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ON THE DESCRIPTIVE THEORY OF SETS

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Certain structures for topological spaces are introduced and studied. Especially, it is shown that a regular space P is analytic if and only if there exists an analytic structure on P. Spaces possessing complete sequences of countable coverings are examined.

Adopting Choquet's definition of analytic spaces we shall prove that a regular space P is analytic if and only if there exists an analytical structure in P, *i. e.* a determining system with nucleus P, satisfying certain condition of completeness. Further a completely regular space P is a $F_{\sigma\delta}$ in the Čech-Stone compactification $\beta(P)$ of P if and only if there exists a countable collection of countable closed coverings of P which satisfies certain conditions of completeness. Finally, B-spaces are introduced (there exists a countable collection of countable coverings which is complete in a certain sense) and studied.

1. NOTATION AND TERMINOLOGY

The notation and terminology of J. KELLEY, General Topology, will be used throughout. For convenience, some further not quite usual symbols and terms (listed below) will be used.

1.1. If $\mathscr{A}$ is a family of sets and if M is a set, then the symbol $\mathscr{A} \cap M$ $(= M \cap \mathscr{A})$ will be used to denote the family consisting of all $A \cap M$ with $A \in \mathscr{A}$.

1.2. A centered family of sets is a family $\mathscr{M}$ with the finite intersection property, *i. e.* such that the intersection of every finite subfamily of $\mathscr{M}$ is non-void.

1.3. Let $\mathscr{M}$ be a family of sets. There exists the smallest family $\mathscr{N}$ containing $\mathscr{M}$ and closed under countable intersections and unions, *i. e.* such that if $\mathscr{A}$ is a countable subfamily of $\mathscr{N}$ then both $\cap\mathscr{A}$ and $\cup\mathscr{A}$ belong to $\mathscr{N}$. The sets from $\mathscr{N}$ will be called $\mathscr{M}$-Borelian sets.

1.4. If $\mathscr{M}$ is a family of sets, then $\mathscr{M}_\sigma$ and $\mathscr{M}_\delta$ will be used to denote the family of all countable unions and intersections, respectively, of sets from $\mathscr{M}$. Instead of $(\mathscr{M}_\sigma)_\delta$ we shall write $\mathscr{M}_{\sigma\delta}$.

1.5. The closure of a subset M of a space P will be denoted by $\overline{M}^P$ or merely by $\overline{M}$. If $\mathcal{M}$ is family of subsets of a space P then the family consisting of all $\overline{M}^P$, $M \in \mathcal{M}$, will be denoted by $\overline{\mathcal{M}}^P$ or merely by $\overline{\mathcal{M}}$. If $\mathcal{M}$ is a family of sets and f is a mapping, then $f[\mathcal{M}]$ denotes the family of all $f[M]$, $M \in \mathcal{M}$.

1.6. Let P be a space. The symbol $\mathsf{F}(P)$ denotes the family of all closed subsets of P and $\mathsf{K}(P)$ denotes the family of all compact subsets of P. Of course, $\mathsf{F}_\sigma(P)$ is used instead of $(\mathsf{F}(P))_\sigma$. K denotes the class of all compact spaces.

1.7. A space P is said to be topologically complete (in the sense of E. Čech) if P is completely regular and P is a G_δ in the Čech-Stone compactification $\beta(P)$ of P. A space P will be called complete metrizable if there exists a metric ϱ generating the topology of P and such that (P, ϱ) is a complete metric space. Of course, a metrizable space P is topologically complete if and only if P is complete metrizable.

1.8. A space P will be called σ-compact if P is the union of a countable number of its compact subspaces, that is, if $P \in \mathsf{K}_\sigma(P)$.

1.9. A mapping f from a space P onto a space Q will be called closed (open), if the images under f of closed sets (open sets) are closed (open, respectively). A mapping f will be called perfect, if f is both continuous and closed and the inverse images of points (that is, the sets of the form $f^{-1}[y]$, $y \in Q$, are compact).

1.10. $\beta(P)$ will be always used to denote the Čech-Stone compactification of a completely regular space P.

1.11. A zero-set in a space P is any set of the form $\{x; x \in P, f(x) = 0\}$ where f is any continuous real-valued function on P.

2. PRELIMINARIES

Definition 1. A space P will be called *analytic in the classical sense* if P is metrizable and there exists a continuous mapping from the space of all irrational numbers of the unit interval $\langle 0, 1\rangle$ of real numbers onto P.

It is well known that every separable complete metrizable space is analytic; moreover, every Borelian subset (in our terminology $\mathsf{F}(P)$-Borelian subset) of a separable complete metrizable space P is analytic. Since the space of all irrational numbers is complete metrizable, analytic spaces in the classical sense are coincident with the metrizable continuous images of $\mathsf{F}(P)$-Borelian subsets of separable complete metrizable spaces P.

In [1], analytic spaces in the classical sense are called "les espaces de Souslin".

G. Choquet introduced (in [2]) the following generalization of spaces analytic in the classical sense:

Definition 2. A space P will be called *analytic* if P is completely regular and if P is the image under a continuous mapping of a space Q which is the intersection of a countable number of countable unions of compact subspaces of a space $R \supset Q$; in other words, if there exists a continuous mapping of a space Q onto P and a space R containing Q as a subspace and such that Q is a $\mathsf{K}_{\sigma\delta}(R)$.

Our definition is more restrictive than Choquet's, because we assume P is completely regular but Choquet does not. If a continuous image P of a $\mathsf{K}_{\sigma\delta}(R)$ is a regular space, then P is normal, because every $\mathsf{K}_{\sigma\delta}$ is a Lindelöf space and every continuous image of a Lindelöf space is a Lindelöf space.

G. Choquet showed ([2], [3]) that analytic spaces do have some useful measure-theoretic and other properties similar to those of spaces analytic in the classical sense. In the present note an internal characterization of analytic spaces is given and some theorems about them are proved or reproved.

Let us remark that in the present note only topological properties of analytic spaces are studied.

In section 3 the general concept of a complete collection of families of subsets of a space is introduced and some basic results proved. However, it seems that it may be possible to build up a theory containing a wide class of concepts and results of general topology.

In section 4, complete countable collections (sequences) of countable coverings of a space are studied. A space having a complete sequence of (closed) countable coverings is called a B-space (a space from E). It is shown that a completely regular space P belongs to E if and only if P is $F_{\sigma\delta}$ in the Čech-Stone compactification $\beta(P)$ of P.

Thus analytic spaces ate precisely those completely regular spaces which are continuous images of spaces from E. Every space from E is a B-space and every B-space is an analytic space. A simple though interesting result is that B-spaces are invariant under one-to-one continuous mappings. The family of all subspaces of a space P which are B-spaces is closed under countable unions and intersections. These properties are similar to those of Borel subsets of separable complete metrizable spaces.

In section 5 analytic spaces are investigated. There are given internal characterizations of analytic spaces (existence of an analytical structure) and a necessary and sufficient condition for a regular space to be the image under a continuous mapping of the space of all irrational numbers.

3. COMPLETE COLLECTIONS

In this section we shall define complete collections of families of subsets of space P and prove some theorems about them.

Definition 1. Let $\mu = \{\mathscr{M}\}$ be a collection consisting of families of subsets of space P. A *μ-Cauchy family* is a centered family $\mathscr{B}$ of subsets of P satisfying the following condition:

For each $\mathscr{M}$ in μ some $B \in \mathscr{B}$ is contained in a $M \in \mathscr{M}$. A collection μ will be called complete if

$$\bigcap \overline{\mathscr{B}} \neq \emptyset \tag{1}$$

for every μ-Cauchy family $\mathscr{B}$. A *maximal μ-Cauchy family* is a maximal centered family which is a μ-Cauchy family.

Example 1. If μ is the collection of all uniform coverings of a uniform space P, then P is a complete uniform space if and only if μ is complete in the sense of Definition 1.

Example 2. For every real-valued continuous function f on a given completely regular space P put

$$\mathscr{V}(f) = \{\{x; |f(x)| < n\}; \ n = 1, 2, \ldots\} .$$

Then P is a realcompact space (in the original terminology of E. Hewitt, a Q-space) if and only if the collection of all $\mathscr{V}(f)$ is complete in the sense of Definition 1. (Cf. [6] and [7].)

Example 3. Let $\gamma(P)$ be the collection of all countable open coverings of a Hausdorff space P. A space P is said to be almost-realcompact if the collection $\gamma(P)$ is complete. Every realcompact space is almost realcompact and the properties of almost realcompact spaces are similar to those of realcompact spaces, but the proofs are simpler. (Cf. [7] and [8].)

Example 4. An interesting class of spaces is formed by those spaces P for which the collection of all countable closed coverings is complete (Cf. [7]).

Example 5. (See Remark 2 of Section 3.)

In the present note we shall study countable collections of countable coverings. In this section we deduce general properties of complete collections.

