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PROBABILITIES ON NON-COMMUTATIVE SEMIGROUPS

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Let $\mathfrak{M} = \mathfrak{M}(S)$ be the semigroup of normalized non-negative measures on a finite semigroup S. The purpose of this paper is the study of the structure of $\mathfrak{M}$, in particular, the determination of all subgroups of $\mathfrak{M}$, the emphasis being on the non-commutative case.

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a finite semigroup. By a measure μ we shall denote a non-negative additive set function defined on the subsets of S such that $\mu(S) = 1$. The set of all measures on S will be denoted by $\mathfrak{M} = \mathfrak{M}(S)$.

If $\nu_1, \nu_2 \in \mathfrak{M}$, we define the product $\nu_1 * \nu_2$ by the relation

$$\nu_1 * \nu_2(x) = \sum_{uv=x} \nu_1(u)\, \nu_2(v)\,.$$

With this multiplication $\mathfrak{M}$ becomes a semigroup.

Let $\mathfrak{A}(S)$ be the semigroup algebra of S, i.e. the set of all formal real linear forms of the form $\sum_{x_i \in S} t_{x_i} \,.\, x_i$ with coordinate-wise addition, obvious scalar multiplication and the multiplication defined by

$$(1) \qquad \Big(\sum_{i=1}^{n} t'_{x_i} \,.\, x_i\Big)\Big(\sum_{k=1}^{n} t''_{x_k} \,.\, x_k\Big) = \sum_{i=1}^{n} \sum_{k=1}^{n} t'_{x_i} \,.\, t''_{x_k} x_i x_k\,.$$

Denote by $\mathfrak{F} = \mathfrak{F}(S)$ the subset of all elements $\in \mathfrak{A}(S)$ for which $0 \leqq t_{x_i} \leqq 1$ and $\sum_{x_i \in S} t_{x_i} = 1$. It is well known and easy to prove that

$$(2) \qquad \nu(x) \leftrightarrow \nu(x_1)\, x_1 + \nu(x_2)\, x_2 + \ldots + \nu(x_n)\, x_n \in \mathfrak{F}(S)$$

is an isomorphic mapping of the semigroup $\mathfrak{M}$ onto the semigroup $\mathfrak{F}(S)$.

Since in our case it is often convenient to deal with the elements $\in \mathfrak{F}(S)$ rather then with the elements $\in \mathfrak{M}(S)$ we shall use the following notation: The element $\nu(x_1)\, x_1 + \ldots + \nu(x_n)\, x_n \in \mathfrak{F}(S)$ corresponding in the isomorphism (2) to the element $\nu \in \mathfrak{M}$ will be denoted by the symbol ν^*. Hence $\nu^* \in \mathfrak{F}(S)$ and $\nu \in \mathfrak{M}(S) \leftrightarrow \nu^* \in \mathfrak{F}(S)$ is a semigroup isomorphism.

Since there can arise no misunderstanding we shall write $\nu \,.\, \mu$ instead of $\nu * \mu$ (i.e. we shall use the same symbol for multiplication in $\mathfrak{M}(S)$ and $\mathfrak{F}(S)$). Further without fear of misunderstanding we shall suppose S imbedded in $\mathfrak{F}(S)$.

Introduce in the set $\mathfrak{F}(S)$ a topology by the requirement $v_\alpha^* = t_{x_1}^{(\alpha)}x_1 + \ldots + t_{x_n}^{(\alpha)}x_n \to \to v^* = t_{x_1}x_1 + \ldots + t_{x_n}x_n$ if and only if $t_{x_i}^{(\alpha)} \to t_{x_i}$ $(i = 1, 2, \ldots, n)$. In this topology $\mathfrak{F}(S)$ becomes clearly a compact Hausdorff semigroup. The same is true for $\mathfrak{M}(S)$ if $v_\alpha \to v \in \mathfrak{M}(S)$ is defined by $v_\alpha^* \to v^* \in \mathfrak{F}(S)$.

The purpose of this paper is to study the structure of $\mathfrak{M}(S)$ and its relation to the structure of S.

In the last few years various contributions to this problem have been given in several special cases by various authors. Н. Н. Воробьев [24] treated the case in which S is a finite abelian group. E. HEWITT and H. S. ZUCKERMAN [9] have given a detailed treatment of the case of a finite abelian semigroup admitting relative inverses. The case of compact groups has been treated by Б. М. Клосс [11], [12], J. GLICKSBERG [6], K. STROMBERG [23] and H. S. COLLINS [4]. Б. М. Клосс [12] and J. Glicksberg [6] have also studied the case of a compact abelian semigroup. The essential novelty of the present paper is that it goes beyond the restriction of commutativity even in the non-group case. We restrict ourselves to the case of finite semigroups though a number of results can be transferred to the compact case as well.

