] \cup [\Psi] = [\Phi \cup \Psi]$ (finite joins),
$[\Phi] \cap [\Psi] = [\Phi \,.\, \Psi]$ (finite meets),
$[\Phi]' = [\sim \Phi]$ (complement; $\sim$ is the negation sign),
$[\Lambda] = [\Phi \,.\, \sim \Phi] = 0$ (the zero of $\mathbf{L}_\Sigma$; Φ is an arbitrary, Λ an arbitrary contradictory prop. function from Σ),
$[V] = [\Phi \vee \sim \Phi] = 1$ (the unit of $\mathbf{L}_\Sigma$; Φ is an arbitrary, V an arbitrary identical prop. function from Σ);
$[\Phi] \subseteq [\Psi]$ (the lattice-ordering) is the case if, and only if, the implication $\Phi \supset \Psi$ is identical. In addition to the already stated finite operations, there are in $\mathbf{L}_\Sigma$ the following two infinite operations:[4])

I. $$\bigcup_\zeta \left[\Phi^*\begin{Bmatrix} x \\ \zeta \end{Bmatrix}\right] = [\exists x \Phi(x)]\,.$$

This is the s. c. *marked* (countably) *infinite join* (i. e. L. U. B.); the symbol $\begin{Bmatrix} x \\ \zeta \end{Bmatrix}$ denotes the replacement of the free individual variable x by an arbitrary (primitive) individual constant or variable ζ of Θ wherever x occurs in Φ; the asterisk denotes the previous convenient change of bounded individual variables if this is necessary in order to avoid ambiguity.

II. $$\bigcap_\zeta \left[\Phi^*\begin{Bmatrix} x \\ \zeta \end{Bmatrix}\right] = [(x)\, \Phi(x)]$$

— the s. c. *marked* (countably) *infinite meet* (i. e. G. L. B.) (with the preceding sense of the symbol).

It is essentially known that a Lindenbaum algebra can be characterized (independently of its construction by the predicate calculus, i. e. disregarding isomorphisms) as follows:[4])

(I) (The generalized σ-algebra.) A Boolean algebra $\mathbf{A}$ may be called a *generalized σ-algebra relatively to the family Ω* of the so called *marked* (multiple)

[3]) This formulation is also essentially due to Tarski [T]. See also [M I] and [K-M]. — So far as we have not the identity, we have not a possibility of introducing a non-primitive individual constant.

[4]) See [R I], [R II] and [R III] for more details.

sequences (more shortly: **A** is a $\Omega\sigma$-algebra) if the following requirements are satisfied:

(i) *(Complement-sequences.)*

If $\{a_{p,q,\ldots,t}\}_{p,q,\ldots,t=1}^{\infty} \in \Omega$, then $\{a'_{p,q,\ldots,t}\}_{p,q,\ldots,t=1}^{\infty} \in \Omega$.

(ii) *(Join- and meet-sequences.)*

a) If $\{a_{p,q,\ldots,t}\}_{p,q,\ldots,t=1}^{\infty} \in \Omega$ and $\{b_{p,q,\ldots,t}\}_{p,q,\ldots,t=1}^{\infty} \in \Omega$, then $\{a_{p,q,\ldots,t} \cup b_{u,v,\ldots,z}\}_{p,q,\ldots,t,u,v,\ldots,z=1}^{\infty} = \{c_{p,q,\ldots,t,u,v,\ldots,z}\}_{p,q,\ldots,t,u,v,\ldots,z=1}^{\infty} \in \Omega$.

b) The same for $\cap$ instead of $\cup$.

(iii) *(Diagonal sequences.)*

If $\{a_{n_1,\ldots,n_k}\}_{n_1,\ldots,n_k=1}^{\infty} \in \Omega$ (k is a fixed integer) and if p, q with $1 \leqq p < q \leqq k$ are fixed chosen integers, then putting $n_p = n_q = n = 1, 2, \ldots$ we get a further (marked) s. c. diagonal sequence

$$\{a_{n_1,\ldots,n_{p-1},n,n_{p+1},\ldots n_{q-1},n,n_{q+1},\ldots n_k}\}_{n,n_1,\ldots n_k=1}^{\infty} \in \Omega.$$

(iv) *(Cylindric sequences.)*

If $\{a_{n_1,\ldots,n_k}\}_{n_1,\ldots,n_k=1}^{\infty}$ (k is a fixed integer) and if p, r are fixed integers, $1 \leqq p \leqq k$, then fixing the value of n_p by r we get a further s. c. cylindric (marked) sequence

$$\{a_{n_1,\ldots,n_{p-1},r,n_{p+1},\ldots n_k}\}_{n,\ldots,n_{p-1},n_{p+1},\ldots n_k=1}^{\infty} \in \Omega .$$

(v) *(Constant sequences.)*

Any $\{a_{p,q,\ldots,t}\}_{p,q,\ldots,t=1}^{\infty}$ with fixed $a = a_{p,q,\ldots,t}$ $(p, q, \ldots, t = 1, 2, \ldots)$ belongs to Ω.

(vi) *(The marked infinite joins and meets.)*

a) Let $\{a_{n_1,\ldots,n_k}\}_{n_1,\ldots,n_k=1}^{\infty} \in \Omega$ (k fixed) be a marked sequence and let p with $1 \leqq p \leqq k$ be a fixed integer. Then to each fixed ordered $(k-1)$-tuple $n_1, \ldots, n_{p-1}, n_{p+1}, \ldots n_k$ of integers there exists the L. U. B. (in the sense of the lattice-ordering of **A**)

$$b_{n_1,\ldots,n_{p-1},n_{p+1},\ldots,n_k} = \bigcup_{n_p=1}^{\infty} a_{n_1,\ldots n_k}$$

and, moreover, the sequence $\{b_{n_1,\ldots,n_{p-1},n_{p+1},\ldots,n_k}\}_{n_1,\ldots,n_{p-1},n_{p+1},\ldots,n_k=1}^{\infty}$ belongs to Ω:

b) The same for $\cap$ instead of $\cup$ (and with G. L. B. instead of L. U. B.).

(**II**) (**The free generalized σ-algebra.**) Any Lindenbaum algebra $\mathbf{L}_\Sigma$ is a generalized σ-algebra relatively to the countable family Ω of multiple marked sequences of the form

$$\left\{\left[\Phi^*\left\{\begin{matrix} x, y, \ldots, z \\ \xi_p, \xi_q, \ldots, \xi_t \end{matrix}\right\}\right]\right\}_{p,q,\ldots,t=1}^{\infty},$$

$(\xi_p, \xi_q, \ldots \xi_t)$ run with $p, q, \ldots, t = 1, 2, \ldots$ over all the free individual variables and constants of Σ, $\Phi(x, y, \ldots, z)$ is any propositional function with the different free individual variables $x, y, \ldots, z$; but in addition, any $\mathbf{L}_\Sigma$ has the following characteristic property of *being free*:

An $\Omega\sigma$-algebra $\mathbf{A}$ is said *to be free with respect to* Ω^G if the family Ω includes an (at most countable) subfamily Ω^G of the s. c. *generating* (multiple marked) *sequences* so that the following is true:

(1) Any element of $\mathbf{A}$ occurs as a member in at most one of the sequences of Ω^G and, moreover, at most at one place in such a sequence.

(2) Ω is the least family with (i)—(vi) containing Ω^G as a subfamily — and $\mathbf{A}$ is the least generalized σ-subalgebra of $\mathbf{A}$ relatively to Ω.

(3) Every mapping of the set G of all the members of the sequences of Ω^G in any σ-algebra $\mathbf{B}$ can be extended to a σ-homomorphic mapping of $\mathbf{A}$ in $\mathbf{B}$. (G is said to be the set of free generators of the $\Omega\sigma$-algebra $\mathbf{A}$.)

It is not difficult to prove the following

Theorem: *Let* $\mathbf{A}_i$ *bc a free* $\Omega_i\sigma$*-algebra* ($i = 1, 2$) *with the generating subfamilies* Ω_i^G. *Let* Ω_i^G *contain* N_{1i} *simple,* N_{2i} *double,* ..., N_{ki} *k-tuple,* ... *sequences* ($i = 1, 2,$ $0 \leqq N_{ki} \leqq \aleph_0$).

Then $\mathbf{A}_1$ *is* σ*-isomorphic to* $\mathbf{A}_2$ *if, and only if,* $N_{11} = N_{12}$, $N_{21} = N_{22}, \ldots,$ $\ldots, N_{k1} = N_{k2}, \ldots$ (The mentioned σ-homomorphisms and σ-isomorphisms are meant in the sense of the marked joins and meets, of course.) Now, the abstract algebraic characterisation of Lindenbaum algebras can be stated as follows:

Any free $\Omega\sigma$*-algebra* $\mathbf{A}$, *the generating family* Ω^G *of which contains* N_1 *simple,* N_2 *double,* ..., N_k *k-tuple sequences, is* σ*-isomorphic with the Lindenbaum algebra* $\mathbf{L}_\Sigma$ *of any theory* ($\Theta = (\Sigma, \Sigma^*)$ *(without logical identity) with* N_1 *unary,* N_2 *binary,* ..., N_k *k-nary primitive predicate constants* ($0 \leqq N_k \leqq \aleph_0$).

We can and will limit ourselves to the case in which the mentioned characteristic sequence $N_1, N_2, \ldots, N_k$ is a finite sequence of (finite) integers.

Now, we are able to restate the *notion of a theory* (at the moment without the logical identity, but compare later) in algebraic terms.[5])

By an *ideal* $\mathbf{I}$ of a generalized σ-algebra $\mathbf{A}$ (with the family Ω of marked (sequences of elements of $\mathbf{A}$) we mean a nonvoid set of elements of $\mathbf{A}$ satisfying the following conditions:

(1) If a and b belong to $\mathbf{I}$, then $a \cap b$ belongs to $\mathbf{I}$.

(2) If a belongs to $\mathbf{I}$ and if $a \subseteq b$, then b belongs to $\mathbf{I}$.

(3) If a_i ($i = 1, 2, \ldots$) belongs to $\mathbf{I}$, then $\bigcap_i a_i$ belongs to $\mathbf{I}$-provided, of course, $\bigcap_i a_i$ is a marked (infinite) join.

[5]) These formulations are essentially due to Mostowski [M I].

It is easy to see that: If $\Theta = (\Sigma, \Sigma^*)$ is a theory, then all the classes of logically equivalent theorems of Θ form an ideal, say $\mathbf{I}_{\Sigma*}$, of the Lindenbaum (generalized free σ-)algebra $\mathbf{L}_\Sigma$. Conversely, every ideal $\mathbf{I}$ of $\mathbf{L}_\Sigma$ produces a set Σ^* of propositional functions, in fact the set-join of classes as elements of $\mathbf{I}$, so that (Σ, Σ^*) defines a theory Θ according to (I) (V).

The theory $\Theta = (\Sigma, \Sigma^*)$ is *consistent* (in the usual sense) if, and only if, $\mathbf{I}_{\Sigma*}$ is a proper ideal, i. e. if $\mathbf{I}_{\Sigma*} \neq \mathbf{L}_\Sigma$.

Remark. The stronger property of being ω-*consistent* (due to Gödel) can also be formulated algebraic by replacing the condition $\mathbf{I}_{\Sigma*} \neq \mathbf{L}_\Sigma$ by the following postulate:

(4) If the classes $\left[\Phi^*\begin{Bmatrix} x \\ \zeta \end{Bmatrix}\right]$ with a fixed x and for every individual constant ζ belong to $\mathbf{I}_{\Sigma*}$, then the class $[(x)\ \Phi(x)]'$ does not belong to $\mathbf{I}_{\Sigma*}$.[6])

It is easy to prove that: $\Theta = (\Sigma, \Sigma^*)$ is *complete* (in the usual sense, i. e. each propositional function or its negation is a theorem) if, and only if, $\mathbf{I}_{\Sigma*}$ is a *prime ideal,* i. e. the following condition holds:

(5) If $a \cup b$ is in $\mathbf{I}_{\Sigma*}$, then a or b is in $\mathbf{I}_{\Sigma*}$. The dual property of the ω-completeness (due to Tarski) is the requirement:

(6) If $[\exists x \Phi(x)]$ belongs to $\mathbf{I}_{\Sigma*}$, then there is an individual constant, say ζ, so that $\left[\Phi^*\begin{Bmatrix} x \\ \zeta \end{Bmatrix}\right]$ also belongs to $\mathbf{I}_{\Sigma*}$.

It is obvious that the ω-consistency together with the completeness implies the ω-completeness — and that the ω-completeness together with the completeness and with consistency implies the ω-consistency.

By a *factor algebra* $\mathbf{A}/\mathbf{I}$ of the generalized σ-algebra formed with the aid of the ideal $\mathbf{I}$ of $\mathbf{A}$ we mean, of course, the generalized σ-algebra of cosets produced by $\mathbf{I}$, a marked sequence of such cosets being one which possesses a marked choice sequence (of elements of $\mathbf{A}$).