A refinement of a family $\mathscr{A}$ of sets is a family $\mathscr{B}$ of sets such that every set from $\mathscr{B}$ is contained in some set from $\mathscr{A}$.

Proposition 1. A collection μ is complete if and only if (1) holds for every maximal μ-Cauchy family $\mathscr{B}$.

Proposition 2. If $\mu = \{\mathscr{M}\}$ is a complete collection, and if for every $\mathscr{M}$ in μ the family $\mathscr{N}(\mathscr{M})$ refines $\mathscr{M}$, then the collection of all $\mathscr{N}(\mathscr{M})$ is complete.

Proofs of Propositions 1 and 2 are obvious.

Theorem 1. *Let $\mu = \{\mathscr{M}\}$ be a complete collection in a space P. For each $\mathscr{M}$ in μ let $\mathscr{N}(\mathscr{M})$ be the family consisting of finite unions of sets from $\mathscr{M}$. Then the collection consisting of all $\mathscr{N}(\mathscr{M})$, $\mathscr{M} \in \mu$,* is complete.

Proof. Let ν be the collection consisting of all $\mathscr{N}(\mathscr{M})$, $\mathscr{M} \in \mu$. Let A be a maximal ν-Cauchy family. To prove ν is complete, it is sufficient to show that $\mathscr{A}$ is also a

μ-Cauchy family. Let $\mathscr{M}$ be a family from the collection μ. There exist an $\mathscr{N}$ in ν such that $\mathscr{N}$ is a family consisting of finite unions of sets from $\mathscr{M}$. Since $\mathscr{A}$ is a maximal ν-Cauchy family, there exists an $N \in \mathscr{N} \cap \mathscr{A}$. By our choice of $\mathscr{N}$, there exists a finite subfamily $\mathscr{M}'$ of $\mathscr{M}$ with $N = \cup \mathscr{M}'$. And finally, $\mathscr{A}$ being a maximal centered family of subsets of P, some $M \in \mathscr{M}'$ belongs to $\mathscr{A}$; this proves $\mathscr{M} \cap \mathscr{A} \neq \emptyset$ and shows that $\mathscr{A}$ is a μ-Cauchy family. The fact that some $M \in \mathscr{M}'$ belongs to $\mathscr{A}$ follows at once from the following well-known result: If $\mathscr{A}$ is a maximal centered family of subsets of a set P and $M \subset P$, then either M or $P - M$ belongs to $\mathscr{A}$.

Theorem 2. *Let f be a continuous one-to-one mapping of a space P onto a space Q. Let μ be a complete collection in P. For each $\mathscr{M}$ in μ put*

$$f[\mathscr{M}] = \{f[M]; M \in \mathscr{M}\} .$$

Then the collection $f[\mu]$ consisting of all $f[\mathscr{M}]$, $\mathscr{M} \in \mu$, is a complete collection in Q.

Proof. Let $\mathscr{A}$ be a family of subsets of P and let $f[\mathscr{A}]$ analogously to $f[\mathscr{M}]$. Then clearly $\mathscr{A}$ is a μ-Cauchy family if and only if $f[\mathscr{A}]$ is a $f[\mu]$-Cauchy family.

Theorem 2'. *If μ is a complete collection in a space P and if P is a subspace of a space R, then μ is complete in R.*

The proof is obvious.

Theorem 3. *Let f be a perfect mapping of a space P onto a space Q. If $\mu = \{\mathscr{M}\}$ is a complete collection in Q, then $f^{-1}[\mu]$ is a complete collection in P, where $f^{-1}[\mu]$ is the collection consisting of all $f^{-1}[\mathscr{M}]$, $\mathscr{M} \in \mu$, and*

$$f^{-1}[\mathscr{M}] = \{f^{-1}[M]; M \in \mathscr{M}\} .$$

Proof. Let $\mathscr{A}$ be a maximal $f^{-1}[\mu]$-Cauchy family. Clearly

$$\mathscr{B} = \{f[A]; A \in \mathscr{A}\}$$

is a μ-Cauchy family. Thus we have

$$\bigcap \overline{\mathscr{B}} \neq \emptyset . \tag{2}$$

Now to prove $\bigcap \overline{\mathscr{A}} \neq \emptyset$, it is sufficient to prove the following general result.

Lemma. *Let f be a perfect mapping of a space P onto a space Q. If $\mathscr{A}$ is a centered family of subsets of P, then*

$$\bigcap \{\overline{f[A]}; A \in \mathscr{A}\} = f[\bigcap \overline{\mathscr{A}}] . \tag{3}$$

Proof. From the continuity of f it follows at once that the inclusion $\supset$ holds. Let y be an element of

$$M = \bigcap \{\overline{f[A]}; A \in \mathscr{A}\} .$$

The set $K = f^{-1}[y]$ is a compact subspace of P. Since the mapping f is both closed and continuous, we have

$$L \subset P \Rightarrow f[\overline{L}] = \overline{f[L]} . \tag{4}$$

From (4) it follows at once that $K \cap \overline{\mathscr{A}}$ is a centered family of sets. K being compact, we have

$$K \cap \bigcap \overline{\mathscr{A}} = \bigcap (\overline{\mathscr{A}} \cap K) \neq \emptyset .$$

Let us choose a point x in this intersection. Clearly $f(x) = y$, which proves the converse inclusion $\subset$.

Remark 1. In Theorem 3, if the $\mathscr{M}$'s consist of open or closed sets, then $f^{-1}[\mathscr{M}]$'s consist of open or closed sets, respectively.

Theorem 4. *Let P be a dense subspace of a Hausdorff space R. Let $\mu = \{\mathscr{M}\}$ be a complete collection of open coverings of P. For every open subset M of P let M' be an open subset of R such that $M' \cap P = M$. Then*

$$\bigcap_{\mathscr{M} \in \mu} \bigcup \{M'; M \in \mathscr{M}\} = P .$$

In particular, if μ is countable, then P is a G_δ in R.

Proof. For every $\mathscr{M}$ in μ put

$$V(\mathscr{M}) = \bigcup \{M'; M \in \mathscr{M}\} .$$

Further, put

$$Q = \bigcap \{V(\mathscr{M}); \mathscr{M} \in \mu\} .$$

The inclusion $Q \supset P$ is obvious. To prove the converse inclusion, let us suppose that there exists a point x in $Q - P$. Let $\mathscr{A}$ be the family consisting of all neighborhoods of the point x in R. Since R is a Hausdorff space, we have

$$\bigcap \overline{\mathscr{A}} = (x) . \tag{5}$$

Put $\mathscr{B} = \mathscr{A} \cap P$. Since P is a dense subspace of R, x belongs to the closure of P, and consequently, $\mathscr{B}$ is a centered family of sets. Since x belongs to Q, from the definitions of Q and the $V(\mathscr{M})$'s we obtain at once that $\mathscr{M} \cap \mathscr{B} \neq \emptyset$ for each $\mathscr{M}$ in μ. Thus $\mathscr{B}$ is a μ-Cauchy family. It follows that

$$\bigcap \overline{\mathscr{B}}^P \neq \emptyset . \tag{6}$$

But (6) contradicts (5). Indeed, by (5),

$$\bigcap \overline{\mathscr{B}}^P \subset \bigcap \overline{\mathscr{B}}^R = (x) \subset R - P .$$

This contradiction shows that $Q \subset P$. The proof is complete.

Remark 2. If P is a dense subspace of compact Hausdorff space K and if

$$P = \bigcap \{V; V \in \mathscr{V}\}$$

where $V \in \mathscr{V}$ are open in K, then it is easy to construct a complete collection $\{\mathscr{M}(V); V \in \mathscr{V}\}$ of open coverings of P. Indeed, it is sufficient to put

$$\mathscr{M}(V) = \{U \cap P; U \text{ open in } R, \overline{U} \subset V\} .$$

Thus we have proved: The following properties of a completely regular space are equivalent ($\mathfrak{m}$ is a cardinal):

(a) There exists a complete collection μ of open coverings of P such that the potency of μ is at most $\mathfrak{m}$.

(b) P is the intersection of $\mathfrak{m}$ open sets in every Hausdorff space R which contains P as a dense subspace.

(c) P is the intersection of $\mathfrak{m}$ open sets in some compact Hausdorff space K which contains P as a dense subspace.

Further information may be found in [4] and [5].