In section 1 we recall some known results concerning compact semigroups.

In section 2 we construct all idempotents $\in \mathfrak{M}(S)$. Every simple subsemigroup of S is a support of idempotents and in Theorem 2,3 an explicit formula for these idempotents is given.

In section 3 the primitive idempotents $\in \mathfrak{M}(S)$ are identified. Their union is the kernel of $\mathfrak{M}(S)$. The structure of this kernel is fully described.

Then our attention is concentrated to the subgroups of $\mathfrak{M}(S)$. First in section 5 we deal with simple semigroups without zero. The subgroups of $\mathfrak{M}(S)$ are studied in detail. Theorem 5,3 gives an explicit description of any measure μ contained in a subgroup of $\mathfrak{M}(S)$. The description of the support $C(\mu)$ of such a $\mu \in \mathfrak{M}(S)$ leads to a double coset decomposition of a simple subsemigroup of S modulo an other simple subsemigroup of S. Though $\mathfrak{M}(S)$ is infinite every group contained in $\mathfrak{M}(S)$ is finite. Also some information concerning the elements μ which are not contained in a subgroup of $\mathfrak{M}(S)$ have been given (Theorem 5,2). Shortly to say the group-membership of an element $\mu \in \mathfrak{M}(S)$ identifies μ to a large extent.

Now if S is a general (finite) semigroup we decompose S into a union of the so called Green's F-classes (which are well-known in the theory of abstract semigroups, and appear in this connection for the first time). Every F-class containing an idempotent contains (in general not a unique) maximal simple subsemigroup without zero of S. These are those simple semigroups which are in some sense decisive for the existence, location and structure of the subgroups of $\mathfrak{M}(S)$. If μ is contained in a subgroup of $\mathfrak{M}(S)$, Theorem 7,1 locates, so to speak, $C(\mu)$. In fact, $C(\mu)$ remains in the same F-class as $C(\varepsilon)$, ε being the unit element of the group containing μ. If μ is not contained in a subgroup of $\mathfrak{M}(S)$ and belongs to a given idempotent ε some restriction as to the location of $C(\mu)$ with respect to $C(\varepsilon)$ is possible (see Theorem 7,3) but the extreme complexity seems not to allow any more precise description.

In section 9 and 10 some rather special cases of the general theory are discussed.

A limit theorem concerning the sequence $\mu, \mu^2, \mu^3, \dots$ is given in Theorem 8,1. The existence of the limit is reduced to some algebraic properties of the semigroup S. In Theorem 8,6 $\lim_{n=\infty} (1/n) \sum_{i=1}^{n} \mu^i$ is studied.

In section 11 we deal with maximal idempotents $\in \mathfrak{M}(S)$.

In spite of the large series of theorems a number of problems remains open. Nevertheless it seems that by means of the results proved below some new problems become accessible, f.i. the question of infinite products (see [9], [23], [24]) which is not considered in this paper.

1. PRELIMINARIES

For the convenience of the reader we recall in this section some known results on compact semigroups which we shall freely use in all the paper.

1,1. Let S be a compact Hausdorff semigroup and $a \in S$. Denote by $\Gamma_n(a)$, $n \geqq 1$, the closure of the set $\{a^n, a^{n+1}, a^{n+2}, \dots\}$. It is known that $\Gamma_1(a)$ contains a unique idempotent e. In fact $\bigcap_{n=1}^{\infty} \Gamma_n(a)$ is a (closed commutative) group containing e as its unit element. We shall say that *a belongs to e*. The set of all elements belonging to e will be denoted by $K(e)$. Since every element $\in S$ belongs exactly to one idempotent e_α, the semigroup S can be written in the form of a union $S = \bigcup_{e_\alpha \in E} K(e_\alpha)$, where $E = \{e_\alpha \mid \alpha \in \Lambda\}$ is the set of all idempotents $\in S$. The set $K(e_\alpha)$ need not be a semigroup.