If a generalized σ-algebra $\mathbf{B}$ is a σ-homomorphic image of another generalized σ-algebra $\mathbf{A}$, then there is an ideal $\mathbf{I}$ of $\mathbf{A}$ so that $\mathbf{A}/\mathbf{I}$ and $\mathbf{B}$ are σ-isomorphic. $\mathbf{I}$ is the set of the counterimages of $1 \in \mathbf{B}$ in the homomorphism in question. (This is the s. c. *first lemma on isomorphism for generalized σ-algebras.*)

The algebraic description of a formalized theory with identity. A theory $\Theta = (\Sigma, \Sigma^*)$ with the s. c. logical identity can be algebraically described as follows (as a theory in the previous identity-free sense):

1. Between the primitive predicates of Θ, there is an additional binary predicate constant, say ‚=‘; the corresponding formation of the Lindenbaum algebra $\mathbf{L}_\Sigma$ is as usual.

[6]) See [H-B] II, p. 274.

2. Between the theorems of Θ, i. e. in Σ^*, there are the following propositional functions:

$$\left.\begin{array}{l} (x)\,(x \,,\!=\!`\, x)\,, \quad (x)(y)(x \,,\!=\!`\, y \supset y \,,\!=\!`\, x)\,, \\ (x)(y)(z)((x \,,\!=\!`\, y \,.\, y \,,\!=\!`\, z) \supset x \,,\!=\!`\, z)\,, \\ \text{and all the prop. functions of the form} \\ (x)(y)\left((x \,,\!=\!`\, y \,.\, \Phi(x)) \supset \Phi^*\begin{Bmatrix} x \\ y \end{Bmatrix}\right), \end{array}\right\} \qquad (^+)$$

where Φ is an arbitrary prop. function containing x freely.

The corresponding property of the ideal $\mathbf{I}_{\Sigma^*}$ of $\mathbf{L}_\Sigma$ is obvious.

The introduction of a new (not primitive) individual constant, say ζ_1, in a theory with identity is obviously to be performed as follows:

Let us have in Σ^* of $\Theta = (\Sigma, \Sigma^*)$ a fixed theorem of the form

$$(\exists x \Phi(x)) \,.\, (y)(z)(\Phi(y) \,.\, \Phi(z) \supset y \,,\!=\!`\, z)\,.$$

Then adjoining the fixed sign ζ_1 to the given individual constants of Θ we extend Σ to Σ_1 and $\mathbf{L}_\Sigma$ to $\mathbf{L}_{\Sigma_1}$ (up to a trivial σ-isomorphism).

Now, adjoining any propositional function of the form

$$\Psi(\zeta_1) \equiv \exists u(\Psi(u) \,.\, \Phi(u))$$

to Σ^* we get Σ_1^* and from $\mathbf{I}_{\Sigma^*}$ we get $\mathbf{I}_{\Sigma_1^*}$, $\quad (\Sigma^* \subseteq \Sigma_1^*, \mathbf{I}_{\Sigma^*} \subseteq \mathbf{I}_{\Sigma^*_1})$.

It is noteworthy that then

$$\mathbf{L}_\Sigma / \mathbf{I}_{\Sigma^*} \cong \mathbf{L}_{\Sigma_1} / \mathbf{I}_{\Sigma^*_1}\,.$$

Moreover, it can be proved that to any theory with identity $\Theta = (\Sigma, \Sigma^*)$ there is another theory $\overline{\Theta} = (\overline{\Sigma}, \overline{\Sigma}^*)$ without identity so that $\mathbf{L}_{\overline{\Sigma}}$ can be σ-isomorphically immerged in $\mathbf{L}_\Sigma$ and the factor algebras $\mathbf{L}_\Sigma/\mathbf{I}_{\Sigma^*}$ and $\mathbf{L}_{\overline{\Sigma}}/\mathbf{I}_{\overline{\Sigma}^*}$ are σ-isomorphic. (Note that $\overline{\Theta}$ can essentially differ from the theory resulting from Θ by the simple omission of the identity-sign, the latter can be finitely axiomatisable while the former need not be.)

We need not enter into more details here, for the use of the logical identity in Gödel's formalized set theory (see [G]) can be avoided in a known manner[7]) as follows:

1. The axiom (theorem) of *extensionality* A 3 of [G] converts into the *definition of the identity — predicate*, in the form

$$X \,,\!=\!`\, Y \underset{\mathrm{df}}{\equiv} (z)(z \in X \equiv z \in Y)\,.$$

2. We add the *following basic substitution-axiom* (theorem) A* 3

$$(x)(y)(Z)((x \,,\!=\!`\, y \,.\, x \in Z) \supset y \in Z)\,.$$

[7]) Comp. [W-N].

Then the above "logical" theorems ($^+$) (on identity) become provable theorems [of such an axiomatic (formalized) set theory] and the described introduction of new individual constants is to be performed just as if the identity ‚=' were a primitive logical sign.

Hereby, the needed algebraic description of the notion of a theory is concluded.

Let us add some criticism. Of course, the described notions (of a theory) are not constructive metamathematical conceptions, since they involve a certain part of the intuitive set theory (of sets and of sets of sets of natural numbers, if we denumerate the signs of a formalized (investigated mathematical) theory).

But by specialization to concrete formalized theories, as a rule, these notions, especially that of the ideal of classes of theorems, become available to the recursive arithmetics. Moreover, when a concrete finite fragment of a given mathematical theory is examined then almost all the general metamathematical considerations can be eliminated by replacing metamathematical proofs and notions by the direct demonstration of concrete needed examples of formalized mathematical proofs.

In all cases, the described (algebraic) metamathematical notions may be understood as a "systematical heuristics" of the constructive metamathematics even by a rigorous constructivist. This remark holds for further algebraic terms of metamathematics as well.

Models and interpretations. In the present paper there is a frequent use of the *notion of a model* of a formalized set theory and of the closely related notion of the *interpretation* (of a formalized theory in another formalized theory).

We do not deal with the s. c. absolute (semantical) notion of the model of a nonformal axiomatic theory, i. e. with the model as a "nonvoid set of real things and of relations between them (in the sense of the nonformal set theory) which satisfy the axioms of the theory in question". Not denying the importance of this absolute notion of a model we are convinced that it cannot be examined by mathematical tools (of mathematical logic) alone. We thus limit ourselves to the s. c. relative and formalizable side of the notions of a model and of interpretation, as this is obvious in mathematical logic. We state these notions in extenso also in algebraic terms and also for nonaxiomatizable theories; this may be useful for further purposes.

(1) Let $\Theta_i \doteq (\Sigma_i, \Sigma_i^*)$, $\mathbf{L}_{\Sigma_i}, \mathbf{I}_{\Sigma_i^*}$ (for $i = 1, 2$) be respectively a theory, its Lindenbaum algebra and the related ideal (of classes of theorems). When there is no danger of ambiguity, let the logical constants and individual variables of both

the Θ_1 and Θ_2 be the same signs; especially, let x ,=' y be in Σ_1^* if, and only if, the same identity lies in Σ_2^*.

The theory Θ_1 will be called the *interpreting*, while Θ_2 the *interpreted theory*.

(2) Let $\varphi_1, \varphi_2, \ldots, \varphi_k, \ldots$ be the primitive (mathematical) predicate constants and let $\zeta_1, \zeta_2, \ldots, \zeta_m, \ldots$ be the primitive (mathematical) individual constants of the interpreted theory.

Let to any $\varphi_k(x, y, \ldots, t)$ (of Θ_2) correspond one-to-one a propositional function $\bar{\varphi}_k(x, y, \ldots, t)$ of Θ_2 (with the same free individual variables) and let to any ζ_m (of Θ_2) correspond one-to-one an individual constant $\bar{\zeta}_m$ of Θ_1.

(It has been noted that any introduction of a well defined individual constant does not change the factor algebra of the Lindenbaum algebra given by the ideal of classes of theorems of a theory — an obvious σ-isomorphism disregarded.)

We say that the φ_k and the ζ_m of Θ_2 are interpreted by the corresponding $\bar{\varphi}_k$ and $\bar{\zeta}_m$ of Θ_1.

(3) Let the following *condition of correctness of the interpretation* hold:

Every theorem Φ of Θ_2 goes into a theorem $\bar{\Phi}$ of Θ_1 whenever each primitive constant occuring in Φ has been replaced by its corresponding interpreting sign (in the previous sense, and under a suitable change of bounded individual variables in order to prevent possible ambiguity).

In this case we say that *the theory Θ_2 has been correctly interpreted in the theory Θ_1*.

Let us restate the concept of "correct interpretation" in algebraic terms:

Let $\bar{\mathbf{L}}_1$ denote the generalized σ-algebra as the subalgebra of $\mathbf{L}_{\Sigma_1}$ generated by the "interpreting" classes of the form $[\bar{\varphi}_k(\ldots)]$.

Note that the family of infinite marked sequences of elements of $\bar{\mathbf{L}}_1$ clearly is the minimal family with (i)—(vi) (of p. 326) containing all the sequences (of classes of equivalent propositional functions, i. e. of elements of $\mathbf{L}_{\Sigma_1}$) of the form

$$\left\{\left[\bar{\varphi}_k^*\left(\ldots\begin{Bmatrix}t\\ \xi_1\end{Bmatrix}\ldots\right)\right],\ \left[\bar{\varphi}_k^*\left(\ldots\begin{Bmatrix}t\\ \xi_2\end{Bmatrix}\ldots\right)\right]\ldots\right\} \quad \text{(see p. 326)}.$$

(Note further, that $\bar{\mathbf{L}}_1$ is not more free, in general.)

Then the set $\bar{\mathbf{L}}_1 \cap \mathbf{I}_{\Sigma_1^*} = \bar{\mathbf{I}}_1$ is an ideal of $\bar{\mathbf{L}}_1$ consisting of all the classes of logically equivalent propositional functions of Θ_1 formed with the help of interpreting signs and proved in Θ_1.

Now take account of the fact that $\mathbf{L}_{\Sigma_2}$ is a free generalized σ-algebra.

This means that the already described interpretation correspondence transfered to the related classes of logically equivalent propositional functions by the correspondence $[\varphi_k(\ldots)] \to [\bar{\varphi}_k(\ldots)]$ defines the s. c. *interpreting σ-homomorphic mapping*, say ι, of the algebra $\mathbf{L}_{\Sigma_2}$ into the algebra $\mathbf{L}_1$.

Clearly then the set of images $\iota''\mathbf{I}_{\Sigma_2^*}$ of elements of the ideal $\mathbf{I}_{\Sigma_2^*}$ is an ideal of $\bar{\mathbf{L}}_1$.

Now, we can state: *The interpretation $\varphi_k \to \bar{\varphi}_k$, $\zeta_m \to \bar{\zeta}_m$ of Θ_2 in Θ_1 is correct if, and only if,*

$$\iota''\mathbf{I}_{\Sigma_1^*} \subseteq \bar{\mathbf{L}}_1 \cap \mathbf{I}_{\Sigma_2^*} = \bar{\mathbf{I}}_1 .$$

Let this condition be fulfilled. Let $\bar{\Sigma}_1$ and $\bar{\Sigma}^*$ denote the sets of propositional functions with classes in $\bar{\mathbf{L}}_1$ and in $\bar{\mathbf{I}}_1$ respectively. (Of course, there is $\bar{\Sigma}_1 \subseteq \Sigma_1$, $\bar{\Sigma}_2^* \subseteq \Sigma_1^*$.)

Then we define: The ordered pair $(\bar{\Sigma}_1, \bar{\Sigma}_1^*)$ is the *model of the theory Θ_2 in the theory Θ_1*, defined by the *given correct interpretation*.

(This concept of a model differs somewhat from that of Mostowski (comp. [K.-M.] p. 258) but it seems to be in accordance with that of [G], where an often implicit use is made of the (undefined) termini such as "holding in the model", "an ordinal number of the model in the model" etc.).

The set $\bar{\Sigma}_1^*$ is said to be the *set of theorems of the model* $(\bar{\Sigma}_1, \bar{\Sigma}_1^*)$, $\bar{\mathbf{L}}_1$ is said to be *the algebra* and $\bar{\mathbf{I}}_1$ is said to be *the ideal* (of classes of theorems) *of it*, while $\bar{\mathbf{L}}_1/\bar{\mathbf{I}}_1$ is the s. c. *factor-algebra of* the model $(\bar{\Sigma}_1, \bar{\Sigma}_1^*)$.