Theorem 5. *Let φ be a collection consisting of families of closed subsets of a Hausdorff space P. The collection φ is complete in P if and only if the following two conditions are fulfilled*:

(a) *If $F(\mathscr{F}) \in \mathscr{F}$, then the intersection of all $F(\mathscr{F})$, $\mathscr{F} \in \varphi$, is a compact subspace of P.*

(b) *If $M(\mathscr{F}) \subset F(\mathscr{F}) \in \mathscr{F}$ and the family of all $M(\mathscr{F})$, $\mathscr{F} \in \varphi$, is centered, then the intersection of the closures of $M(\mathscr{F})$ is non-void.*

Proof. First let us suppose that φ is a complete collection. The condition (b) is obviously satisfied because under the assumptions of (b) the family of all $M(\mathscr{F})$, $M \in \varphi$, is a φ-Cauchy family. To prove (a), let $\mathscr{M}$ be a maximal centered family of subsets of

$$K = \bigcap\{F(\mathscr{F});\ \mathscr{F} \in \varphi\}\,. \tag{7}$$

Let $\mathscr{N}$ be a maximal centered family of subsets of P containing $\mathscr{M}$. From (7) it follows that $F(\mathscr{F}) \in \mathscr{N}$ for every $\mathscr{F}$ in φ. Thus $\bigcap\overline{\mathscr{M}}^P \neq \emptyset$. On the other hand, clearly $\bigcap\overline{\mathscr{M}}^P \neq \bigcap\overline{\mathscr{N}}^P$, and finally, K being closed, $\bigcap\overline{\mathscr{M}}^K \neq \emptyset$.

Conversely, let us suppose (a) and (b) are true. Let $\mathscr{A}$ be a maximal φ-Cauchy family. Let us choose $F(\mathscr{F}) \in \mathscr{A} \cap \mathscr{F}$, $\mathscr{F} \in \varphi$, and let us consider the set K from (7). By (a), K is a compact subspace of P. Thus to prove $\bigcap\overline{\mathscr{A}} \neq \emptyset$, it is sufficient to show that the family $\overline{\mathscr{A}} \cap K$ is centered, and $\mathscr{A}$ being finitely multiplicative, it is sufficient to show that $\bar{A} \cap K \neq \emptyset$ for every A in $\mathscr{A}$. Put

$$M(\mathscr{F}) = A \cap F(\mathscr{F})\,. \tag{8}$$

By condition (b) in view of (7) and (8) we have

$$\emptyset \neq \bigcap\overline{M(\mathscr{F})} \subset \bigcap F(\mathscr{F}) = K\,,$$

which completes the proof of Theorem 5.

Theorem 5′. *Let μ be a collection of families of subsets of a regular space P. The collection μ is complete if and only if the following two conditions are fulfilled*:

(a) *If $M(\mathscr{M}) \in \mathscr{M} \in \mu$, then the intersection of closures of all sets of the form*

$$M(\mathscr{M}_1) \cap \ldots \cap M(\mathscr{M}_n)\,, \quad \mathscr{M}_i \in \mu\,, \quad n = 1, 2, \ldots\,, \tag{9}$$

is a compact subspace of P.

(b) *The condition* (b) *from Theorem* 5.

Proof. First let us suppose that μ is complete. The proof of condition (b) is the same as that of condition (b) in Theorem 5. To prove (a), let us consider the intersection K of closures of all sets of the form (9), and a maximal centered family $\mathcal{N}$ of subsets of K. Let $\mathcal{A}$ be the family of open subsets of P containing the closure of some $N \in \mathcal{N}$. Since P is regular we have

$$\bigcap \overline{\mathcal{A}} = \bigcap \overline{\mathcal{N}} . \tag{10}$$

Thus to prove $\bigcap \overline{\mathcal{N}} \neq \emptyset$, it is sufficient to show that $\bigcap \overline{\mathcal{A}} \neq \emptyset$. Let us consider the family $\mathcal{B}$ consisting of all $A \in \mathcal{A}$ and all sets $M(\mathcal{M})$, $\mathcal{M} \in \mu$. To show $\mathcal{B}$ is a μ-Cauchy family it is sufficient to prove that $\mathcal{B}$ is a centered family. But this is obvious because the closures of sets of the form (9) contain K. Since the collection μ is complete we have

$$\bigcap \overline{\mathcal{B}} \neq \emptyset . \tag{11}$$

Finally, $\mathcal{A} \subset \mathcal{B}$ and hence $\bigcap \overline{\mathcal{A}} \supset \bigcap \overline{\mathcal{B}}$, which proves $\bigcap \overline{\mathcal{A}} \neq \emptyset$, and by (10) $\bigcap \overline{\mathcal{N}} \neq \emptyset$. Thus K is a compact space.

Conversely, suppose (a) and (b). Let $\mathcal{N}$ be a maximal μ-Cauchy family. For every $\mathcal{M}$ in μ choose a $M(\mathcal{M}) \in \mathcal{N} \cap \mathcal{M}$. Let K be the intersection of closures of all sets of the form (9). As in the proof of Theorem 5 one can prove that $\overline{\mathcal{N}} \cap K$ is a centered family of sets.

Theorem 6. *Let φ be a complete collection of closed coverings of a normal space P. Let R be the Čech-Stone compactification of P. Then*

$$P = \bigcap_{\mathcal{F} \in \varphi} \bigcup \{\overline{F}^R; F \in \mathcal{F}\} . \tag{12}$$

Proof. Let Q denote the right-hand side of (12). Clearly $Q \supset P$. To prove the converse inclusion, suppose that there exists a point x in $Q - P$. Let $\mathcal{A}$ be the family consisting of all closed neighborhoods of x in R. Let $\mathcal{B}$ be the family consisting of all closed subsets F of P which contain the point x in their closure in R. Clearly $\mathcal{A} \cap P$ is a sub-family of $\mathcal{B}$ and $\mathcal{B} \cap \mathcal{F} \neq \emptyset$ for every $\mathcal{F} \in \varphi$. Now let us recall that the Stone-Čech compactification R of a normal space P has the following property:

If F_1, F_2 are closed in P and $x \in \overline{F}_i^R$ $(i = 1, 2)$ then $x \in \overline{F_1 \cap F_2}^R$.

From this property it follows at once that the family $\mathcal{B}$ is centered. Thus we have proved that $\mathcal{B}$ is a φ-Cauchy family. Since φ is complete, we have

$$\bigcap \overline{\mathcal{B}}^P \neq \emptyset .$$

But this is impossible because

$$\bigcap \overline{\mathcal{B}}^P \subset \bigcap \overline{\mathcal{B}}^R \subset \bigcap \overline{\mathcal{A}}^R = (x) \subset R - P .$$

This contradiction shows that $P = Q$.

Corollary. If P is a normal space and if the collection of all closed countable coverings of P is complete, then P is the intersection of σ-compact subspaces of $\beta(P)$, and consequently, P is a realcompact space. Conversely, if P is a realcompact space, then the collection of all closed countable coverings is complete because the collection of all $\mathscr{V}(f)$ from Example 2 (or the collection of all countable coverings consisting of zero-sets) is complete. (Cf. [7].)

4. COMPLETE SEQUENCES OF COUNTABLE COVERINGS

A sequence $\{\mathscr{M}_n\}$ of families of subsets of a space P will be called complete if the collection consisting of all $\mathscr{M}_n$; $n = 1, 2, \ldots$, is complete.

Proposition 1. Let $\mu = \{\mathscr{M}_n\}$ be a complete sequence of families of subsets of a regular space P. If $M_n \in \mathscr{M}_n$ and if K_n denotes the closure of the set $M_1 \cap \ldots \cap M_n$, and finally, if U is an open set containing

$$K = \bigcap\{K_n;\ n = 1, 2, \ldots\},$$

then some K_n is contained in U.

Proof. Let us suppose that every K_n meets $P - U$. Let $\mathscr{A}$ be the family consisting from all open sets containing some $K_n - U$. Since P is a regular space, we have

$$(1)\qquad \bigcap\overline{\mathscr{A}} = \bigcap\{K_n - U;\ n = 1, 2, \ldots\} = \emptyset.$$

Let $\mathscr{B}$ be the family consisting of all $A \in \mathscr{A}$ and all M_k, $k = 1, 2, \ldots$ Clearly $\mathscr{B}$ is a centered family of sets, and hence clearly $\mathscr{B}$ is a μ-Cauchy family. Thus

$$\bigcap\overline{\mathscr{B}} \neq \emptyset,$$

which contradicts (1) because $\mathscr{B} \supset \mathscr{A}$. The proof is complete.

Proposition 2. *Let* $\mu = \{\mathscr{M}_n\}$ be a complete sequence of coverings of a space P and let a Hausdorff space R contain P as a subspace. For every $M_i \in \mathscr{M}_i$ let $F(M_1, \ldots, M_n)$ be the closure in R of the set $M_1 \cap \ldots \cap M_n$. Then

$$(2)\qquad \bigcup_{\{M_i\}} \bigcap_{n=1}^{\infty} F(M_1, \ldots, M_n) = P.$$

(Of course $\{M_i\}$ runs over all sequences $\{M_i\}$ where $M_i \in \mathscr{M}_i$.)