To every $e_\alpha \in E$ there exists a unique *maximal group* $G(e_\alpha)$ such that $G(e_\alpha)$ is the largest subgroup of S containing e_α as a unit element. Clearly $G(e_\alpha) \subseteqq K(e_\alpha)$. The group $G(e_\alpha)$ is closed and for $e_\alpha \neq e_\beta$ we have $G(e_\alpha) \cap G(e_\beta) = \emptyset$. The group $G(e)$ belonging to e can be described also by the relation $G(e) = eSe \cap \{x \mid e \in Sx \cap xS\}$. For every $e_\alpha \in E$ we have $e_\alpha K(e_\alpha) = K(e_\alpha)\, e_\alpha = G(e_\alpha)$. An element $a \in S$ will be called *regular* if $a \in \bigcup_{e_\alpha \in E} G(e_\alpha)$. An element $a \in K(e_\alpha)$ is regular if and only if $ae_\alpha = e_\alpha a = a$ holds.

The proofs of these statements can be found in the papers K. Numakura [16], A. D. Wallace [25], [26], [27], Š. Schwarz [22].

1,2. Let S be *any* semigroup. A subset $L \subset S$ is called a left ideal of S if $SL \subset L$. A right ideal is defined analogously. A subset which is both a left and a right ideal is called a two-sided ideal.

An ideal (left, right, two-sided) of S is called minimal if it does not contain as a proper subset a (left, right, two-sided) ideal of S.[1])

[1]) If S has a zero element, minimal (left, right, two-sided) ideals are often defined as minimal but not zero. In this paper we use the word minimal in the strict sense, i.e. if S has a zero element, minimal ideals reduce to the zero element itself.

Two minimal left (right) ideals (if such exist) have an empty intersection. If the intersection N of all two-sided ideals of a semigroup S is non-vacuous it is the unique minimal two-sided ideal of S. N is called the *kernel* of S.

It is known (see A. H. CLIFFORD [3] or E. C. Ляпин [15]): If S contains at least one minimal left and at least one minimal right ideal, then S contains a kernel. Hereby $N = \bigcup_{\alpha \in \Lambda_1} R_\alpha$, where R_α runs through all minimal right ideals of S and $N = \bigcup_{\alpha \in \Lambda_2} L_\alpha$, where L_α runs through all minimal left ideals of S. Further $R_\alpha \cap L_\beta = R_\alpha L_\beta = G_{\alpha\beta}$ is a group and $N = \bigcup_{\alpha \in \Lambda_1} \bigcup_{\beta \in \Lambda_2} G_{\alpha\beta}$ is a union of disjoint isomorphic groups. Finally for any $\alpha \in \Lambda_1$, $\beta \in \Lambda_2$ we have $L_\beta R_\alpha = N$. The groups $G_{\alpha\beta}$ will be called the *group-components* of N.

1,3. Let now S be a *compact* semigroup. K. Numakura [16] was the first who proved that such a semigroup contains at least one minimal left and at least one minimal right ideal. Every minimal left ideal L is closed and it can be written in the from $L = Se$, where e is an idempotent $\in L$. An analogous statement holds for every minimal right ideal. This implies that a compact Hausdorff semigroup has always a kernel N, which is a closed subset of S. If e_α, $e_\beta \in N$, we have either $Se_\alpha \cap Se_\beta = \emptyset$ or $Se_\alpha = Se_\beta$. The set $e_\alpha S e_\beta$ is one of the group-components of the decomposition $N = \bigcup_{\alpha \in \Lambda_1} \bigcup_{\beta \in \Lambda_2} G_{\alpha\beta}$. The groups $G_{\alpha\beta}$ are closed, in fact all being maximal subgroups of S.

1,4. A semigroup S is called to be *simple* if it does not contain any two-sided ideal $\neq S$. In this case the kernel N is identical with S.[2]) The semigroup S is simple if and only if $SaS = S$ for every $a \in S$. The kernel of any semigroup (if it exists) is a simple semigroup.

If, in particular, S is a *compact simple* semigroup it follows from 1,2 that S can be written in the form $S = \bigcup_{\alpha \in \Lambda_1} R_\alpha = \bigcup_{\beta \in \Lambda_2} L_\beta$. We have $L_\alpha R_\beta = S$ and $R_\alpha \cap L_\beta = R_\alpha L_\beta = G_{\alpha\beta}$ is a closed group for every couple $\alpha \in \Lambda_1$, $\beta \in \Lambda_2$. Clearly $R_\alpha = \bigcup_{\beta \in \Lambda_2} G_{\alpha\beta}$, $L_\beta = \bigcup_{\alpha \in \Lambda_1} G_{\alpha\beta}$. If, for instance, S is a finite simple semigroup having s minimal right ideals and r minimal left ideals the situation can schematically be described by the table

	L_1	L_2	$\dots$	L_l
R_1	G_{11}	G_{12}	$\dots$	G_{1r}
R_2	G_{21}	G_{22}	$\dots$	G_{2r}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
R_s	G_{s1}	G_{s2}	$\dots$	G_{sr}

[2]) It should be emphasized that in sections 1—5 we use the words "simple semigroup" in the sense just introduced. Beginning with section 6 we also introduce "simple semigroups with zero" and the simple semigroups will often be called "simple semigroups without zero". (These two notions coincide if and only if S contains a single element.)