There are two further important ideals given by the correct interpretation, i. e.

a) the set of counterimages $\mathbf{I}_2 = \iota^{-1}{}''\bar{\mathbf{I}}_1$ of $\bar{\mathbf{I}}_1 = \bar{\mathbf{L}}_1 \cap \mathbf{I}_{\Sigma_1^*}$. Of course, $\mathbf{I}_{\Sigma_2^*} \subseteq \mathbf{I}_2$, $\mathbf{I}_2$ being an ideal of $\mathbf{L}_{\Sigma_2}$.

b) the ideal $\mathbf{I}_1$ of $\mathbf{L}_{\Sigma_1}$ generated by the set $\iota''\mathbf{I}_{\Sigma_2^*}$. Of course, there is $\mathbf{I}_1 \supseteq \mathbf{I}_{\Sigma_1^*}$.

Denoting by Θ_1^* and Θ_2^* the theories with the given Lindenbaum algebras $\mathbf{L}_{\Sigma_1}$ and $\mathbf{L}_{\Sigma_2}$ respectively, but with the new ideals $\mathbf{I}_2$ and $\mathbf{I}_1$ respectively, we can call Θ_2^* the *secondary interpreted*, and Θ_1^* the *secondary interpreting theory*. *Θ_1^* is weaker than Θ_1, Θ_2^* is stronger than Θ_2.*[9])

Clearly $(\bar{\Sigma}_1, \bar{\Sigma}_1^*)$ is also a model of the secondary interpreted theory Θ_2^*. Analogously, the given interpretation induces a model, say $(\bar{\Sigma}_1, \bar{\Sigma}_1^{**})$, of Θ_2 in the secondary interpreting theory Θ_1^*.

Let us now define: *A correct interpretation and the related model* are said to be *true* if the *factor algebras* of the model and of the interpreted theory are *σ-isomorphic* (the isomorphism being induced by the interpreting homomorphism).

Then (by Lemma I on isomorphism) the following statements are easily proved:

[9]) In the sense of inclusion of corresponding ideals (the identity not excluded).

The model $(\bar{\Sigma}_1, \bar{\Sigma}_1^*)$ *is a true model of* Θ_2 *in* Θ_1 *if, and only if*, $\mathbf{I}_2 = \mathbf{I}_{\Sigma_2^*}$.

The secondary model $(\bar{\Sigma}_1, \bar{\Sigma}_1^{**})$ *is always a true model of the* (given) *interpreted theory* Θ_2.

The (given) *model* $(\bar{\Sigma}_1, \bar{\Sigma}_1^*)$ *is always a true model of the secondary interpreted theory* Θ_2^*.

In the present paper, we have the following special situations only:

Neither Θ_1 nor Θ_2 have a primitive individual constant.

Θ_i $(i = 1, 2)$ both have three primitive predicate constants, i. e. two unary, $\mathbf{M}_i(.)$ ("to be a set") and $\mathbf{Cls}_i(.)$ ("to be a class") — and one binary $(.) \in (.)$ (". belongs (= is an element of) to .").

(Note that $\mathbf{M}_i$ and $\mathbf{Cls}_i$ can be made superfluous by a slight modification of axioms of [G], see e. g. [M II], but we will maintain the original version of [G] since this simplification is unessential).

The ideals $\mathbf{I}_{\Sigma_i^*}$ $(i = 1, 2)$ always contain the classes of propositional functions of the form of axioms sub A, B, C of [G]. But note that while the same is true for the ideal $\mathbf{I}_2$, this need no longer be true for the ideal $\mathbf{I}_1$.

We say that the theories Θ_i $(i = 1, 2)$ under the already stated assumptions are *Gödelian set theories*. It is noteworthy that while the secondary interpreted theory is, of course, a Gödelian as well, the secondary interpreting theory need not be.

Sometimes, both the Θ_i will be axiomatic theories and to the basic axioms sub A, B, C some further axioms will be added, e. g. the axiom D or E (the strong Göde'ls axiom of choice) or the s. c. Gödel's axiom of constructivity (see [G]) or also ev. an axiom requiring the existence of predicative sets (of Russell), or the generalized Continuum-Hypothesis — etc.

In the case of the original Gödel's model Δ of [G], we have the following situation:

The interpreting theory Θ_1 (by Gödel called the "system Σ") is an axiomatic theory with axioms sub A, B, C, D of [G].[10])

The interpreted theory Θ_2 is an axiomatic theory with axioms sub A, B, C, D and with the s. c. axiom of constructivity V = L as an additional axiom.

The interpretations are as follows: $\mathbf{M}_2(X)(\mathbf{Cls}_2(X))$ of Θ_2 are respectively interpreted by the propositional functions expressing that "X is a constructive set (class) of Θ_1". $X \in_2 Y$ is interpreted by $X \in_1 Y$ with constructive X, Y.

It is noteworthy (and easy to see from [G]) that Gödel's Δ *is a true model of* Θ_2 *in* Θ_1.

In our § 2 we will encounter the following generalization of Gödel's situation:

Θ_1 is an arbitrary Gödelian set theory.

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[10]) With the already stated modification, if wished.

Θ_2 is an axiomatic set theory with the axioms sub A, B, C of [G] (but not stating D as an axiom) and with an appropriate generalization of the axiom of constructivity.

The interpretations are the same as in Gödel's Δ (with the difference, of course, that the concepts "constructive set (class)" must be now defined somewhat more generally, i. e. without use of the axiom D).

The resulting model *need no longer be true.*

In our § 3, using the results of § 2, we deal with quite another kind of models (and the need of the notions already introduced in § 1 will then be clearer).

Let us conclude this introductory § with a semantical note. Concerning the relation of the absolute and the relative notion of a model (of a formalizable theory), we can, of course, speak about the absolute model of the interpreted theory whenever the interpreting theory has such a model; but such an introduction of the "absolute" model of the interpreted theory, of course, is not "absolutely absolute" since it refers to another absolute model.

What is now the basic theory possessing a "really absolutely given" absolute model?

Some mathematicians and logicians are convinced that without any reference to empirical sciences, the s. c. intuitive theory of ("absolute") natural numbers is such a basic theory; the well known theorem of Skolem-Löwenheim would then ensure an absolute model of each formalizable consistent theory (in the domain of absolute natural numbers). The autor's opinion is that this point of view can be criticized and that the important but unclear notion of the absolute model cannot be clarified without the aid of real sciences on the one hand and scientific philosophy on the other.

2. The avoidance of the axiom D

In the preceding §, there has been stated what is the aim of § 2.

Let us consider an interpreting Gödelian set theory Θ_1.

First, following step by step Chapters I—IV of [G], we shall build a certain portion of Θ_1 as based on theorems of the form of the axioms sub A, B, C of [G] alone, i. e. without any use of the axiom D. (This possibility has been already stated without proof by Gödel himself in [G].)

After having introduced the suitably generalized notion of constructivity (of sets and classes) in Θ_1 (i. e. after this notion has been made independent of the axiom D) we can take the axiomatic set theory, say Θ_2, with axioms sub A, B, C and with the generalized axiom of constructivity alone, for an interpreted theory. Then we form the generalized Gödels model Δ_{Θ_1} (of constructive sets and classes of Θ_1) and we verify its main properties as in Chapters V—VII of [G].

Since it follows easily that (in Θ_2) the axiom D is a consequence of the generalized axiom of constructivity, the main result of Gödel (i. e. the deduction of $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ from A, B, C, D completed by the axiom of constructivity V = L) remains analogous in Θ_2 and in Δ_{Θ_1} as well.

Both of these preliminary results, i. e. the constructed part of a Gödelian set theory without axiom D and the slightly generalized Gödels model will be applied in the sequel.

For the sake of brevity, not every obvious step of this program will be performed in detail.

The first (immediate) consequence of D in [G] is the

Theorem 1.6. $(x) \sim (x \in x)$, (i. e. a set is never an element of itself).

This theorem cannot be admitted to be a theorem of Θ_1 so far it has not been deduced from the theorems (axioms) sub A, B ,C alone; it turns out later that this deduction is impossible.

The second (also immediate) consequence of D in [G] is the

Theorem 1.7. $\sim (\exists x, y)(x \in y)(y \in x)$, (i. e. there are no two sets so that the first is an element of the second, whereas the second is an element of the first).

The same remark as to 1.6 applies to 1.7.

Hereby the examination of Chapter I of [G] is finished, since other theorems and definitions of this chapter do not depend on axiom D.

Let us proceed to Chapter II of [G].

The next theorem proved with use of D or of 1.6 or of 1.7 is the following general metamathematical *existence theorem* M 1:[11])

If the sign $\varphi(x_1, \dots x_n)$ denotes a primitive propositional function (without quantified class-variables, see [G]) *not containing free set variables different from $x_1, \dots x_n$* (not necessarily containing all of them), *then there exists a class A so that for arbitrary sets $x_1, \dots x_n$, the prop. function ,$\langle x_1, \dots x_n\rangle \in A$' and the prop. function denoted by the sign $\varphi(x_1, \dots x_n)$ are equivalent* (i. e. their equivalence is a theorem).

Surveying the proof given in [G], we see what follows:

Of the Theorems D, 1.6 and 1.7, Theorem 1.6 alone has been used exactly once, i. e. in the first step of the inductive argument of the proof — and concretely in the exclusive case of $\varphi(x_1, \dots x_n) =$,$x_r \in x_r'$ $(1 \leq r \leq n)$ only. Then the void class $\emptyset$ is taken for A of the theorem, in view of 1.6 and Def. 2.1.

[11]) In its original form M 1 merely is a theorem-scheme (since '$\varphi(x_1, \dots, x_n)$' is not a sign of the theory, but of the metatheory).

In the remaining cases of $\varphi(x_1, \ldots x_n) =$ ‚$x_p \in x_q$ with $p \neq q$, $1 \leqq p \leqq n$, $1 \leqq q \leqq n$, the two subcases $p < q$, $q < p$ are carried out with the help of axioms sub B (with B 1 mainly) and without any use of D or 1.6 or 1.7.

But consider that x_p ‚=‘ x_q with $p \neq q$ has never been excluded, and moreover, that in B 1 of [G], i. e. in

$$\exists A(x, y)(\langle xy \rangle \in A \equiv x \in y)$$

there is no exclusion of x ‚=‘ y. Hence the exclusivity of the case $p = q$ in

$$\varphi(x_1, \ldots x_n) = ,x_p \in x_q` \quad (1 \leqq p \leqq n,\ 1 \leqq q \leqq n)$$

must be apparent only.

Indeed, let the case $p \neq q$ be the only initial case of the inductive argument of the proof of M 1 of [G]. Then the case $p = q$ is automatically carried out in a further step of the induction, namely with

$$\varphi(x_1, \ldots, x_n) = ,(x_p \in x_q) \,.\, [(x)(x \in x_p \equiv x \in x_q)]`$$

using the equivalence (in view of A 3)

$$[(x_p \in x_q) \,.\, (x)(x \in x_p \equiv x \in x_p)] \equiv x_p \in x_p \,.$$

Hereby, theorem M 1 remains true for our Θ_1.

Theorems M 2—M 6 then follow by purely metamathematical reasons and remain valid for our Θ_1.

Examining in detail the remaining text of Chapter II of [G] we see that, of the statements D, 1.6, 1.7, the Theorem 1.6 alone has been used once only, namely in the proof of Theorem 5.31 $\sim \mathbf{M}(V)$ (i. e. the universal class V of all sets is not itself a set).

Let us give *another proof* not using D or 1.6 or 1.7.

Suppose $\sim \mathbf{M}(V)$ is not a theorem of Θ_1. Then $\mathbf{M}(V)$ would be a theorem of a consistent (Gödelian set) theory $\hat{\Theta}_1$ stronger than Θ_1.

Since $B \subseteq V$ for every class B, hence every class is a set in $\hat{\Theta}_1$, by theorem 5.12 (proved in [G] without any use of D or 1.6 or 1.7), i. e. proper classes do not exist in $\hat{\Theta}_1$.

Now, the class P defined by the equivalence $z \in P \equiv \sim (z \in z)$ exists in $\hat{\Theta}_1$ in view of the already proved theorem M 1.

But since $\mathbf{M}(P)$ holds in $\hat{\Theta}_1$, hence setting $z = P$ we get $P \in P \equiv \sim(P \in P)$ i. e. the well known s. c. Russell's contradiction. (But compare Theorem IV.) This proves Theorem 5.31. (Note that no use is made of the consistency of the axioms sub A, B, C of [G].)

Hereby, all the theorems (and the related definitions) of Chapters I and II of [G] can be seen as valid in our Θ_1.

Proceeding to Chapter III of [G], we first need a new, more general notion of ordinal (and of ordinal number) than the Def. 6.6 of [G] gives, i. e. we need a formulation not based on the axiom D, or on 1.6 or on 1.7.