Proof. The inclusion $\subset$ is obvious. Let us suppose that there exists a point x in

$$\bigcap\{F(M_1, \ldots, M_n);\ n = 1, 2, \ldots\} - P$$

where $M_i \in \mathscr{M}_i$. Let $\mathscr{A}$ be the family of all neighborhoods of the point x (in R). Let $\mathscr{B}$ be the family consisting of all $A \cap P$, $A \in \mathscr{A}$ and all M_i, $i = 1, 2, \ldots$ It is easy to see that $\mathscr{B}$ is a centered family of sets, and consequently, $\mathscr{B}$ is a μ-Cauchy family. μ being complete, we have

$$\bigcap\overline{\mathscr{B}}^P \neq \emptyset;$$

but this is impossible, since $\mathscr{A} \subset \mathscr{B}$ and $\bigcap\overline{\mathscr{A}}^R = (x) \subset R - P$. This completes the proof.

Proposition 3. Let μ be a complete sequence of countable coverings of a space P. If P is regular, then P is a Lindelöf space, and consequently P is a normal space.

Proof. Let us suppose that there exists an open covering $\mathscr{U}$ of P containing no countable subcovering. The coverings $\mathscr{M}_n \in \mu$ being countable, we can define by induction sets $M_n \in \mathscr{M}_n$ such that no countable subfamily of $\mathscr{U}$ covers the closure $F(M_1, \ldots, M_n)$ of $M_1 \cap \ldots \cap M_n$, $n = 1, 2, \ldots$ Clearly the sets $M_1 \cap \ldots \cap M_n$ are non-void. Thus by Theorem 5′ of section 3, the set

$$K = \bigcap_{n=1}^{\infty} F(M_1, \ldots, M_n)$$

is a compact subspace of P. It follows that some finite subfamily $\mathscr{V}$ of $\mathscr{U}$ covers K. Put $V = \bigcup \mathscr{V}$. By Proposition 1 some $F(M_1, \ldots, M_n)$ is contained in V, which contradicts the definition of M_i. This completes the proof.

Theorem 7. *Given a completely regular space P, there exists a complete sequence of closed countable coverings of P if and only if P is a $F_{\sigma\delta}$ in the Čech-Stone compactification βP of P.*

Proof. First suppose there exists a complete sequence of closed countable coverings of a completely regular space P. By Proposition 4, P is a normal space and finally, by Theorem 6 of section 3 P is a $F_{\sigma\delta}$ in the Čech-Stone compactification of P. The converse implication is an immediate consequence of the following result.

Proposition 4. If P is the intersection of a countable number of σ-compact subspaces of a space $R \supset P$, then there exists a complete sequence of closed countable coverings of P.

Proof of Proposition 4. Let $R \supset P$, and let

$$P = \bigcap \{P_n;\ n = 1, 2, \ldots\} \tag{3}$$

where P_n are σ-compact subspaces of R. For every n there exists a countable family $\mathscr{K}_n$ consisting of compact subspaces of R such that

$$P_n = \bigcup \mathscr{K}_n \,. \tag{4}$$

Put $\mathscr{F}_n = P \cap \mathscr{K}_n$, $\varphi = \{\mathscr{F}_n\}$. We shall prove that φ is complete. Let $\mathscr{A}$ be a maximal φ-Cauchy family. Thus $\mathscr{A} \cap \mathscr{F}_n \neq \emptyset$ for every n, and consequently, in view of (3) and (4) we have

$$\bigcap \overline{\mathscr{A}}^R \subset P \,. \tag{5}$$

On the other hand, $\mathscr{A} \cap K$ is a centered family in a compact space $K \in \mathscr{K}_n$. Thus

$$\bigcap \overline{\mathscr{A}}^R \neq \emptyset \,. \tag{6}$$

Now from (5) it follows at once that $\bigcap \overline{\mathscr{A}}^P = \bigcap \overline{\mathscr{A}}^R$, and in view of (6) we have finally $\bigcap \overline{\mathscr{A}}^P \neq \emptyset$. This completes the proof.

Proposition 5. Given a space P, if there exists a complete sequence of (closed) countable coverings of P, then there exists a complete sequence $\{\mathscr{M}_n\}$ of (closed) countable coverings of P such that every $\mathcal{M}_n$ is well-ordered by inclusion, and if $M_n \in \mathcal{M}_n$, then $\{M_n\}$ is a centered sequence.

Proof. Let $\{\mathcal{L}_n\}$ be a complete sequence of (closed) countable coverings of a space P. Every $\mathcal{L}_n$ can be arranged in a sequence $\{L(n, k)\}_{k=1}^{\infty}$. For every n and k put

$$M(n, k) = \bigcup_{i=1}^{k} L(n, i) .$$

By Proposition 3 of section 3 the sequence $\{\mathcal{M}'_n\}$ is complete, where $\mathcal{M}'_n$ is the family of all sets $M(n, k)$, $k = 1, 2, \ldots$ Now let $P \neq \emptyset$. Put $\mathcal{M}_1 = \mathcal{M}'_1 - (\emptyset)$. Let M_1 be the least set from $\mathcal{M}_1$. Let M_2 be the least set from $\mathcal{M}'_2$ for which $M_1 \cap M_2 \neq \emptyset$. Let $\mathcal{M}_2$ be the family consisting of all $M \in \mathcal{M}'_2$, $M \supset M_2$. Proceeding by induction, we can define $M_n \in \mathcal{M}'_n$ such that

$$\bigcap_{k=1}^{n} M_k \neq \emptyset \quad (n = 1, 2, \ldots) .$$

Put

$$\mathcal{M}_n = \{M;\, M \in \mathcal{M}'_n,\, M \supset M_n\} .$$

Clearly every $\mathcal{M}_n$ is well-ordered by inclusion. Now if $L_k \in \mathcal{M}_k$, then

$$\bigcap_{k=1}^{n} L_k \supset \bigcap_{k=1}^{n} M_k \neq \emptyset .$$

Finally, the sequence $\{\mathcal{M}_n\}$ is complete because every $\mathcal{M}_n$ is a refinement of $\mathcal{M}'_n$ and the sequence $\{\mathcal{M}'_n\}$ is complete. Of course, if $\mathcal{L}$ are closed coverings then $\mathcal{M}_n$ are also closed coverings.

Proposition 6. Given a space P, if there exists a complete sequence of countable coverings of P, then there exists a complete sequence $\{\mathcal{M}_n\}$ of countable coverings of P such that every $\mathcal{M}_n$ is disjoint and $\mathcal{M}_{n+1}$ is a refinement of $\mathcal{M}_n$.

Proof. Let us suppose that $\{\mathcal{L}_n\}$ is a complete sequence of countable coverings of a space P. As in the proof of Proposition 5, we construct the sets $M(n, k)$. Now let $\mathcal{M}'_n$ be the family of all $M(n, k + 1) - M(n, k)$. Finally, let $\mathcal{M}_n$ be the family of all sets of the form $M_1 \cap \ldots \cap M_n$, where $M_i \in \mathcal{M}'_i$. It is easy to see that $\{\mathcal{M}_n\}$ has the desired properties.

Definition 4. A Hausdorff space P will be called a B-space, if there exists a complete sequence of countable coverings of P. The family of all subspaces R of a given space P which are B-spaces will be denoted by B(P).

Theorem 8. *The class* B *of all* B-*spaces has the following properties.*

(1) *If a space P is the image under a one-to-one continuous mapping of a space from* B, *then P belongs to* B.

(2) *If a space P is the inverse image under a perfect mapping of a space from* B, *then P belongs to* B.

(3) *If P is a closed subspace of a space from* B, *then P belongs to* B.

(4) *Every regular $P \in$* B *is normal and Lindelöf.*

(5) *The topological product of a countable number of spaces from* B *is a space from* B.

Proof. The assertions (1) and (2) are corollaries of Theorems 1 and 2 of Section 3. The assertion (3) follows from the fact that if $\{\mathscr{M}\}$ is a complete collection of coverings of a space P and F is closed in P, then $\{\mathscr{M} \cap F\}$ is a complete collection of coverings of F. The assertion (4) is Proposition (4). The assertion (5) follows from the following result.

Proposition 7. For every $n = 1, 2, \ldots$, let μ_n be a complete sequence of (closed) countable coverings of a space P_n. Let P be the topological product of all P_n. For every n let π_n be the projection of P onto P_n. Put

$$\mu = \{\pi_n^{-1}[\mathscr{M}];\ \mathscr{M} \in \mu_n,\ n = 1, 2, \ldots\}$$

where

$$\pi_n^{-1}[\mathscr{M}] = \{\pi_n^{-1}[M];\ M \in \mathscr{M}\}\ .$$

Then μ is a complete countable collection of countable (closed) coverings of P.

Proof. Let $\mathscr{A}$ be a maximal μ-Cauchy family. Clearly the family

$$\mathscr{A}_n = \{\pi_n[A];\ A \in \mathscr{A}\}$$

is a μ_n-Cauchy family. Hence there exists a point x_n in $\bigcap \overline{\mathscr{A}_n}^{P_n}$.

Clearly the point

$$x = \{x_1, x_2, \ldots\} \in P$$

belongs to $\bigcap \overline{\mathscr{A}}$. This completes the proof.