Now we recall some relations which will often be used in the following. Hereby we use the notations: $g_{\alpha\beta}$ denotes an element $\in G_{\alpha\beta}$, $e_{\alpha\beta}$ is the unit element of $G_{\alpha\beta}$.

a) $L_\gamma g_{\alpha\beta} = L_\beta$, $g_{\alpha\beta} R_\gamma = R_\alpha$.

b) $\{e_{\alpha\beta} \mid \alpha \in \Lambda_1\}$ is the set of all idempotents contained in L_β. Each of these elements is a right unit of the semigroup L_β. The set $\{e_{\alpha\beta} \mid \beta \in \Lambda_2\}$ is the set of all idempotents $\in R_\alpha$. Each of them is a left unit of R_α.

c) Any two minimal left ideals L_α, L_β are isomorphic. The corresponding mapping can be realised by $x \in L_\alpha \to x e_{\alpha\beta} \in L_\beta$. The inverse mapping is $y \in L_\beta \to y e_{\beta\alpha} \in L_\alpha$. If, in particular, S is finite, all minimal left ideals have the same number of elements.

d) $g_{\alpha\beta} L_\gamma = G_{\alpha\gamma}$, $R_\gamma g_{\alpha\beta} = G_{\gamma\beta}$.

e) $G_{\alpha\beta} g_{\gamma\delta} = G_{\alpha\delta}$, $g_{\gamma\delta} G_{\alpha\beta} = G_{\gamma\beta}$.

f) Finally $G_{\alpha\beta} G_{\gamma\delta} = G_{\alpha\delta}$.

Remark. Note that we have $e_{\alpha\beta} e_{\gamma\delta} \in G_{\alpha\delta}$ but $e_{\alpha\beta} e_{\gamma\delta} = e_{\alpha\delta}$ – in general – need not hold since the product of two idempotents need not be an idempotent. Of course it follows from b) that we have always $e_{\gamma\beta} e_{\alpha\beta} = e_{\gamma\beta}$ and $e_{\alpha\beta} e_{\alpha\gamma} = e_{\alpha\gamma}$.

1,5. We have remarked above that the kernel N of a compact semigroup is a simple semigroup. It is worthy of notice that S can contain also other subsemigroups which themselves are simple semigroups. F.i. every subgroup of S has this property. The "maximal" simple subsemigroups of S will play an essential role in this paper.

1,6. A semigroup T is called *left simple* if it does not contain any left ideal $\neq T$. Every left simple semigroup is simple. A semigroup T is left simple if and only if $Ta = T$ for every $a \in T$, i.e. the equation $xa = b$ has a solution $x \in T$ for every couple $a, b \in T$.

Every minimal left ideal of any semigroup S (if such exists) is a left simple semigroup. Note again that a semigroup can contain also other left simple subsemigroups.

If T is compact, we can use the results of 1,4 which in this case take a simple form. Let $E = \{e_\alpha \mid \alpha \in \Lambda\}$ be the set of all idempotents $\in S$. Every $e_\alpha \in E$ is a right unit of T. Further $G_\alpha = e_\alpha T$ is a group and $G_\alpha \cap G_\beta = \emptyset$ for $\alpha \neq \beta$. We have $T = \bigcup_{\alpha \in \Lambda} G_\alpha$. For every $x \in G_\alpha$ we have $xT = G_\alpha$ and for every $y \in T$ we have $G_\alpha y = G_\alpha$. Each of the groups G_α is a minimal right ideal of T. The semigroup T is isomorphic with the direct product $G \times E_0$ where $G \cong G_\alpha$ and E_0 is a semigroup in which $xy = x$ for every couple $x, y \in E_0$.

1,7. Finally we mention that we shall use some results proved in the paper [18] concerning subsemigroups of a completely simple semigroup.

A semigroup is called *completely simple* if it contains at least one right and at least one left minimal ideal. In this paper all simple semigroups which will occur will be completely simple. This follows from the result mentioned above (see 1,3) since S and $\mathfrak{M}(S)$ are compact, and we shall have to deal only with closed simple subsemigroups.