This is reached by the following essentially well known **Definition I** (comp. e. g. [K-M], pp. 247—):

In Θ_1, a class X is said to be an *ordinal*, if the following conditions a:, b:, c: are satisfied:

a: If $x \in X$, then $x \subseteq X$

b: If $x \in X$, $y \in X$, then at least one of the cases $x = y$ or $x \in y$ or $y \in x$ is true (in Θ_1).

c: To each set $x \neq \emptyset$ with $x \subseteq X$ there exists a set y so that $y \in x$ and $x \,.\, y = \emptyset$ is true (in Θ_1).

In order to complete the definition, let us add the following conventions:

d: (The same as 6.61 of [G]). An ordinal which is a set is called an *ordinal number*.

e: (The same as 6.62 of [G].) The *class of all the ordinal numbers* (existing by Theorem M 1) may be denoted by the sign *On*; i. e. $x \in On \equiv x$ is an ordinal number in the sense of the previous items a: b: c. d:.

Let us use in Θ_1 all the definitions and signs of [G] based on Definitions 6.61 and 6.62 alone.

It is easy to see the equivalence of Definition I and of Definition 6.6 of [G] when the axiom D is assumed in the theory Θ_1 in question.

Not so evident is the fact that Definition I suffices to ensure, in view of A, B, C of [G] alone, all the theorems on ordinals of Chapter III of [G] needed to the notion of constructivity and to the construction of the (generalized) model Δ of [G].

This may be shown in the sequel; unnecessary lemmas of [G] will, of course, be disregarded.

Theorem 6.7. *Let X be an ordinal. Then the following statements are theorems of* Θ_1.

1: *X is well ordered* (in the usual sense) *by the relation* $\in$.

2: *Every set u with* $u \in X$ *is identical with the set of all the sets x preceding* (in the sense of 1:) *the u.*

Proof. Suppose $X \neq \emptyset$ for the case $X = \emptyset$ is trivial.

(i) There is $\sim (x \in x)$ for each $x \in X$, since otherwise $x \in x$, i. e. $\{x\} \subseteq x \subseteq X$ (by a:) for a certain $x \in X$, which would give $\emptyset \neq \{x\} = y \,.\, \{x\}$ for every $y \in \{x\}$ — in contradiction to c:

Hence the relation $\in$ is irreflexive on any ordinal X.

(ii) If $x \in X$, $y \in X$, then $x \in y$ is incompatible with $y \in x$. Indeed, otherwise by a: $\{x\} \subseteq y \subseteq X$ and $\{y\} \subseteq x \subseteq X$ would imply $\{x\} \subseteq y \,.\, \{x\}$ and $\{y\} \subseteq x \,.\, \{xy\}$, (by meeting with $\{xy\}$ both the inclusions).

Hence $\emptyset \neq z \,.\, \{xy\}$ for each $z \in \{xy\} \subseteq X$ — in contradiction to c:

Therefore by b: we have the trichotomy: either $x \in y$ or $y \in x$ or $x = y$ whenever $x \in X$, $y \in X$ and X is an ordinal, in view of (i) and (ii).

(iii) Suppose $x \in X$, $y \in X$, $z \in X$, $x \in y$, $y \in z$, where X is an ordinal.

By (ii), $x = z$ is excluded; we shall exclude $x \in z$ too. Indeed, $z \in x$ would lead to the following contradiction:

Form the set $\{x\} + \{y\} + \{z\} = \{xyz\} \subseteq X$.

By supposition, we have $\{x\} \subseteq y$, $\{y\} \subseteq z$, $\{t\} \subseteq x$, whence

$$\emptyset \neq \{z\} = \{z\} \,.\, \{xyz\} \subseteq x \,.\, \{xyz\} ,$$
$$\emptyset \neq \{x\} = \{x\} \,.\, \{xyz\} \subseteq y \,.\, \{xyz\} ,$$
$$\emptyset \neq \{y\} = \{y\} \,.\, \{xyz\} \subseteq z \,.\, \{xyz\} .$$

Therefore $u \,.\, \{xyz\} \neq \emptyset$ for an arbitrary $u \in \{xyz\}$ in contradiction to c:

Hence the relation $\in$ is transitive on X.

Hereby, the ordinal X is shown to be ordered by the relation $\in$.

From a: (of Definition I) we immediately infer (by M 1) the statement 2: of the theorem.

In order to finish the proof of the well-ordering of X by $\in$, suppose $\emptyset \neq Y \subseteq X$. Take an arbitrary $y \in Y$. If $y \,.\, T = \emptyset$ then all is proved. Let $y \,.\, Y \neq \emptyset$. Since $y \,.\, Y$ is a set (on account of theorem 5.12 holding in Θ_1 including its proof), there exists, by c:, a set z with $z \in y \,.\, Y$, $z \,.\, y \,.\, Y = \emptyset$. By the proved 2: of the theorem, it is clear that this z is exactly the first z with $z \in Y$ in the sense of the ordering in question.

Hereby, Theorem 6.7 is proved in Θ_1.

Corrollary of Thm. 6.7. *Every element of an ordinal is an ordinal number.*

Theorem 7.12. *If X and Y are ordinals, then exactly one of the cases $X \in Y$, $X = Y$, $Y \in X$ is true.*

Proof. (i) At least one of the cases must occur:

Clearly $X \,.\, Y$ is an ordinal by Definition I.

By B 2, B 3, $X = X \,.\, Y + (X - Y)$ with disjoint summands. Without loss of generality, suppose $X - Y \neq \emptyset$ if $X \neq Y$. [Otherwise (i) is proved.]

Then the first $y \in X - Y$ is an ordinal number so that $y = X \,.\, Y$. Writing $Y = X \,.\, Y + (Y - X)$ with disjoint summands, we see that $Y - X = \emptyset$ since otherwise the first $z \in Y - X$ would be identical with the already defined $y \in X - Y$, which contradicts the relation $(Y - X) \,.\, (X - Y) = \emptyset$.

Hence $Y = X \,.\, Y = y$ is an ordinal number and $y = Y \epsilon X$ is true in Θ_1.

(ii) At most one of the cases occur; this is clear by thm 6.7.

Theorem 7.14. *The relation $Y \subseteq On$ holds for every ordinal Y.*

Proof. The corrollary of Thm. 6.7.

Theorem 7.16. *The class On is itself an ordinal.*

Proof. The requirement a: of Def. I is given by Thm. 7.14. The requirement b: of Def. I is given by Thm. 7.12. The requirement c: follows by the argument used at the end of the proof of Thm. 6.7.

Theorem 7.161. *The class On* (and hence any class of ordinal numbers) *is well ordered by the relation* ϵ.

Proof. Theorems 7.16 and 6.7.

Remark. By 7.161, the s. c. principle of transfinite induction is ensured in Θ_1 if we understand the proof by transfinite induction to be a reductio ad absurdum of the supposed existence of an ordinal number violating the statement to be proved.

A corresponding remark is valid concerning the principle of transfinite construction in Θ_1. (We do not state them explicitly — comp. 7.5 of [G].)

Theorem 7.17. *The class On is a proper class* (i. e. it is not a set.)

Proof: If *On* were a set, then by Thm. 7.16 it would be an ordinal number, whence $On \epsilon On$, which is impossible by 7.16.

Theorem 7.2. *On is the only ordinal not being an ordinal number.*

Proof. If $X \neq On$ is an ordinal, then $X \epsilon On$ by Thms. 7.12 and 7.16, since $On \epsilon X$ is excluded by Thm. 7.17 (and by A 2). Therefore X is an ordinal number, q. e. d.

Observing now which of the statements about ordinals is in fact used in the sequel of [G], we see that these are the already (in Θ_1) proved theorems alone.

Especially, the definition of Gödel's "enumeration-function" F and the related definitions 9.4 and 9.41 (of "constructive set" and of "constructive class") remain valid in Θ_1 and the whole following construction of Gödel's model Δ (see chp V) can be reproduced word for word in Θ_1 and gives the model called Δ_{Θ_1}.

Now, take the axiomatic set theory, say Θ_2, with axioms sub A, B, C and with the axiom of constructivity (in the already generalized sense, see Defs. 9.4 and 9.41 of [G]) as an additional axiom.

The verification (in Θ_1) of axioms of Θ_2 for the (interpreting) termini of Δ_{Θ_1} (the interpretation being that of [G]) remains also almost exactly the same as in Chapter VI of [G], the axiom D disregarded, as far as it is not supposed in Θ_1.

The only difference is in the verification of the (generalized) axiom of constructivity; more precisely, we have only to give another proof of the decisive theorem 11.31, as stated as follows.

Theorem 11.31. *The "class" On_{Δ_Θ} of "constructive ordinal numbers" (i. e. of "ordinal numbers" of the model Δ_Θ in this model) equals to the class On of the original ordinal numbers of Θ, i. e. $On_{\Delta_\Theta} = On$ holds in Θ.*[12])

Proof. Denoting (as in [G]) constructive sets and classes with the bar, we have the following definition of the class On_{Δ_Θ} (see Def. I).

A constructive class $\overline{X}$ by definition is a constructive ordinal if the following implications can be proved in Θ:

a_{Δ_Θ}: $\bar{x} \in \overline{X} \supset \bar{x} \subseteq \overline{X}$.

b_{Δ_Θ}: $\bar{x}, \bar{y} \in \overline{X} \supset (\bar{x} \in \bar{y}) \vee (\bar{y} \in \bar{x}) \vee (\bar{x} = \bar{y})$.

c_{Δ_Θ}: $\emptyset \neq \bar{x} \subseteq \overline{X} \supset \exists \bar{y}[(\bar{y} \in \bar{x})(\bar{x} \,.\, \bar{y} = \emptyset)]$.

Then $z \in On_{\Delta_\Theta} \equiv$ (z is a constructive ordinal number) $\equiv$ (z is a set and a constructive ordinal) (whence $z = \bar{z}$ is a constructive set, see [G]).

First, prove $On_{\Delta_\Theta} \subseteq On$: Indeed, the conditions a_{Δ_Θ}: and b_{Δ_Θ}: imply the corresponding conditions with arbitrary x and y instead of $\bar{x}$ and $\bar{y}$, since the implicans in both cases is wrong if x or y respectively are nonconstructive sets.

In order to verify the condition corresponding to c_{Δ_Θ}: though with x, y instead of $\bar{x}, \bar{y}$, we take an y with $y \in x$ and with as small an α as possible so that $F'\alpha = y$. (Such an α exists, since by $x \subseteq \overline{X}$ every element of x is constructive.) Then clearly $x \,.\, y = \emptyset$, since otherwise there would be a constructive u with $u \in x \,.\, y$ and therefore with $F'\beta = u$, $\beta < \alpha$, $u \in x$.

Second, suppose $On_{\Delta_\Theta} \subset On$ and draw a contradiction.

Let ν be the smallest nonconstructive ordinal in the class $On - On_{\Delta_\Theta}$. Since every β with $\beta \in \nu$ is constructive, we get $\nu = On_{\Delta_\Theta}$. Hence On_{Δ_Θ} is a set. This leads to a contradiction, since (by § 2 reproduced in the model Δ_Θ) On_{Δ_Θ} needs to be a "proper class" (in the sense of the model Δ_Θ) and therefore a proper class (in the sense of Θ) also.

The (slightly generalized) theorem 11.31 of [G] is thus proved.

In order to verify $V = L$ (i. e. the generalized axiom of constructivity) for Δ_Θ, we have only to observe that Gödel's generalized function F_{Δ_Θ} (as redefined now in the model Δ_Θ) indeed does not differ from the original F (as defined in the theory Θ).

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[12]) The original version of 11.31 uses the notion of "to be absolute" which is not needed explicitely here.

Hence the (generalized) axiom of constructivity is a theorem of the model Δ_{Θ}, q. e. d.

Now it is easy to prove the following

Theorem I. *The axiom* D *is a consequence of the axioms* sub A, B, C *and of the (generalized) axiom of constructivity.*

Proof. Let D not hold in a Gödelian set theory Θ (no matter whether axiomatizable or not) and let the generalized axiom of constructivity (in the already described sense) be true in Θ.

Then there exists a constructive class $A \neq \emptyset$ so that for each $u \in A$ we have $u \, . \, A \neq \emptyset$.

By the (generalized) axiom of constructivity, there exist a set v and a smallest ordinal number α so that $F'\alpha = v \in A$, where F is the (generalized) Gödel's function (of [G], see 9.3).

Since $v \, . \, A \neq \emptyset$, there exists a further set w and an ordinal number β as small as possible, so that $F'\beta = w \in v \, . \, A$.

In view of Theorem 9.5 (valid in Θ) by $w \in v$ we get $\beta < \alpha$ in contradiction with the definition of α.

Hereby the model Δ_{Θ_1} satisfying the (generalized) axiom of constructivity satisfies the axiom D too.