Theorem 9. *If P is a regular space, then the family* B(P) *is closed under countable intersections and unions. Thus* B(P) *contains the family of all* K*-Borelian subsets of P; if P is a* B*-space, then* B(P) *contains all* F(P)*-Borelian subsets of P.*

The proof follows at once from the following result.

Proposition 8. Let R_n be a subspace of a given regular space P. Put

$$R = \bigcap_{n=1}^{\infty} R_n\,, \tag{7}$$

$$S = \bigcup_{n=1}^{\infty} R_n\,. \tag{8}$$

If there exists a complete sequence of countable coverings of all R_n, then there exists a complete sequence of countable coverings of R and S.

Proof. Let μ_n be a complete countable collection of countable coverings of R_n. Let μ be the collection of all $\mathscr{M} \cap R$, where $\mathscr{M} \in \mu_n$ and $n = 1, 2, \ldots$ It is easy to see that μ is a complete countable collection of countable coverings of R. Indeed, if $\mathscr{A}$ is a maximal μ-Cauchy family, then $\mathscr{A}$ is a μ_n-Cauchy family for every $n = 1, 2, \ldots$ By Proposition 2, we have

$$(9) \qquad \emptyset \neq \bigcap \overline{\mathscr{A}}^P \subset R_n .$$

It follows that

$$(10) \qquad \bigcap \overline{\mathscr{A}}^P \subset \bigcap_{n=1}^{\infty} R_n = R .$$

Since $\mathscr{A}$ is a family of subsets of R, from (10) there follows $\bigcap \overline{\mathscr{A}}^P = \bigcap \overline{\mathscr{A}}^R$, and by (9) we have $\bigcap \overline{\mathscr{A}}^R \neq \emptyset$. Now we shall prove that there exists a complete sequence of countable coverings of S. Let $\mu_k = \{\mathscr{M}_n^k\}$ be a complete sequence of countable coverings of R_k, $k = 1, 2, \ldots$ Put

$$(11) \qquad S_i = \bigcup_{k=1}^{i} R_k , \quad S_0 = \emptyset$$

and

$$(12) \qquad \mathscr{M}_n = \bigcup \{\mathscr{M}_n^k \cap (R_k - S_{k-1});\ k = 1, 2, \ldots\} .$$

It is easy to see that $\mu = \{\mathscr{M}_n\}$ is a complete sequence of countable coverings of S. Indeed, if $\mathscr{A}$ is a μ-Cauchy family, then from (11) and (12) it follows at once that $\mathscr{A}$ is a μ_k-Cauchy family for some k, and consequently,

$$\emptyset = \bigcap \overline{\mathscr{A}}^P \subset R_k \subset S$$

and finally, $\bigcap \overline{\mathscr{A}}^S = \bigcap \overline{\mathscr{A}}^P$. This completes the proof.

Theorem 10. *Let* E *be the class of all regular spaces for which there exists a complete sequence of closed countable coverings. The class* E *has the following properties*:

(1) *If a regular space* P *is the inverse image under a perfect mapping of* P *onto a* $Q \in \mathsf{E}$, *then* P *belongs to* E.

(2) *Every space from* E *is a normal Lindelöf space.*

(3) *Every closed subspace of every space from* E *belongs to* E.

(4) *The topological product of a countable number of space from* E *belongs to* E.

(5) *A space* P *belongs to* E *if and only if* P *is completely regular and* P *is a* $F_{\sigma\delta}$ *in the Čech-Stone compactification* $\beta(P)$ *of* P.

Theorem 11. *Let* P *be a regular space. The family* $\mathsf{E}(P)$ *of all subspaces* R *of* P *which belong to* E *is finitely additive and countably multiplicative.*

Proof of Theorems 10 and 11. Assertion (1) follows from Theorem 9. Assertion (2) is Proposition 4. Assertion (3) is obvious. Assertion (4) follows from Proposition 7. Assertion (5) is Theorem 7. The proof of Theorem 10 is complete. The proof of Theorem 11 is elementary and may be left to the reader (*cf.* Proposition 8).

Theorem 12. *A space* P *belongs to* E *if and only if* P *is homeomorphic to a closed subspace of the topological product of a countable number of* σ-*compact regular spaces.*

Proof. Clearly every regular σ-compact space belongs to E. Thus by Theorem 10 every closed subspace of the topological product of a countable number of σ-compact regular spaces is a space from E.

Conversely, let a space P belong to E. By Theorem 10, P is a $F_{\sigma\delta}$ in the Čech-Stone compactification $\beta(P)$ of P. Hence there exist F_σ-subsets R_n of $\beta(P)$ such that

$$P = \bigcap\{R_n;\ n = 1, 2, \ldots\}\,. \tag{13}$$

Clearly all spaces R_n are σ-compact. Let us consider the topological product R of R_n's. For every x in P let $f(x)$ be the point of R all of whose coordinates are x, that is,

$$f(x) = \{x, x, \ldots\}\,.$$

Clearly f is a homeomorphic mapping and in view of (13), $f[P]$ is a closed subset of R. This completes the proof.

Remark. The assertion (4) from Theorem 4 is an immediate consequence of Theorem 6.

Theorem 13. *Every space from* E *is the image under a continuous mapping of a topologically complete space (in the sense of E. Čech) belonging to* E.

Proof. Let P be a space from E. There exists a complete sequence $\varphi = \{\mathscr{F}_n\}$ of closed countable coverings of P. Let us define a new topology for the set P such that the family of all open subsets of P and all sets from $\mathscr{F} = \bigcup\{\mathscr{F}_n;\ n = 1, 2, \ldots\}$ form a sub-base for open sets. The set P with this topology will be denoted by R. It is easy to prove that R is a completely regular space. Indeed, if V is a neighborhood of a point x in R, then there exists a neighborhood U of x in P and an intersection F of a finite number of sets from $\mathscr{F}$ such that

$$x \in U \cap F \subset V\,.$$

Since P is completely regular, there exists a real-valued continuous function f on P such that $f(x) = 1$ and $f[P - U] = (0)$. Put

$$g(y) = \begin{cases} f(x) & \text{for } x \in U \cap F\,, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly g is a continuous function on R, $g(x) = 1$ and $g[R - V] = (0)$. Let $\beta(R) = R_1$, $\beta(P) = P_1$. Put

$$Q_n = \bigcup\{\overline{F}^{R_1};\ F \in \mathscr{F}_n\}\,,$$
$$Q = \bigcap\{Q_n;\ n = 1, 2, \ldots\}\,.$$

Since R_1 is the Čech-Stone compactification and all $F \in \mathscr{F}$ are both closed and open in R, the closures of $F \in \mathscr{F}$ in R_1 are closed and open. Thus every Q_n is σ-compact and open in R_1, and consequently, Q is both a G_δ and a $F_{\sigma\delta}$ in R_1. Let g be the restriction to the set Q of the Čech-Stone mapping f of $R_1 = \beta(R)$ onto $P_1 = \beta(P)$. To prove Theorem 6 it is sufficient to show that

$$f[Q] \subset P\,. \tag{14}$$

If $x \in Q$, we can choose $F_n \in \mathscr{F}_n$ such that

$$x \in \bar{F}_n^{R_1} \quad (n = 1, 2, \ldots) .$$

Since f is a continuous mapping, we have

$$f(x) \in \bigcap_{n=1}^{\infty} \bar{F}_n^{P_1} .$$

On the other hand, since φ is a complete sequence of closed covering, we have by Theorem 13 that

$$\bigcap_{n=1}^{\infty} \bar{F}_n^{P_1} \subset P .$$

Thus $f(x) \in P$ and (14) is proved.

Problem 1. To describe the class of all completely regular spaces P which are $F_{\sigma\delta}$ in every completely regular space $R \supset P$.

Problem 2. Is every regular B-space the image under a one-to-one continuous mapping of a space from E?

Problem 3. By Theorems 3 and 4, if P is a separable complete metrizable space, then $\mathsf{B}(P)$ contains all F-Borelian subsets of P. Is there a $B \in \mathsf{B}(P)$ which is not an F- Borelian subset of P?

5. ANALYTICAL STRUCTURES

We have proved that if f is a one-to-one continuous mapping of a space P onto a space Q, and if $\{\mathscr{M}_n\}$ is a complete sequence of coverings of P, then $\{f[\mathscr{M}_n]\}$ is a complete sequence of coverings of the space Q (*cf.* Section 3, Theorem 2). The following example shows that the assumption "f is one-to-one" is essential.

Example. Let P be the countable infinite discrete space and let $\{P_n\}_{n=1}^{\infty}$ be a decomposition of P such that all P_n's are infinite. The set P can be arranged into a sequence $\{p_n\}$ such that $n \neq m$ implies $p_n \neq p_m$. Let f_n be a one-to-one map of P_n onto $P - \{p_1, \ldots, p_{n-1}\}$. Clearly, f is a continuous mapping of P onto P and the inverses $f^{-1}[p]$ of points $p \in P$ are finite, and hence compact. For every $n = 1, 2, \ldots$, let $\mathscr{M}_n$ be the family of all sets $P_1, \ldots, P_n$ and all the one-point sets (x), where $x \in \bigcup_{k=n+1}^{\infty} P_k$. It is easy to see that $\mu = \{\mathscr{M}_n\}$ is a complete sequence of coverings of P. Indeed, if $\mathscr{A}$ is a μ-Cauchy family, then $\mathscr{A}$ contains a one-point set (x), and consequently, $\bigcap \mathscr{A} \neq \emptyset$. By construction,

$$(P - \{p_1, \ldots, p_{n-1}\}) \in f[\mathscr{M}_n]$$

for every $n = 1, 2, \ldots$ It follows that every centered family of subsets of P is a $\{f[\mathscr{M}_n]\}$-Cauchy family. Since P is not compact, the sequence $\{f[\mathscr{M}_n]\}$ is not complete.

We want to find a structure on a space such that a space P is the continuous image of an E-space if and only if there exists such a structure on P.

Let $\{\mathcal{M}_n\}$ be a complete sequence of countable coverings of a space P. Every $\mathcal{M}_n$ can be arranged in a sequence $\{M_n^k\}_{k=1}^{\infty}$. For every $k_1, \ldots, k_n$ put

$$M(k_1, \ldots, k_n) = \bigcap_{i=1}^{n} M_i^{k_i} .$$

Let $\mathcal{N}_n$ be the family of all $M(k_1, \ldots, k_n)$; every $\mathcal{N}_n$ refines $\mathcal{M}_n$, and hence $\{\mathcal{N}_n\}$ is a complete sequence. Now let f be a continuous mapping of P onto a space Q, and put

$$F(k_1, \ldots, k_n) = f[M(k_1, \ldots, k_n)] .$$

Let us consider a centered family $\mathcal{L}$ of subsets of Q which contain all sets $F(k_1, \ldots, k_n)$, $n = 1, 2, \ldots$, where $\{k_i\}$ is a sequence of positive integers. It is easy to prove (*cf.* Proposition 1) that $\bigcap \overline{\mathcal{L}} \neq \emptyset$.

This leads us to the following definitions:

Let S be the set of all finite sequences of positive integers and let Σ be the set of all infinite sequences of positive integers. If $s \in S$ and $\sigma \in \Sigma$, then we shall write $s = \{s_1, \ldots, s_n\}$, $\sigma = \{\sigma_i\}_{i=1}^{\infty}$. The number n will be called the length of s. Next $\sigma \succ s$ means that s is a section of σ, that is, $\sigma_i = s_i$ for all $i \leqq n$.

A determining system in a family of sets $\mathcal{M}$ is a map M from S to $\mathcal{M}$. We shall write

$$M = \{M(s)\} = \{M(s_1, \ldots, s_n)\} .$$

The nucleus of a determining system M is the set

$$\mathcal{A}(M) = \bigcup_{\sigma \in \Sigma} \bigcap_{s \prec \sigma} M(s) . \tag{1}$$

The operation leading from sets $M(s)$ to the set $\mathcal{A}(M)$ is said to be the Souslin operation or the operation $(\mathcal{A})$.

If $\mathcal{M}$ contains the intersection of every countable subfamily of $\mathcal{M}$, then every determining system M in $\mathcal{M}$ can be extended to a mapping of Σ to $\mathcal{M}$ such that

$$M(\sigma) = \bigcap_{s \prec \sigma} M(s) . \tag{2}$$

Then

$$A(M) = \bigcup_{\sigma \in \Sigma} M(\sigma) . \tag{3}$$

Every set of the form $\mathcal{A}(M)$, where M is a determining system in $\mathcal{M}$, will be called a Souslin set with respect to $\mathcal{M}$, or merely $\mathcal{M}$-Souslin. The family of all $\mathcal{M}$-Souslin sets will be denoted by $\mathcal{A}[\mathcal{M}]$. It is easy to prove that

$$\mathcal{A}[\mathcal{A}[\mathcal{M}]] = \mathcal{A}[\mathcal{M}] . \tag{4}$$

A determining system is said to be regular if

$$M(s_1, \ldots, s_{n+1}) \subset M(s_1, \ldots, s_n) . \tag{5}$$

Definition 5. Let P be a space and let M be a determining system in $\exp P$. We shall say that M is a determining system in the space P. An M-Cauchy family is a centered family $\mathscr{M}$ of subsets of P such that for some σ in Σ every $M(s)$, $s \prec \sigma$, is contained in a set from $\mathscr{M}$. A determining system M in a space P will be called complete if

$$\bigcap \overline{\mathscr{M}} \neq \emptyset \tag{6}$$

for every M-Cauchy family $\mathscr{M}$.

Definition 6. An analytical structure in a space P is a complete regular determining system M in P such that $\mathscr{A}(M) = P$.

Proposition 1. Analytical structures are invariant under continuous mappings; more precisely, if M is an analytical structure in a space P and if f is a continuous mapping of P onto a space Q, then

$$f[M] = \{f[M(s)]\} \tag{7}$$

is an analytical structure in Q.

Proof. Let $\mathscr{L}$ be a maximal $f[M]$-Cauchy family. There exists a $\sigma \in \Sigma$ such that

$$s \prec \sigma \Rightarrow f[M(s)] \in \mathscr{L} .$$

Let $\mathscr{K}$ be the family consisting of all sets of the form $f^{-1}[L]$, $L \in \mathscr{L}$, and all $M(s)$, $s \prec \sigma$. Clearly $\mathscr{K}$ is an M-Cauchy family and

$$\mathscr{L} = \{f[K];\ K \in \mathscr{K}\} . \tag{8}$$

M being an analytical structure, we have $\bigcap \overline{\mathscr{K}} \neq \emptyset$ and by continuity of f, from (8) we have at once $\bigcap \overline{\mathscr{L}} \neq \emptyset$. This completes the proof.

Proposition 2. Let M be a regular determining system in a regular space P and let $\mathscr{A}(M) = P$. Then M is complete if and only if the following two conditions are satisfied:

(I) The sets $F(\sigma) = \bigcap_{s \prec \sigma} \overline{M(s)}$ are compact.

(II) If $\sigma \in \Sigma$ and $\{N(s)\}_{s \prec \sigma}$ is a centered family of sets such that $M(s) \supset N(s)$, then $\bigcap_{s \prec \sigma} \overline{N(s)} \neq \emptyset$.

Proof. First, let us suppose that M is complete. Clearly (II) holds. To prove (I), let $\mathscr{M}$ be a maximal centered family of subsets of a $F(\sigma)$. Let $\mathscr{L}$ be the family of all open sets which contain the closure of some set from $\mathscr{M}$. Since P is regular, we have

$$\bigcap \overline{\mathscr{M}} = \bigcap \overline{\mathscr{L}} . \tag{9}$$

On the other hand, the family $\mathscr{N}$ consisting of all $L \in \mathscr{L}$ and all $M(s)$, $s \prec \sigma$, is clearly an M-Cauchy family. Thus $\bigcap \overline{\mathscr{N}} \neq \emptyset$, and hence $\bigcap \overline{\mathscr{L}} \neq \emptyset$, and finally by (9) $\bigcap \overline{\mathscr{M}} \neq \emptyset$.

Conversely, let us suppose (I) and (II). Let $\mathscr{M}$ be a maximal M-Cauchy family. There exists a $\sigma \in \Sigma$ such that all $M(s)$, $s \prec \sigma$, belong to $\mathscr{M}$. Now from condition (II) it follows at once that $F(\sigma) \cap \overline{\mathscr{M}}$ is a centered family of subsets of $F(\sigma)$, and from condition (I) it follows that

$$\bigcap(F(\sigma) \cap \overline{\mathscr{M}}) \neq \emptyset,$$

and consequently $\bigcap\overline{\mathscr{M}} \neq \emptyset$. This completes the proof.

Remark. From the proof it is clear that even without the assumption of regularity of P the conditions (I) and (II) imply completeness of M.

Proposition 3. The condition (II) in Proposition 2 may be replaced by the following condition

(III) If $\sigma \in \Sigma$ and $\{N(s)\}_{s \prec \sigma}$ is a centered family of sets such that $\overline{M(s)} \supset N(s)$, then $\bigcap_{s \prec \sigma} \overline{N(s)} \neq \emptyset$.

Proof. It is sufficient to prove that if M is complete then (III) holds. Let $\mathscr{L}$ be the family of all open sets which contain some $\overline{M(s)}$, $s \prec \sigma$. Since P is regular, we have

$$\bigcap_{s \prec \sigma} \overline{N(s)} = \bigcap\overline{\mathscr{L}}. \tag{10}$$

From (III) it follows at once that the family $\mathscr{N}$ consisting of all $L \in \mathscr{L}$ and all $M(s)$, $s \prec \sigma$ is an M-Cauchy family. Thus $\bigcap\overline{\mathscr{N}} \neq \emptyset$. Since $\bigcap\overline{\mathscr{N}} \subset \bigcap\overline{\mathscr{L}}$, we have by (10) $\bigcap\overline{N(s)} \neq \emptyset$.