The model Δ_{Θ_1}[13]) *need no longer be a true model* of Θ_2 (in the sense of our § 1), since e. g. if there are inaccessible ordinals in the interpreting Θ_1, then, as it is easy to see, there are inaccessible ordinals in Δ_{Θ_1} as well, whereas in the interpreted Θ_2 no inaccessible ordinals need occur. (Indeed the hypothesis of inaccessibility is independent of axioms A, B, C completed by the axiom of constructivity; this is easy to see from [K]. Of course, to this purpose we have assumed the consistency of this hypothesis.) Hence we conclude (in view of [G]) with

Theorem II. *The (generalized) Gödel's model Δ_{Θ_1} can be formed in an arbitrary Gödelian set theory Θ_1. In Δ_{Θ_1}, axioms sub* A, B, C, D, E *and the Generalized Continuum Hypothesis hold.*

Corrollary. *If the axioms sub* A, B, C *are consistent, then they remain so after axiom* D, *Gödel's* (strong) *axiom of choice* E *and the Generalized Continuum Hypothesis have been added.*

(This includes the consistency of the axiom D with A, B, C (see [N]) due to v. Neumann.)

Hereby, our preliminary considerations are completed. Let us return to the main matter of the present paper.

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[13]) With the ideal $\bar{\mathbf{L}}_1 \cap \mathbf{I}_{\Sigma_1^*}$ of classes of theorems (of the model), see § 1.

3. Index-models (T-models) and their first applications

Stating the concept of the model of a (formalized) set theory, we have limited ourselves to both the interpreting theory Θ_1 and the interpreted theory Θ_2 as Gödelian set theories.

This limitation may hardly cause any essential loss of generality in formalizable set theory.

The only essential (but very natural) limitation we assume lies in that, forming the predicate $\tilde{\mathbf{M}}(.)$ (of "to be a set of the model" in question), we will suppose there is (in the interpreting theory Θ_1) a class C so that

$$\tilde{\mathbf{M}}(X) \equiv X \in {}_1C \quad (\text{in } \Theta_1) .$$

Under these assumptions, a certain "normalization" of models is possible i. e. a *reduction of any model to a model of certain standard type* to be called an *index-model*. Let us briefly trace this reduction.

Suppose $\tilde{\mathbf{M}}$, $\tilde{\mathbf{C}}ls$, $\tilde{\in}$ are the predicate constants (of the interpreting theory Θ_1) giving the interpretation of the primitive predicates $\mathbf{M}_2$, $\mathbf{C}ls_2$, $\in_2$ of the interpreted theory Θ_2. Suppose C is the class of all the "sets" of the model, i. e. $\tilde{\mathbf{M}}(X) \equiv X \in_1 C$.

Let us distinguish any conception of the model in question by the sign ‚$\sim$' from the same conception of Θ_1 (or of Θ_2); we thus write

$$X \in_1 C \equiv X \tilde{\in} \tilde{V} \equiv \tilde{\mathbf{M}}(X) .$$

($\tilde{V}$ is the "universal class" *of* the model *in* the model.)

By our supposition, every "set" of the model is a set of the interpreting theory. (We can always assume that every "class" of the model is always a class of the interpreting theory.)[14]) Moreover, by the satisfaction of B 1 in the model, the predicate $\tilde{\in}$ (of "to be an element" in the model) can be seen as a relation in the interpreting theory, i. e. we can write $\tilde{\in} \subseteq C \times C$ in Θ_1 provided $\tilde{\in}$ has been limited to "sets" of the model.

Now, to each "class" X of the model (i. e. with $\tilde{\mathbf{C}}ls(X)$) denote by X^* the class of all the $z \in C$ with $z \tilde{\in} X$; especially $\tilde{V}^* = C$.

Then we get (by Thm. M 1 and by the extensionality of $\tilde{\in}$) a *one-to-one mapping, say M*, of the class C on to a certain subclass $\tilde{C} = M''C$ of the potency class $\mathbf{P}(C)$, with $M'x = x^*$. Clearly then

$$X \tilde{\subseteq} Y \equiv X^* \subseteq Y^*$$

14) For the notion "to be a class" is essentially superfluous (comp. p. 333) since in [G], every individuum to be considered is a class.

(provided $\tilde{\mathbf{Cls}}(X)$, $\tilde{\mathbf{Cls}}(Y)$); moreover, writing $T = M^{-1}$, we get $T''\tilde{C} = C$ and

$$X \tilde{\epsilon} Y \equiv X \epsilon Y^* \equiv T'X^* \epsilon Y^*$$

whenever $X \epsilon C$, $\tilde{\mathbf{Cls}}(Y)$.

Therefore let us introduce the following new interpretation:

$$\mathbf{M}_T(Z) \equiv Z \epsilon \tilde{C}\,, \quad \mathbf{Cls}_T(Z) \equiv (Z \subseteq C)\,.\,\tilde{\mathbf{Cls}}(T^{-1''}Z)\,,$$
$$X \epsilon_T Y \equiv \mathbf{M}_T(X)\,.\,\mathbf{Cls}_T(Y)\,.\,(T'X \epsilon Y)\,.$$

Clearly $\mathbf{Cls}_T(Z) \supset Z \subseteq C$, $\mathbf{M}_T(Z) \equiv Z \epsilon_T C$, and we repeat with emphasis that $\tilde{C} \subseteq \mathbf{P}(C), T''\tilde{C} = C$.

It can be proved that *if* $\tilde{\mathbf{M}}$, $\mathbf{Cls}$, $\tilde{\epsilon}$ *give a correct interpretation of* $\mathbf{M}_2$, $\mathbf{Cls}_2$, ϵ_2 (of the theory Θ_2 in the theory Θ_1) *then the corresponding* $\mathbf{M}_T$, $\mathbf{Cls}_T$, ϵ_T *do so as well.*

Moreover, the given model and the already introduced model can be seen to be *equivalent* in the sense of that the it *corresponding factor-algebras are σ-isomorphic.* — Hence we can call the new model the *T-reductum* of the given one. (Note that "isomorphism between two models" (in the sense of a function) can be defined only when all the "classes" of the models are sets of the interpreting theory; therefore "equivalent" means not "isomorphic".)

We do not enter into the exact treatement of these statements here, since they play, in the sequel, a heuristic role only in that they show the great generality of models of a certain relatively simple kind; let us define this kind of models explicitely:

Definition II. a) A model $(\tilde{\Sigma}, \tilde{\Sigma}^*)$ of a Gödelian set theory Θ_2 in another Gödelian set theory Θ_1, given by the predicates $\mathbf{M}_T$, $\mathbf{Cls}_T$, ϵ_T as interpreting the $\mathbf{M}_2$, $\mathbf{Cls}_2$, ϵ_2 (of theory Θ_2) is said to be an *index-model*, or (more precisely) a *T-model* if the following conditions hold:

There is in Θ_1 a class $\tilde{C}$ and a one-to-one mapping (-class) T of $\tilde{C}$ on to a further class C so that $C \subseteq \mathbf{P}(C)$ and $\mathbf{M}_T(X) \equiv X \epsilon \tilde{C}$, $\mathbf{Cls}_T(Y) \supset Y \subseteq C$, $X \epsilon_T \epsilon_T Y \equiv (T'X \epsilon Y)\,.\,\mathbf{M}_T(X)\,.\,\mathbf{Cls}_T(Y)$.

The class C is called the *class of indices*, each $T'\,x \epsilon C$ is called the *index* of the "set" x (of the T-model).

b) A T-model is said to be *complete*, if $\mathbf{Cls}_T(Y) \equiv Y \subseteq C$ and $\tilde{C} = \mathbf{P}(C)$.

In the rest of this paper, we shall mainly be concerned with complete index-models. For deeper questions, of course, incomplete index-models need to be considered;[15]) this is a further task.

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[15]) E. g. Gödel's Δ_{Θ_1} (of § 2 of this paper, in general) is an incomplete T-model with the (partial) identity-mapping taken for the T, with $C = \tilde{C} = L$ (= the class of constructive sets) and with $Cls_T(Y) \equiv (X)(X \epsilon L \supset X\,.\,Y \epsilon L)\,.\,(Y \subseteq L)$.

Theorem III. *Let Θ_1 be an arbitrary, Θ_2 an axiomatic Gödelian set theory with the axioms sub* A, B, C *alone.*

Let T in Θ_1 be a one-to-one mapping of the potency-class $\mathbf{P}(C)$ on to the proper class C (i. e. $\mathbf{Pr}(C)$ is assumed).

Put

$$\mathbf{M}_T(X) \equiv X \in \mathbf{P}(C) , \quad \mathbf{Cls}_T(Y) \equiv Y \subseteq C ,$$
$$X \in_T Y \equiv (T'X \in Y) . (\mathbf{M}_T(X)) . (\mathbf{Cls}_T(Y)) .$$

Then $\mathbf{M}_T$, $\mathbf{Cls}_T$, $\in_T$ give a correct interpretation of the primitive predicates $\mathbf{M}_2$, $\mathbf{Cls}_2$, $\in_2$ of Θ_2, i. e. we get a complete T-(index) *-model of Θ_2 in Θ_1, say Δ_T. If the axiom of choice* E *is assumed in Θ_2, then it is satisfied in the index-model in question too.*

Proof. I. The following almost trivial (metamathematical) remark may be useful and often tacitly used:

Let $\Phi(x)$ be a propositional function of Θ_1 with the free set-variable 'x'. Then we have in Θ_1 the following equivalences (as theorems):

$$(x)(x \in \mathbf{P}(C) \supset \Phi(x)) \equiv (x)[x \in C \supset \Phi(T'x)] ,$$
$$\exists x(x \in \mathbf{P}(C) . \Phi(x)) \equiv \exists x((x \in C) . \Phi(T'x)) .$$

In words: A propositional function with quantifiers relativized to the class $\mathbf{P}(C)$ can be replaced without loss of equivalence by the same propositional function with quantifiers relativized to the class C if each of the bound variables ζ has been simultaneously replaced by the term $T'\zeta$ (in the scope of the quantifier in question) — and vice versa.

II. Proceeding in the verification of the axioms sub A, B, C in their ordering in [G] we tacitly introduce notions and symbols of the model based on the already verified axioms, by replacing $\mathbf{M}_2$, $\mathbf{Cls}_2$, $\in_2$ by $\mathbf{M}_T$, $\mathbf{Cls}_T$, $\in_T$ respectively. We distinguish these notions of the model by the subscript '$_T$' from the corresponding notions of Θ_2 (and of Θ_1).

III. The verification of the axioms. Axioms A 1, A 2 are trivially fulfilled in Δ_T. The axiom A 3 or A′ 3 (of extensionality) is immediately verified since T is a one-to-one mapping. The pair-set axiom A 4 for Δ_T is the following theorem of Θ_1:

$$(x)(y) \exists z(u)[T'u \in z \equiv (u = x \vee u = y)]$$

with the set variables limited to the class $\mathbf{P}(C)$.

Putting $z = \{T'xT'y\}$ we prove this theorem in view of A 4 in Θ_1 immediately.

Hence we denote by

$$\{xy\}_T = \{T'xT'y\} \quad (x \in \mathbf{P}(C), y \in \mathbf{P}(C))$$

the "pair-set" of the model Δ_T.

Especially, $\{xx\}_T = \{x\}_T = \{T'x\}$.

Further by

$$\langle xy\rangle_T = \{\{x\}_T\{xy\}_T\}_T = \{T'\{T'x\}\ T'\{T'xT'y\}\}$$

we denote the "ordered pair" of Δ_T with the uniquely determined ordered pair $\langle xy\rangle$ (of Θ_1) (by theorem 1.13 of [G] based on the axioms sub A alone).

It is obvious that the equivalence $X \subseteq_T Y \equiv X \subseteq Y$ holds in Θ_1 whenever the left side is meaningful, i. e. whenever $X \subseteq C$, $Y \subseteq C$. (Therefore we shall omit the subscript '$_T$' in any "inclusion" of the model.) More generally, if in this sense a certain conception of the model Δ_T will be equivalent to the corresponding conception of Θ_1 (with unchanged free variables), then we say that such a conception is *invariant*[16]); denoting such conceptions (of the model) we can — if no danger of ambiguity arises — omit the subscript '$_T$'. — So e. g. the concept of "being an element" is not invariant, whereas the concept of "being a subclass" is invariant; "being a set" and "being a class" are also invariant concepts. The concepts based on inclusion (as e. g. the disjunctivity of two classes and the property of being empty) obviously are invariant; therefore $\emptyset_T = \emptyset$. But e. g. "ordinal numbers" are not invariant (in general).

Let us return to the verification of axioms sub B.

B 1 (the axiom of the ϵ-relation) in Δ_T changes into the theorem

$$\exists C_T(x)(y)[T'\langle xy\rangle_T \in C_T \equiv T'x \in y]$$

with the limitations $C_T \subseteq C$, $x \in \mathbf{P}(C)$, $y \in \mathbf{P}(C)$ in Θ_1.