As immediate consequences of Propositions 2 and 3 we have the following two results.

Proposition 4. (a) Let M be a regular determining system in a regular space P such that $\mathscr{A}(M) = P$. For every $s \in S$ put

$$F(s) = \overline{M(s)}. \tag{11}$$

Then M is an analytical structure in P if and only if F is such.

(b) Let M be an analytical structure in a regular space P, $\sigma \in \Sigma$ and let U be an open set containing $F(\sigma) = \bigcap_{s \prec \sigma} \overline{M(s)}$. Then there exists an $s \prec \sigma$ such that $\overline{M(s)} \subset U$.

Proposition 5. Let f be a continuous mapping of a regular space P onto a Hausdorff space Q. Let M be an analytical structure in P. For every $\sigma \in \Sigma$ we have

$$f[\bigcap_{s \prec \sigma} \overline{M(s)}] = \bigcap_{s \prec \sigma} \overline{f[M(s)]}.$$

Proof. Let $L$and R denote the left and right sides, respectively, of the above equality. Clearly $L \subset R$. On the other hand, let y be a point of $Q - L$. Since $L$is compact (as the image under f of a compact subspace $K = \bigcap_{s \prec \sigma} \overline{M(s)}$ of P) and Q is Hausdorff, there exists a closed neighborhood F of y with $F \cap L = \emptyset$. The set $P - f^{-1}[F]$ is an open set containing K. By the preceding Proposition 4, part (b), there exists an $s \prec \sigma$ with

$$\overline{M(s)} \subset P - (P - f^{-1}[F]).$$

It follows that

$$f[\overline{M(s)}] \subset Q - F$$

and finally, F being a neighborhood of y,

$$y \notin \overline{f[\overline{M(s)}]} .$$

Thus $y \notin R$, which proves $R \subset L$.

Proposition 6. Let F be a regular determining system in a space R and let all $F(s)$ be closed. Put

$$P = \mathscr{A}(F) , \tag{12}$$

$$M(s) = P \cap F(s) . \tag{13}$$

Then M is a regular determining system in P, $\mathscr{A}(M) = P$, and if F is complete in R, then M is complete in P, and consequently M is an analytical structure in P.

Proof. The first two assertions are obvious. To prove the third one, let us suppose that $\mathscr{L}$ is a maximal M-Cauchy family in P. Let $\mathscr{M}$ be a maximal centered family of subsets of R containing $\mathscr{L}$. There exists a σ in Σ such that all $M(s)$, $s \prec \sigma$, belong to $\mathscr{L}$, and consequently,

$$s \prec \sigma \Rightarrow F(s) \in \mathscr{M} . \tag{14}$$

Thus $\mathscr{M}$ is an F-Cauchy family, and hence

$$L = \bigcap \overline{\mathscr{M}}^R \neq \emptyset .$$

But by (14), $L \subset F(s)$ for all $s \prec \sigma$, and hence

$$L \subset \bigcap_{s \prec \sigma} F(s) \subset P .$$

It follows that

$$\bigcap \overline{\mathscr{L}}^P \supset L \neq \emptyset ,$$

which proves M is complete in P.

Proposition 7. If K is a determining system in $\mathsf{K}(P)$, where P is a space, then K is a complete determining system in P.

Proof. If $\mathscr{M}$ is a maximal K-Cauchy family, then $\mathscr{M}$ contains a compact subspace of P, in fact a $K(s)$, and consequently, $\bigcap \overline{\mathscr{M}} \neq \emptyset$.

Proposition 8. If M is an analytical structure in a space P and P is a subspace of a Hausdorff space R, then

$$\mathscr{A}(F) = P , \tag{15}$$

where

$$F(s) = \overline{M(s)}^R . \tag{16}$$

Proof. Let us suppose that R is a Hausdorff space and let x be a point of $\mathscr{A}(F)$. Let $\mathscr{L}$ be the family of all neighborhoods of x in R and let σ be an element of Σ such that

$$s \prec \sigma \Rightarrow x \in F(s)\,. \tag{17}$$

Finally, let $\mathscr{M}$ be the family consisting of all $L \cap P$, $L \in \mathscr{L}$, and all $M(s)$, $s \prec \sigma$.

By (16) and (17), $\mathscr{M}$ is a centered family of subsets of P (because $\{M(s);\, s \prec \sigma\}$ is a monotone family). Thus $\mathscr{M}$ is a M-Cauchy family. M being complete, we have

$$\bigcap \overline{\mathscr{M}}^P \neq \emptyset$$

and hence,

$$(x) = \bigcap \overline{\mathscr{L}}^R \supset \bigcap \overline{\mathscr{M}}^P \subset P$$

so that $x \in P$.

Proposition 9. If P is a regular space and if there exists an analytical structure in P, then P is a Lindelöf space, and consequently a normal space.

Proof. By the preceding Proposition 6 there exists an analytical structure F such that all $F(s)$ are closed. Let us suppose that there exists an open covering U of P which contains no countable subcovering of P.

By induction we can construct a $\sigma \in \Sigma$ such that no countable subfamily of U covers any $F(s)$, $s \prec \sigma$. By Proposition 5, the set

$$K = \bigcap_{s \prec \sigma} F(s)$$

is compact. There exists a finite subfamily V of U which covers K. Put $V = \bigcup V$. By Proposition 4, there exists an $s \prec \sigma$ such that

$$F(s) \subset V\,,$$

which contradicts the definition of σ and proves P is a Lindelöf space.

Remark. If M is a determining system then, in general,

$$\mathscr{A}(M) \neq \bigcap_{n=1}^{\infty} \bigcup_{s \in S_n} M(s) \tag{18}$$

where S_n denotes the set of all $s \in S$ of length n. However, if M is regular and

$$s \neq s',\ s, s' \in S_n \Rightarrow M(s) \cap M(s') = \emptyset\,, \tag{19}$$

then equality holds,

$$\mathscr{A}(M) = \bigcap \bigcup M(s)\,. \tag{20}$$

Remark. Till now the letter Σ has been used to denote the set of all infinite sequences of positive integers, or equivalently, the cartesian product of a countably infinite number of copies of the set of all positive integers. From now on we shall also use Σ to denote the topological product of a countably infinite number of copies of the discrete space of all positive integers, in other words, to denote the set Σ with pointwise convergence. Finally, for every $s \in S$ put

$$\Sigma(s) = \{\sigma;\ \sigma \in \Sigma,\ s \prec \sigma\}\,, \tag{21}$$

$$\Sigma = \{\Sigma(s)\}\,. \tag{22}$$

Thus Σ will also be used to denote the regular determining system (22). It is easy to show that every $\Sigma(s)$ is homeomorphic with Σ and that (19) is true (reading Σ instead of M). It is well known that the space Σ is homeomorphic with the space of all irrational numbers from the unit interval of real numbers (to a $\sigma \in \Sigma$ there corresponds an irrational number with continuous fraction expansion $\{0; \sigma_1, \sigma_2, \ldots\}$). Since the space Σ is the topological product of σ-compact spaces, Σ belongs to E.

Now we are prepared to prove the following result. It may be noticed that the idea of the proof is very old.

Proposition 10. If a space P is Souslin with respect to compact subspaces of a space R, then there exists a subspace Q of the topological product $R \times \Sigma$ such that the image of Q under the projection of $R \times \Sigma$ onto R is P and $Q \in \mathsf{K}_{\sigma\delta}(R \times \beta(\Sigma))$.

Proof. Let K be a regular determining system in $\mathsf{K}(R)$, such that

$$P = \mathscr{A}(K)\,. \tag{23}$$

Let us consider the determining system M in $R \times \Sigma$, where

$$M(s) = K(s) \times \Sigma(s)\,. \tag{24}$$

M is a regular determining system satisfying (19) because Σ is such. Put

$$Q_n = \bigcup\{M(s);\ s \in S_n\}\,, \tag{25}$$

$$Q = \bigcap\{Q_n;\ n = 1, 2, \ldots\}\,. \tag{26}$$

It is easy to see that

$$Q_n \in \mathsf{K}_{\sigma\delta}(R \times \beta(\Sigma))\,.$$

Indeed, because Σ belong to $\mathsf{K}_{\sigma\delta}(\beta(P))$, it is sufficient to prove that the family consisting of closures in $\beta(\Sigma)$ of all $\Sigma(s)$, $s \in S_n$, is disjoint; but this is obvious. It remains to prove that P is the projection of Q. If y is a point of Q, then there exists a $\sigma \in \Sigma$ such that $y \in M(s)$ for every $s \prec \sigma$, and consequently, the projection x of y onto R belongs to every $K(s)$, $s \prec \sigma$, and thus finally, x belongs to $\mathscr{A}(K) = P$. This completes the proof.