It is proved on setting

$$z \in C_T \equiv \exists x\, \exists y[(x \in \mathbf{P}(C))\,.\,(y \in \mathbf{P}(C))\,.\,(T'x \in y)\,.\,(z = T'\langle xy\rangle_T)],$$

where C_T exists by Theorem M 3 of [G].

B 2 (the axiom of the class-meet) is obviously fulfilled in Δ_T; moreover $A\,._T B = A\,.\,B$ (whenever $A \subseteq C$, $B \subseteq C$, i. e. the class-meet is invariant).

B 3 (the axiom of complement) is immediately verified on setting $-_T A = C - A$ (provided $A \subseteq C$); the "complement" is not invariant.

B 4 (the axiom of domain): By M 3 of [G] to every A with $A \subseteq C$ we have the class B with

$$x \in B \equiv \exists y[(T'\langle yx\rangle_T \in A)\,.\,(y \in \mathbf{P}(C))]\,.$$

This B is the "domain" of A in our model Δ_T. — The "domain" $\mathbf{D}_T(A) = B$ is not invariant.

B 5 (the axiom of direct product): The class $A \subseteq C$ being given, put

$$z \in B \equiv \exists x\, \exists y[T'x \in A)\,.\,(T'\langle yx\rangle_T = z)\,.\,(x \in \mathbf{P}(C))\,.\,(y \in \mathbf{P}(C))]$$

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[16]) "Invariant" transferred to Gödel's model Δ would be the same as Gödel's "absolute"; the inclusion is invariant in every index-model.

(B exists by M 3 of [G]). Then $B = C \times_T A$ is the "direct product" in Δ_T of the "universal class" $C = V_T$ (*of* the model *in* the model) — with the "class" A (in Δ_T). The "direct product" is not invariant.

B 6, B 7, B 8 (axioms of inversion) are verified in the same manner by

$$z \in B \equiv \exists x \, \exists y [(T'\langle xy \rangle_T = z) \, . \, (T'\langle yx \rangle_T \in A] \, ,$$

$$v \in B \equiv \exists x \, \exists y \, \exists z [(T'\langle T'x \; T'\langle yz \rangle_T \rangle_T = v) \, . \, (T'\langle T' \, y \, T'\langle zx \rangle_T \rangle_T \in A)] \, ,$$

$$v \in B \equiv \exists x \, \exists y \, \exists z [(T'\langle T'x \; T'\langle yz \rangle_T \rangle_T = v) \, . \, (T'\langle T'z \; T'\langle xy \rangle_T \rangle_T \in A)]$$

respectively — always under the limitation $x, y, z \in \mathbf{P}(C)$.

Axioms sub C: C 1 (the axiom of infinity) changes into the theorem (in Θ_1)

$\exists a \{(a \neq \emptyset) \, . \, (a \in \mathbf{P}(C)) \, . \, (x)[(x \in \mathbf{P}(C)) \, . \, (T'x \in a) \supset \exists y \; (y \in \mathbf{P}(C))(y \in a)(x \subset y)\}$.

Prove it by induction over ω_0 in Θ_1 (in the sense of 8.4 and 8.45 of [G]). Put $a_0 = \emptyset$, $a_{n+1} = a_n + \{T'a_n\}$ for $n \in \omega_0$.

Clearly then $a_n \in \mathbf{P}(C)$ and $a_n \subseteq a_{n+1}$ for each $n \in \omega_0$.

Prove that $a_n \subset a_{n+1}$ for each $n \in \omega_0$.

Otherwise $a_m = a_{m+1}$ for a minimal $m \in \omega_0$, while $a_0, a_1, \ldots, a_m$ are mutually different sets.

Hence by $a_{m+1} = a_m + \{T'a_m\}$ we get $\{T'a_m\} \subseteq a_m$ i. e. $T'a_m \in a_m$ — and $\sim (T'a_{m-1} \in a_{m-1})$. (Note that $m > \emptyset$ by $a_0 = \emptyset$.) Therefore we have $a_m = a_{m+1} + \{T'a_{m-1}\} = a_m + \{T'a_m\}$.

Since $a_{m-1} \neq a_m$, hence $T'a_{m-1} \neq T'a_m$ (T being a one-to-one mapping).

Now, by $a_{m-1} \, . \, \{T'a_{m-1}\} = \emptyset$ and by $T'a_m \in a_m$ we get $T'a_m \in a_{m-1} \neq \emptyset$ i. e. $m - 1 \neq \emptyset$ — and $T'a_m \in a_{m-2} + \{T'a_{m-2}\}$ with $a_{m-2} \, . \, \{T'a_{m-2}\} = \emptyset$.

Repeating this argument, we get $T'a_m \in a_{m-2}$, $T'a_m \in a_{m-3}$, etc.; after m such steps we get the contradiction $T'a_m \in a_0 = \emptyset$.

Therefore indeed $a_m \subset a_{m+1}$ for each $n \in \omega_0$.

Finally, put $a = \sum_{n \in \omega_0} a_n$.[17])

Then $a \neq \emptyset$, $a \subseteq C$ and by 5.1 of [G], a is a set such that $a \in \mathbf{P}(C)$, hence a "set" of the model.

If $x \in \mathbf{P}(C)$, $T'x \in a$ then there exists a minimal $m \in \omega_0$ so that $T'x \in a_m$. Hence $T'x = T'a_{m-1}$ (by the already proved result), i. e. $x = a_{m-1}$. Therefore putting $y = a_m$ we have $T'y \in a$, i. e. this $y = a_m$ is the desired y with $x \subset y$, $y \in_T a$.

Axiom C 1 is thus verified.

C 2 (the sum-axiom): To any $x \in \mathbf{P}(C)$ we have to find an $y \in \mathbf{P}(C)$ so that

$$(T'u \in v) \, . \, (T'v \in x) \supset T'u \in y \, .$$

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[17]) "Σ" is the usual set-join (= set-sum).

Indeed, replacing ‚$T'v \in x$' by the equivalent ‚$v \in T^{-1}{}''x$' we observe that $y = \mathbf{S}(T^{-1}{}''x)$ is such an y.

Moreover, we can write $\mathbf{S}_T(x) = \mathbf{S}(T^{-1}{}''x) \quad (x \in \mathbf{P}(C))$.

Hence the forming of a "set-sum" is not invariant.

C 3 (the potency-set axiom): To every $x \in \mathbf{P}(C)$ we have to find a $y \in \mathbf{P}(C)$ so that

$$u \subseteq x \supset T'u \in y .$$

Indeed, $y = T''\mathbf{P}(x)$ is such a y. Moreover, we can write $\mathbf{P}_T(x) = T''\mathbf{P}(x)$. The "potency-set" is not invariant.

C 4 (the axiom of substitution): Let us define

$$\mathbf{U}n_T(A) \equiv (A \subseteq C) \,.\, (u)(v)(w)[(u,\, v,\, w \in \mathbf{P}(C)) \supset [(T'\langle uv\rangle_T \in A) \,.\, (T'\langle wv\rangle_T \in A) \supset (v = w)]] \,.$$

Then we have to prove the following theorem (in Θ_1):

$$(x)(A)\{\mathbf{U}n_T(A) \supset [\exists y(t)(T't \in y \equiv \exists s(T's \in x)(T'\langle ts\rangle_T \in A))]\}$$

provided $A \subseteq C,\ x, y, s, t \in \mathbf{P}(C)$.

To any $A \subseteq C$ put

$$z \in A^* \equiv \exists t\, \exists s[(t \in \mathbf{P}(C)) \,.\, (s \in \mathbf{P}(C)) \,.\, (z = \langle T'tT's\rangle \,.\, (T'\langle ts\rangle_T \in A)] \,,$$

the class A^* being given in view of M 3 of [G].

Clearly $A^* \subseteq C \times C$ and $\mathbf{U}n(A^*) \equiv \mathbf{U}n_T(A)$ (see the def. 1.3 of [G]). For an arbitrary x we have by C4 the following theorem in Θ_1:

$$\mathbf{U}n(A^*) \supset \exists y^*(t^*)[t^* \in y^* \equiv \exists s^*(s^* \in x) \,.\, (\langle t^*s^*\rangle \in A^*)] \,.$$

Assuming $x \in \mathbf{P}(C)$, $t^* = T't \in C$, $s^* = T's \in C$ we then easily obtain

$$\mathbf{U}n(A^*) \supset \exists y(t)[T't \in y \equiv \exists s(T's \in x) \,.\, (\langle T'tT's\rangle \in A^*)]$$

provided $s,\ t,\ y \in \mathbf{P}(C)$.

Replacing here the propositional functions $\langle T'tT's\rangle \in A^*$ and $\mathbf{U}n(A^*)$ by equivalent propositional functions $T'\langle ts\rangle_T \in A$ and $\mathbf{U}n_T(A)$ respectively, we get the desired result.

Axiom of choice E: Suppose the axiom of choice E in Θ_1. Let A be the universal choice-class of Θ_1. Setting

$$z \in A_T \equiv \exists u\, \exists v[(z = T'\langle uv\rangle_T) \,.\, (\langle T'u\ v\rangle \in A) \,.\, (u,\, v \in \mathbf{P}(C))]$$

(in view of M3) we observe that $A_T \subseteq C$, $\mathbf{U}n_T(A_T)$ and that A_T indeed is the "universal choice-class" *of* the model Δ_T. Hereby the proof of Theorem III is completed.

Remark I. All the definitions and theorems of [G] are (mutatis mutandi) valid in the complete T-model Δ_T, except those which depend on the axiom D or on certain of its consequences (not proved without using D — see § 2).

Remarks II. Our tools of the verification of axioms in Δ_T fail in the case of the axiom D.

The satisfaction of D in Δ_T is equivalent to the theorem

$$(A)[(A \subseteq C) \,.\, (A \neq \emptyset) \supset \exists u[(u \in \mathbf{P}(C)) \,.\, (T'u \in A) \,.\, (u \,.\, A \neq \emptyset)]] \,.$$

But in view of D in Θ_1 we only get

$$(A)[(A \subseteq C) \,.\, (A \neq \emptyset) \supset \exists u[(u \in \mathbf{P}(C) \,.\, (T'u \in A) \,.\, (T'u \,.\, A \neq \emptyset)]] \,.$$

Indeed, the axiom D as not requiring the existence of a class or of a set, but rather excluding some classes and sets, cannot be satisfied in each Δ_T in general, even if it is true in Θ_1.

Moreover, we can prove the following

Theorem IV. *In any Gödelian set theory, a complete index-(T-)model Δ_{T_0} can be formed so that there exist in Δ_{T_0}* "predicative sets" (in the sense of Russell), i. e. ‚$\exists x(x \in_{T_0} x)$‘ *is a theorem of the model Δ_{T_0}.*

In other words: *Setting an arbitrary Gödelian set theory Θ_1 as the interpreting theory and the axiomatic Gödelian set theory Θ_2 with the axioms sub* A, B, C *completed by an additional axiom requiring the existence of predicative sets (and moreover of the class of such sets) as the interpreted theory, we always have a model of Θ_2 in Θ_1.*

If the axiom of choice E *holds in Θ_1, then it can be assumed in Θ_2 as well since it is then satisfied in Δ_{T_0} too.*

Corrollary. *The axiom* D — *and the more so*(by Theorem I) *the (generalized) axiom of constructivity are independent of the remaining axioms*, i. e. *of the axioms sub* A, B, C *and* E. (This result is essentially due to Bernays, [B]). Indeed, the existence of predicative sets is incompatible with the axiom D and the more so with the generalized axiom of constructivity.

Proof of Theorem IV. Put $C = V$ (— the universal class of Θ_1). Then clearly $\mathbf{P}(C) = C$.

Define T_0 as follows:

$$T_0'\emptyset = \{\emptyset\},\ T_0'\{\emptyset\} = \emptyset \quad \text{and} \quad T_0'y = y \quad \text{if} \quad \emptyset \neq y \neq \{\emptyset\} \,.$$

Then $T_0, C, \mathbf{P}(C)$ have the properties required in Theorem III, whence we have the complete index-model Δ_{T_0} in Θ_1.

But $T_0'\{\emptyset\} \in \{\emptyset\}$ i. e. $\{\emptyset\} \in_{T_0} \{\emptyset\}$ holds by definition in Θ_1, i. e. in Δ_{T_0}, q. e. d.

The remaining statements of Theorem IV obviously follow by the second statement of Theorem III.

Remark I (on the s. c. Russell's antinomy). It may be noteworthy to clarify (in view of Thm. IV) why the existence of predicative sets cannot cause any contradiction in Gödelian set theory (without the axiom D, of course). Moreover, no contradiction arises by the existence of the class of all the predicative sets as well as by its complement, i. e. by the class of all "impredicative" (normal) sets.

Indeed, in Δ_{T_0} of Thm. IV, the set $\{\emptyset\}$ is the unique "predicative set" (of the model) and hence this set is identical with the "class of all the predicative sets" (this "class" being ,moreover, a "set"), for $\{\{\emptyset\}\}_{T_0} = \{T'_0\{\emptyset\}\} = \{\emptyset\}$.

The forming of the contradiction is simply stopped by the fact that the "class of all the impredicative sets", i. e. the "complement" $-_{T_0}\{\emptyset\} = V - \{\emptyset\} = - \{\emptyset\}$ of the "class of all the predicative sets" is a "proper class" and hence "apriori impredicative", though not a "set".

Thus the s. c. Russell's antinomy seems to be a purely logical failure: it rests in requiring an individuum, say y, to a (nonvoid) binary relation R so that yRx is to be true with an arbitrary x if (and only if) $\sim (xRx)$ is true. But such a y does not exist, no matter whether R is the relation of "to be an element" or e. g. the relation of "to shave" (the "barber-antinomy") or any other of this kind.

On the other hand, of course, all the s. c. diagonal arguments (like the famous Cantor's one) are indirectly based in this way, i. e. the corresponding "reductio ad absurdum" always goes into a construction of a binary relation R with the y in question.

Remark II. a) It is not difficult to give a complete index-model Δ_T such that for a given $n \in \omega_0$, there is *a* "cyclical set" x_1 so that

$$x_1 \in_T x_2 \in_T \ldots \in_T x_n \in_T x_1 \qquad (\text{in } \Delta_T)$$

with suitable "sets" x_i of the model, $x_i \neq x_j$ provided $i \neq j$, $1 \leqq i \leqq n$, $1 \leqq j \leqq n$.

It suffices e. g. to put $C = V = \mathbf{P}(C)$,

$$T'\emptyset = \{\{\ldots\{\{\emptyset\}\}\ldots\}\}, \quad T'\{\{\ldots\{\{\emptyset\}\}\ldots\}\} = \emptyset$$

with n parentheses ,}' and ,{' respectively, and to complete the definition of T by $T'x = x$ if $\emptyset \neq x \neq \{\{\ldots\{\{\emptyset\}\}\ldots\}\}$. Then since $T'\{\{\ldots\{\{\emptyset\}\}\ldots\}\} \in \{\emptyset\}$ we get

$$\{\emptyset\} \in_T \{\{\emptyset\}\} \in_T \{\{\{\emptyset\}\}\} \in_T \ldots \in_T \{\{\ldots\{\{\emptyset\}\}\ldots\}\} \in_T \{\emptyset\}.$$

(The parentheses are meant in the sense of Θ_1, of course.)

b) It is also not difficult to give (in an arbitrary Gödelian Θ_1) a complete index-model and an infinite sequence of mutually different "sets" (of the model) $x_1, x_2, \ldots$ so that

$$\ldots \in_T x_j \in_T x_{j-1} \in_T \ldots \in_T x_2 \in_T x_1 .$$

Indeed, provided $C = V$ as before, define T by the identities

$$\left.\begin{array}{ll} T'\{2\} = 1 & T'1 = \{2\}, \\ T'\{3\} = 2 & T'2 = \{3\}, \\ \ldots\ldots & \ldots\ldots \\ T'\{n\} = n-1 & T'(n-1) = \{n\}, \\ \ldots\ldots & \ldots\ldots \end{array}\right\} (n \in \omega_0)$$

while $T'x = x$ if $x \neq j \in \omega_0, x \neq \{j+1\}$.

Then we get $\ldots \{j\} \epsilon_T \{j-1\} \epsilon_T \ldots \epsilon_T \{3\} \epsilon_T \{2\} \epsilon_T \{1\}$.

c) We cannot believe that with $C = V$ and with a nonidentical T, the axiom D or the stronger axiom of constructivity (in the generalized sense, see § 2) must be disprovable in Δ_T.[18]) This need not be the case even if T has finite cycles.

Indeed, put e. g. (provided $C = V$)

$$T'\emptyset = \{\emptyset\},\ T'\{\emptyset\} = \{\{\emptyset\}\},\ T'\{\{\emptyset\}\} = \emptyset;\ T'x = x \quad \text{if} \quad x \neq \emptyset,\ \{\emptyset\},\ \{\{\emptyset\}\}.$$

Then in the corresponding Δ_T, the (generalized) axiom of constructivity — and (by Theorem I) even more so the axiom D is satisfied, assumed it is true in the interpreting Θ_1.

The proof (somewhat tedious in details, but easy in principle) will only be outlined.

The relation ϵ_T can differ from ϵ only in the case that the left hand member is one of the sets $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}\}$. Especially, we have

$$\emptyset \epsilon_T \{\{\emptyset\}\}\ , \quad \{\{\emptyset\}\} \epsilon_T \{\emptyset\}\ , \quad \{\emptyset\} \epsilon_T \{\{\{\emptyset\}\}\}\ .$$

Therefore, as it is not difficult to prove, the "ordinal numbers" of the model Δ_T are as follows:

$$0_T = 0\ , \quad 1_T = \{\{\emptyset\}\} \neq 1\ , \quad 2_T = \{\{\emptyset\}\ \emptyset\} = \{\emptyset\{\emptyset\}\} = 2$$

and further always $3_T = 3$, $4_T = 4, \ldots$ (i. e. the mapping Z of theorem V below is the identical one from $\alpha = 2$ upwards).

In view of the fact that $0_T = \emptyset$ is "constructive" in Δ_T as well as $1_T = \{\{\emptyset\}\}$ and $2_T = 2 = \{\emptyset\{\emptyset\}\}$, we see that $\{\emptyset\} = 1 = \{\emptyset\{\emptyset\}\} - \{\emptyset\} = \{\emptyset\{\emptyset\}\} - \{\emptyset\}$ is "constructive" in Δ_T too.

Hence all the ordinal numbers of Θ are "constructive sets" of Δ_T which suggests the rest of the proof.

Remark III. It will be obvious (in view of Theorem V below) that the class L_T of "constructive sets" of a complete T-model is always included in the class L of the constructive sets of the interpreting theory. It has been shown by the previous examples it can happen that $L = L_T$, though $L_T \subset L$ is also possible. (See the Δ_{T_0} of Thm. IV, where $\{\emptyset\}$ is not "constructive" (in Δ_{T_0}) but constructive in the interpreting theory.)

Remark IV. Let us consider the notion of the "ordinal number" of an arbitrary index-model explicitely.

By Definition I a:, b:, c: and by § 1, an "ordinal" of a T-model (see Definition II) is any "class" X (with $\mathbf{Cls}_T(X)$) such that the following statements can be proved *in the interpreting* Θ_1:

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[18]) This remark is due to M. KINDLER of my seminarium.

a_T: If $x \epsilon_T X$ i. e. $T'x \epsilon X$, then $x \subseteq X$ (provided $\mathbf{M}_T(x)$).

b_T: If $x \epsilon_T X$, $y \epsilon_T X$ i. e. $T'x \epsilon X$, $T'y \epsilon X$, then $x = y$ or $x \epsilon_T y$ (i. e. $T'x \epsilon y$) or $y \epsilon_T x$ (i. e. $T'y \epsilon x$) (provided $\mathbf{M}_T(x)$, $\mathbf{M}_T(y)$).

c_T: To every x with $x \subseteq X$ there is a y so that $y \epsilon_T x$, i. e. $T'y \epsilon x$ — and $x \,.\, y = \emptyset$ (provided $\mathbf{M}_T(x)$, $\mathbf{M}_T(y)$).

Note that in general the related "ordinals" of the corresponding *secondary* T-model (see § 1) would be quite different; these latter would be def ned in the same manner though under the verifications of the items a_T:, b_T:, c_T: as based *on purely logical consequences of the interpreted axioms* A, B, C of Θ_2[19]) only, i. e. in the *secondary* interpreting theory Θ^*. The nature of these "secondary-model ordinals" (also in the relatively simple case of complete T-models) depends on the concrete definition of the mapping T. The related theory seems to be a difficult task, and is not attempted here.

Let us complete the definition of the "ordinal" of a T-model by the following conventions:

d_T: An "ordinal" X of the T-model in question is said to be an "ordinal number" (of this model) if $\mathbf{M}_T(X)$ (i. e. if $X \epsilon \tilde{C}$, i. e. if $T'X \epsilon C$).

e_T: Denote the "class" of all the "ordinal numbers" (of the T-model in the T-model) by On_T and the class of all the "ordinal numbers" of the model (in the interpreting theory) by ${}^TOn$. — We observe, that they both exist (in view of Thm. M 2 of [G]) in the interpreting theory Θ_1, that On_T moreover exists in the T-model and that

$$T''{}^TOn = On_T \quad (\text{for } x \epsilon {}^TOn \equiv T'x \epsilon On_T)\,.$$

The "ordinal numbers" (as elements of ${}^TOn$) can be "shown" as follows:

$$O_T = \emptyset\,, \quad 1_T = \{\emptyset\}_T = \{T'\emptyset\}\,, \quad 2_T = \{\emptyset_T 1_T\}_T = \{T'\emptyset \;\; T'\{T'\emptyset\}\}\,,$$
$$3_T = \{\emptyset_T 1_T 2_T\}_T = \{T'\emptyset \; T'\{T'\emptyset\} \; T'\{T'\emptyset \;\; T'\{T'\emptyset\}\}\}, \ldots, \alpha_T +_T 1_T =$$
$$= \alpha_T +_T \{\alpha_T\}_T = \alpha_T + \{T'\alpha_T\}, \ldots.$$

Of course, the already used symbol α_T has no meaning for a variable α and the more so it cannot be understood as a mapping, or even an isomorphism of the class On (of ordinal numbers of the interpreting theory) on to the class ${}^TOn$. (But compare theorem V below.) The sign α_T has meaning only if α is a concrete (constant) ordinal number defined in view of axioms (or theorems) holding in both the interpreting and the interpreted theory; then α_T is to be seen as the corresponding concrete (constant) "ordinal number" of the T-model, i. e. defined by the same definition as that of α is but replacing ‚$\mathbf{M}_1'$' by ‚$\mathbf{M}_T'$', ‚$\mathbf{Cls}_1'$' by ‚$\mathbf{Cls}_T'$' and ‚ϵ_1'' by ‚ϵ_T''.

[19]) I. e. without using further theorems of Θ_1 (not being interpretations of theorems of Θ_2) provided Θ_2 is based on the axioms sub A, B, C only.

Nevertheless, in the case of a complete T-model, the "ordinal numbers" of the model have the same structure as the ordinal numbers of the interpreting theory; this is stated by the

Theorem V. *Let Θ be a Gödelian set theory and let Δ_T be the complete T-model given in Θ by the one-to-one mapping T of the proper class $\tilde{C} = \mathbf{P}(C)$ on to the* (proper) *class C.* (I. e. $\mathbf{M}_T(X) \equiv X \in \tilde{C}$ and $\mathbf{Cls}_T(Y) \equiv Y \subseteq C$).

Then the mapping Z, defined inductively by $Z'\emptyset = \emptyset$, for a nonlimit $\beta = \alpha + 1 \in On$, $Z'(\alpha + 1) = Z'\beta + \{T'Z'\alpha\}$- and for a limit $\lambda \in On$, $Z'\lambda = T''Z''\lambda$, is an isomorphism of the class On on to the class ${}^TOn$ (of "ordinal numbers" of the T-model Δ_T) *with respect to the well-ordering relations $\in$ and $\in_T$.*

(See Definitions 4.52 and 4.65 as well as Theorems 7.42, 7.43 of [G] valid in any Θ.)

If T is constructive in Θ then Z is constructive in Θ.

Proof. I. First prove the following two statements by induction (simultaneously:)

A. For every $\beta \in On$ we have $Z'\beta \in {}^TOn$.

B. If $\beta \in \gamma$, then $T'Z'\beta \in Z'\gamma$, $Z'\beta \neq Z'\gamma$.

Clearly $Z'\emptyset = \emptyset \in {}^TOn$ and if $Z'\alpha \in {}^TOn$, then $Z'(\alpha + 1) \in {}^TOn$ (by the definition of "ordinal number" of Δ_T). Note, that by the argument used in the verification of the axiom C1 in Δ_T (Thm. IV), we get $\sim (T'Z'\alpha \in Z'\alpha)$. In view of this argument[20]) it obviously remains to prove the following statements only:

Suppose $Z'\alpha \in {}^TOn$ for each $\alpha \in \lambda$ where λ is a limit ordinal number and let Z map λ on to $Z''\lambda$ isomorphically with respect to $\in$ and $\in_T$. Then $Z'\lambda \in {}^TOn$.

Let us prove this statement, i. e. let us verify the requirements a_T:, b_T:, c_T: of the preceding Remark IV.

First, clearly $Z'\lambda \subset C$ by the inductive assumption, i. e. $\mathbf{Cls}_T(Z'\lambda)$.