From the proof of Proposition 10 it is easy to deduce the following generalization (which will not be used in the sequel).

Proposition 10′. If there exists an analytical structure K in a space P, then there exists a subspace Q of $P \times \Sigma$ and a complete sequence of closed countable coverings of Q such that the projection of Q is P. (Moreover, the sequence $\{\{M(s);\ s \in S_n\}\}_{n=1}^{\infty}$ where $M(s)$ are defined by (24), is complete, and K is the projection (defined in a natural manner) of this sequence.)

Theorem 14. *The following conditions on a completely regular space P are equivalent*:

(a) *P is an analytic space.*

(b) *P is the image under a continuous mapping of a* B*-space.*

(c) *There exists an analytical structure in P.*

(d) *There exists an analytical structure F in P such that all F(s) are closed.*

(e) *P is Souslin with respect to closed subsets of every Hausdorff space R containing P, that is,* $P \in \mathscr{A}(\mathsf{F}(R))$ *for every Hausdorff space* $R \supset P$.

(f) *P is Souslin with respect to compact subspace of a Hausdorff space* $R \supset P$, *that is,* $P \in \mathscr{A}(\mathsf{K}(R))$ *for some Hausdorff space* $R \supset P$.

(g) *P is the image under a continuous mapping of a completely regular topologically complete space Q (in the sense of E. Čech, i. e. Q is both a* $F_{\sigma\delta}$ *and* G_δ *in* $\beta(Q)$) *from* E.

Proof. Clearly (a) implies (b). To prove (b) implies (c), it is sufficient to prove that there exists an analytical structure in every E-space because, by Proposition 1, the presence of an analytical structure is invariant under continuous mappings. Let $\{\mathscr{M}_n\}$ be a complete sequence of countable coverings of a space Q. Arranging the covering $\mathscr{M}_n$ into a sequence $\{M_n^k\}_{k=1}^{\infty}$ and putting

$$M(k_1, \ldots, k_n) = \bigcap_{i=1}^{n} M_i^{k_i},$$

we obtain an analytical structure M in Q. By Proposition 4 the conditions (c) and (d) are equivalent. By Proposition 8, (d) implies (e). Clearly (e) implies (f). By Propositions 6 and 7, (f) implies (c). Thus we have proved that the conditions (c), (d), (e), and (f) are equivalent, and (e) implies (b) and (b) implies (c). Since clearly (g) implies (a), it remains to prove, for example, that (f) implies (g); but this follows from Proposition 10. This completes the proof.

Proposition 11. Let f be a perfect mapping of a space P onto a space Q. If M is an analytical structure in Q and if

$$F(s) = f^{-1}[M(s)],$$

then $F = \{F(s)\}$ is an analytical structure in P.

Proof. Let $\mathscr{L}$ be a maximal F-Cauchy family. There exists a $\sigma \in \Sigma$ with

$$s \prec \sigma \Rightarrow F(s) \in \mathscr{L}.$$

Let $\mathscr{M}$ be the family consisting of all $f[L]$, $L \in \mathscr{L}$. Clearly $M(s) \in \mathscr{M}$ for all $s \prec \sigma$ and hence

$$\bigcap \overline{\mathscr{M}} \neq \emptyset.$$

Now from the Lemma of Section 3 (following Theorem 3) we have

$$\bigcap \overline{\mathscr{M}} = f[\bigcap \overline{\mathscr{L}}]$$

and consequently, $\bigcap \overline{\mathscr{L}} \neq \emptyset$.

Theorem 15. *The class* A *of all analytic spaces has the following properties:*

(1) *Every regular space which is the image under a continuous mapping of a space from* A *belongs to* A.

(2) *Every regular space which is the inverse image under a perfect mapping of a space from* A *belongs to* A.

(3) A *is countably productive, i. e. the topological product of a countable number of spaces from* A *belongs to* A.

(4) A *is* F*-hereditary, i. e. closed subspaces of spaces from* A *belong to* A.

(5) *Every space from* A *is a normal Lindelöf space.*

Proof. The first assertion is obvious and the second follows from Proposition 11. To prove the third, let P_n, $n = 1, 2, \ldots$, be analytic spaces. For every n there exists a continuous mapping f_n from a space $Q_n \in \mathsf{E}$ onto P_n. By Theorem 4 of Section 4, the topological product Q of Q_n's belongs to E. For every $x = \{x_n\} \in Q$ put $f(x) = \{f_n(x_n)\} \in P$, where P is the topological product of P_n's. Clearly f is a continuous mapping of Q onto P. Thus P belongs to A, which proves (3). Finally, the assertion (4) and (5) follows, for example, from the corresponding properties of the class E.

For the sake of completeness we shall prove the following result of M. Sion [8] and G. Choguet [3].

Theorem 16. *A metrizable space* P *is analytic if and only if* P *is analytic in the classical sense.*

Proof. In Section 2 we proved that every analytic space in the classical sense is analytic. Conversely, let P be a metrizable analytic space. By Theorem 15, P is a Lindelöf space. P being metrizable, P is a separable space. Let R be a metrizable topologically complete space containing P as a dense subspace. By Theorems 14 and 15, P is Souslin with respect to closed subspaces of R. By a well-known classical theorem, P is the image under a continuous mapping of the space of all irrational numbers of the unit interval $\langle 0, 1\rangle$, and hence P is an analytic space in the classical sense.

Theorem 17. *A regular space* P *is the image under a continuous mapping of the space of all irrational numbers if and only if there exists an analytical structure* F *in* P *such that the sets*

$$F(\sigma) = \bigcap_{s \prec \sigma} \overline{F(s)} \tag{27}$$

each contain only one point.

Proof. First let F be an analytical structure in P such that all $F(\sigma)$'s are one-point. For every $\sigma \in \Sigma$ let $f(\sigma)$ be the point of $F(\sigma)$. Clearly f is a mapping of Σ onto P. Since the space Σ is homeomorphic with space of all irrational numbers, it is sufficient to prove that f is continuous. Let $x = f(\sigma)$ and let V be a neighborhood of x. By proposition there exists a $s' \prec \sigma$ such that

$$\overline{F(s')} \subset V.$$

But clearly

$$f[\Sigma(s')] \subset \overline{F(s')},$$

and consequently,

$$f[\Sigma(s')] \subset V,$$

which shows that f is continuous at the point σ, because the sets $\Sigma(s)$ are open.

Conversely, let f be a continuous mapping of Σ onto a regular space P. Put

$$M(s) = f[\Sigma(s)], \tag{28}$$

$$F(s) = \overline{M(s)}. \tag{29}$$

By Proposition 1, M is an analytical structure in P. By Proposition 4, F is also an analytical structure. It is easy to see that the sets $F(\sigma)$ are one-point. Indeed, by Proposition 5 we have

$$(f(\sigma)) = \bigcap \overline{f[\Sigma(s)]} = F(\sigma).$$

This completes the proof.

For the sake of completeness we shall prove the following result of G. Choquet:

Theorem 18. *Let* $\mathsf{A}(P)$ *denotes the family of all subspaces of* P *which belong to* A. *If* P *is completely regular, then*

$$\mathscr{A}(\mathsf{A}(P)) = \mathsf{A}(P); \tag{30}$$

moreover, if $P \in \mathsf{A}$, *then*

$$\mathsf{A}(P) = \mathscr{A}(\mathsf{F}(P)). \tag{31}$$

Proof. Let R be a compactification of P. By Theorem 1,

$$\mathsf{A}(R) = \mathscr{A}(\mathsf{K}(R)). \tag{32}$$

Clearly

$$Q \subset P \Rightarrow [Q \in \mathsf{A}(P) \Leftrightarrow Q \in \mathsf{A}(R)]. \tag{33}$$

Since in general $\mathscr{A}(\mathscr{A}(\mathscr{M})) = \mathscr{A}(\mathscr{M})$, the assertion (30) follows from (32) and (33).

By Theorem 14, $\mathsf{A}(P) \subset \mathscr{A}(\mathsf{F}(P))$. If $P \in \mathsf{A}$, then by Theorem 15, $\mathsf{F}(P) \subset \mathsf{A}(P)$ and by (30) $\mathscr{A}(\mathsf{F}(P)) \subset \mathsf{A}(P)$. Thus (31) is proved.

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Резюме

О ДЕСКРИПТИВНОЙ ТЕОРИИ МНОЖЕСТВ

ЗДЕНЕК ФРОЛИК (Zdeněk Frolík), Прага

В статье вводится понятие аналитической структуры, и при помощи него дается внутренняя характеризация вполне регулярных аналитических пространств в смысле Шоке-Шнейдера и регулярных непрерывных образов пространства иррациональных чисел. Далее в статье рассматриваются пространства, на которых существует полная последовательность счетных покрытий или счетных замкнутых покрытий (полнота такой последовательности определяется, грубо говоря, требованием, чтобы пересечение замыканий множеств, взятых по одному из каждого покрытия, было компактным).