Suppose $x \in \mathbf{P}(C)$, $y \in \mathbf{P}(C)$ and $x \in_T Z'\lambda$, i. e. $T'x \in Z'\lambda = T''Z''\lambda$, i. e. $x \in Z''\lambda$, i. e. $x = Z'\alpha$ with a suitable $\alpha \in \lambda$.

Then in view of the inductive assumption, $y \in x$ (i. e. $T'T^{-1\prime}y \in x$) means $T^{-1\prime}y = Z'\gamma$ i. e. $y = T'Z'\gamma$ with a uniquely determined $\gamma \in \alpha$ and $\gamma \subset \lambda$. $Z'\gamma \in Z''\gamma$ implies $y = T'Z'\gamma \in T''Z''\gamma \subset T''Z''\lambda$ so that indeed $x \subset Z'\lambda$. The requirement a_T: is thus verified.

Suppose $x \in_T Z'\lambda$ i. e. $T'x \in Z'\lambda$ and $y \in_T T'\lambda$ i. e. $T'y \in Z'\lambda$.

Then $x \in Z''\lambda$, $y \in Z''\lambda$ i. e. $x = Z'\alpha$, $y = Z'\beta$ for suitable $\alpha \in \lambda$, $\beta \in \lambda$ (by the inductive assumption).

[20]) Not to be repeated here.

If $\alpha = \beta$ then obviously $x = y$ and if $\alpha \in \beta$ then $T'x \in y$ i. e. $x \in_T y$ by the inductive assumption. The requirement b_T: is thus verified.

Suppose $x \in \mathbf{P}(C)$ and $\emptyset \neq x \subseteq Z'\lambda$. In order to verify c_T:, we have to find an $y \in \mathbf{P}(C)$ so that $T'y \in x$ and $y \,.\, x = \emptyset$.

To this purpose (in view of the inductive assumption) we take, for every set of the form $x \,.\, Z'\alpha$ $(\alpha \in \lambda)$, a y_α so that

(i) $T'y_\alpha \in x \,.\, Z'\alpha$,

(ii) $y_\alpha \,.\, x \,.\, Z'\alpha = \emptyset$,

(iii) $y_\alpha = Z'\alpha^*$ with a uniquely determined smallest possible $\alpha^* \in \lambda$.

If $\alpha \in \beta \in \lambda$, then $Z'\alpha \subset Z'\beta$ and $x \,.\, Z'\alpha \subseteq x \,.\, Z'\beta$ by inductive assumption.

Hence we get $\beta^* \in \alpha^* \vee \alpha^* = \beta^*$ whenever $\alpha \in \beta$, i. e. the function B, given by $B'\alpha = \alpha^*$ [with (i), (ii), (iii)] on λ, is a non-increasing function.

Therefore $B'\alpha = (\bar{\alpha})^*$ for every $\beta \in \lambda$ with $\bar{\alpha} \in \beta \in \lambda$, where $\bar{\alpha}$ is a suitable constant ordinal number.

Thus $y = y_{(\bar{\alpha})^*}$ is the desired y.

Indeed, $T'y \in x$ is clear and $y \,.\, x = \emptyset$ is not difficult to prove as follows: Put $x = \sum_{\bar{\alpha} \in \beta \in \lambda} x \,.\, Z'\beta$ (since $Z'\lambda = \sum_{\beta \in \lambda} Z'\beta$ as it is easy to see by the already verified item a_T:). Then $z \in y \,.\, x$ would imply $z \in y_\beta \,.\, x \,.\, Z'\beta$ for every β with $\bar{\alpha} \in \beta \in \lambda$.

Hence $z = T'Z'\gamma \in y_\beta = Z'(\bar{\alpha})^*$ for a suitable $\gamma \in (\bar{\alpha})^*$ (by the inductive assumption) — in contradiction to the definition of $(\bar{\alpha})^*$.

Thus requirement c_T: is verified too.

Therefore $Z'\lambda$ is an "ordinal" of Δ_T. But moreover, since

$$Z'\lambda \in_T Z'\lambda +_T 1_T = Z'\lambda + \{T'Z'\lambda\}$$

is an "ordinal" of Δ_T as well, hence $Z'\lambda$ is an "ordinal number" of Δ_T, i. e. $Z'\lambda \in {}^T On$.

Therefore both the statements A. and B. are proved.

II. Second, prove $Z''On = {}^T On$, i. e. prove that $T''Z''On = T''{}^T On = On_T$. Indeed, $T''Z''On$ is an "ordinal" of Δ_T as it is immediately seen by the already used arguments. (Take $T''Z''On$ instead of $Z'\lambda = T''Z''\lambda$). But $T''Z''On$ is a proper class (of Θ) and the more so a "proper class" of Δ_T. Hence $T''Z''On$ must be the "class of all the ordinal numbers" of Δ_T in Δ_T — by Theorem 7.2 of [G] valid in Θ by § 2.

Since the additional statement (on the constructivity of Z if T is constructive) is almost obvious, hence Theorem V is proved.

Remark I. If we were not interested in the explicit formula of the isomorphism Z (of On on to ${}^T On$), then we could prove in Θ and then apply Theorem 7.7.1 of [G] (in showing that ${}^T On$ is a proper class well ordered by $\in_T$ and such that the class of all the $x \in {}^T On$ preceding a given $y \in {}^T On$ is a set).

Remark II. The argument of the proof of Theorem V obviously fails if the T-model in question is not complete, since $T''Z''\lambda$ with a limit λ need not be a "set" (or a "class") of the model.

We conclude with the

Theorem VI. *The existence of predicative sets* (being an element of itself) *is consistent with the axioms sub* A, B, C *and* E *of* [G] *completed by the Generalized Continuum Hypothesis. Axiom* D (and the more so *the generalized axiom of constructivity*) *is independent of the axioms sub* A, B, C, E *completed by the Generalized Continuum Hypothesis.*

Proof. Suppose a complete T-model Δ_T in any Gödelian set theory. Then $\mathbf{P}_T(x) = T''\mathbf{P}(x)$ by Theorem III — provided $x \in \mathbf{P}(C)$ (see Def. II). Therefore in view of Theorems III (the last statement) and V we observe the following relation between the "power" ("cardinal number") of a "potency-set" in Δ_T and the power of the potency-set of the same set $x \in \mathbf{P}(C)$ (see Definition 8.20 of [G]):

$$\overline{\overline{(\mathbf{P}_T(x))_T}} = \overline{\overline{Z'\mathbf{P}(x)}} .$$

Now, Theorem VI is an immediate consequence of Theorem IV and of the fundamental result of [G].

Remark. All our considerations could be performed mutatis mutandi in another sufficiently powerful formalized set theory, e. g. in the system of Mostowski (see [K-M]). In this system, moreover, some arguments would be simpler, but the results perhaps somewhat weaker in a certain sense.

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Резюме

К ГЕДЕЛЕВСКОЙ АКСИОМАТИЧЕСКОЙ ТЕОРИИ МНОЖЕСТВ, I

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(Поступило в редакцию 18/V 1956 г.)

Настоящая работа тесно примыкает к основной работе Геделя [*G*]. Вступительный параграф 1 содержит необходимые в последствии метаматематические понятия, в частности, понятие формализированной теории и понятие формализированной *интерпретации* (и *модели*, данной этой интерпретацией) одной формализированной теории в другой формализированной теории — независимо от того, идет ли речь о теориях, допускающих аксиоматизацию, или нет. Эти понятия (и основные соотношения между ними) сформулированы *алгебраически*, т. е. при помощи т. наз. *свободных обобщенных* σ*-алгебр* и их *идеалов*. В частности, понятие интерпретации оказывается известного рода σ-гомоморфным отображением одной обобщенной свободной σ-алгебры (принадлежащей к т. наз. интерпретируемой теории) в другую обобщенную свободную алгебру (принадлежащую к т. наз. интерпретирующей теории).

В § 2 (который также носит еще подготовительный характер) прежде всего построена (за счет небольших видоизменений, подобно тому, как и в [*G*]) необходимая нам часть произвольной, т. е. общей *геделевской* формализированной теории множеств (включая порядковые числа); геделевской мы называем всякую формализированную элементарную теорию множеств с теми же тремя примитивными понятиями, как и в [*G*], в которой справедливы хотя бы аксиомы *A*, B, C как теоремы, причем, однако, аксиома D вообще говоря не обязательно справедлива, и сама теория не должна допускать аксиоматизацию.

Далее (независимо от аксиомы D, в отличие от [*G*]) формулируется (обобщенный) закон конструктивности и обобщается построение геделевской модели Δ конструктивных множеств и классов (теорема I) так, чтобы его можно было провести в произвольной геделевской формализированной теории множеств.

В § 3 (составляющем главную часть этой работы) предже всего вводится понятие (в известном смысле универсальной) т. наз. *индекс-модели* согласно следующему

определению II. Модель (геделевской теории множеств Θ_2 в геделевской теории множеств Θ_1) называется индекс-моделью, если т. наз. примитивные понятия моделей $\mathbf{M}_T$ (унавный предикат ,,множество''), $\mathbf{Cls}_T$ (унарный предикат ,,класс'') и ϵ_T (бинарный предикат ,,быть элементом'' в смысле модели) даны так:

В Θ_1 существует класс $\tilde{C}$ и взаимно однозначное отображение (т. е. класс упорядоченных пар) T класса $\tilde{C}$ на другой класс C так, что

1. $\tilde{C}$ есть часть класса $\mathbf{P}(C)$ всех подмножеств класса C,
2. примитивные понятия модели выполняют следующие условия

$$\mathbf{M}_T(X) \equiv X \in \tilde{C}\,, \quad \mathbf{Cls}_T(Y) \supset Y \subseteq C\,,$$
$$X \in_T Y \equiv (T'X \in Y)\,.\,\mathbf{M}_T(X)\,.\,\mathbf{Cls}_T(Y)\,.$$

Класс C называется *классом индексов*, множество $T'X \in C$ (образ множества X при отображении T) есть т. наз. индекс ,,множества модели'' X.

Индекс-модель называется *полной*, если имеет место

$$\mathbf{Cls}_T(Y) \equiv Y \subseteq C\,, \quad \tilde{C} = \mathbf{P}(C)\,.$$

Приведем главные результаты § 3, в котором мы ограничиваемся полными индекс-моделями:

Теорема III. *Пусть Θ_1 — произвольная и пусть Θ_2 — аксиоматическая теория множеств, заданная при помощи аксиом* A, B, C.

Пусть T — взаимно однозначное отображение класса $\mathbf{P}(C)$ всех подмножеств некоторого фиксированного собственного класса C в Θ_1 — на самое C.

Положим

$$\mathbf{M}_T(X) \equiv X \in \mathbf{P}(C)\,,$$
$$\mathbf{Cls}_T(Y) \equiv Y \subseteq C\,,$$
$$X \in_T Y \equiv (T'X \in Y)\,.\,\mathbf{M}_T(X)\,.\,\mathbf{Cls}_T(Y)\,.$$

Тогда $\mathbf{M}_T$, $\mathbf{Cls}_T$, ϵ_T образуют модель теории Θ_2 в теории Θ_1, т. е. полную индекс-модель.

Если же и геделевская аксиома выбора E *имеет место в Θ_2, то* E *выполняется и в этой моделн.*

Теорема IV. *В каждой геделевской теории множеств Θ_1 можно построить полную индекс-модель аксиоматической геделевской теории мно-*

жеств Θ_2 *с аксиомами* A, B, C *так, что в этой модели существуют т. наз. ,,предикативные множества'' (в смысле Ресселя), т. е. множества, содержащие сами себя в качестве элемента. (Другими словами, имеет место* $\exists x(x \in_T x)$.)

Если в Θ_2 справедлива геделевская аксиома выбора E, то E выполняется и в указанной модели. В этой полной индекс-модели существует непустой класс (и даже множество) предикативных (ненормальных) множеств, так же как и класс нормальных (импредикативных) множеств. (Несмотря на это мы не получим известного т. наз. парадокса Ресселя.)

Теорема V утверждает, что *класс порядковых чисел* произвольно заданной геделевской теории множеств и *класс ,,порядковых чисел''* произвольной *полной* индекс-модели, определенной в этой теории, *изоморфны*. На основании § 2 (обобщение основного результата Геделя) отсюда легко вытекает существенное усиление теоремы IV, а именно

Теорема VI. *Существование предикативных множеств* (содержащих самих себя в качестве элемента) и непустого их класса *совместимо* с аксиомами Геделя A, B, C и E, *дополненными обобщенной гипотезой континуума* — если только аксиомы A, B, C сами совместимы. В частности, аксиома Ф. Нейманна D, а тем более и более сильная аксиома конструктивности Геделя — является независимой от геделевских аксиом A, B, C, E, дополненных обобщенной гипотезой континуума.

В подготовляемой второй части автор предполагает заняться более глубокими вопросами *неполных* индекс-моделей.