Special cases of the results of the paper [18] which we shall need in the following are:

Lemma 1,1. *Let S be a compact simple semigroup and H a closed subsemigroup of S. Then H is a completely simple semigroup. If* $S = \bigcup_{\alpha \in \Lambda_2} L_\alpha$ *is the decomposition of S into the union of its minimal left ideals and if* $L'_\alpha = L_\alpha \cap H \neq \emptyset$, *then L'_α is a minimal left ideal of H. Conversely, every minimal left ideal L'_β of H is of the form form* $L'_\beta = L_\beta \cap H$ *with a suitably chosen minimal left ideal L_β of S.*

Lemma 1,2. *Let* $S = \bigcup_{\alpha \in \Lambda_2} L_\alpha = \bigcup_{\beta \in \Lambda_1} R_\beta$ *be the decomposition of a compact simple semigroup into the union of its minimal left and right ideals respectively. Let H be a closed subsemigroup of S containing at least one maximal group of S. Then* $H = [\bigcup_{\alpha \in \Lambda_1'} R_\alpha] \cap [\bigcup_{\beta \in \Lambda_2'} L_\beta]$, *where Λ'_1, Λ'_2 are suitably chosen subsets of Λ_1 and Λ_2 respectively.*

Lemma 1,3. *Let H be a closed subsemigroup of a compact simple semigroup S, which contains all idempotents $\in S$. Then there exists a decomposition of S into pairwise disjoint classes of the form*

$$S = H \cup HaH \cup HbH \cup \ldots$$

Some further facts concerning this decomposition proved in [18] will be mentioned at the appropriate places below.

2. THE IDEMPOTENTS OF $\mathfrak{M}(S)$

Let S be finite and $\mu \in \mathfrak{M}(S)$. The set $C(\mu) = \{x \mid \mu(x) \neq 0\}$ is called the *support* of μ. It is known (see [19], more generally Б. M. Клосс [12]):

Lemma 2,1. $C(\mu\nu) = C(\mu)\, C(\nu)$.

For $\mu^* \in \mathfrak{F}(S)$ we define $C(\mu^*) = C(\mu)$. If $C(\mu)$ contains a single element x_i, we shall say that μ is the *point mass* at x_i. Such a measure will be denoted by ε_{x_i}. Hence

$$\varepsilon_{x_i}(x) = \begin{cases} 0 & \text{for } x \neq x_i, \\ 1 & \text{for } x = x_i, \end{cases}$$

or otherwise $\varepsilon^*_{x_i} = x_i$.

Clearly $\varepsilon_{x_i}\varepsilon_{x_k} = \varepsilon_{x_i x_k}$. If $\mathfrak{S}$ is the set of all such measures, we have clearly $\mathfrak{S} \cong S$. If, in particular, x_i is an idempotent $\in S$, we have $\varepsilon_{x_i}\varepsilon_{x_i} = \varepsilon_{x_i}$. The semigroup $\mathfrak{M}(S)$ contains therefore idempotents. Of course, these are in general not all idempotents $\in \mathfrak{M}(S)$.

If μ is an idempotent $\in \mathfrak{M}$, the relation $\mu^2 = \mu$ implies by Lemma 2,1 $C(\mu)\, C(\mu) = C(\mu)$, i.e. $C(\mu)$ is a semigroup. Б. M. Клосс ([12], Theorem 14) proved the following essentially stronger assertion:

Theorem 2,1. *Let be $\mu \in \mathfrak{M}(S)$ and $\mu^2 = \mu$, then $C(\mu)$ is a simple subsemigroup of S.*

We first prove that conversely every simple subsemigroup of S is the support of some idempotent $\in \mathfrak{M}$.

Lemma 2,2. *Let P be a simple subsemigroup of a finite semigroup S. Let $P = \bigcup_{\alpha=1}^{r} L_\alpha$ be the decomposition of P into the union of its minimal left ideals. Define μ by the requirements $\mu(x) = \mu(y) = t_\alpha > 0$ for every couple $x, y \in L_\alpha$ $(\alpha = 1, 2, \ldots, r)$ and $\sum_{\alpha=1}^{r} \mu(L_\alpha) = 1$. Then μ is an idempotent $\in \mathfrak{M}(S)$ and $C(\mu) = P$.*

Proof. Let be $L_\alpha = \{x_1^{(\alpha)}, x_2^{(\alpha)}, \ldots, x_p^{(\alpha)}\}$. By supposition we have $\mu(x_1^{(\alpha)}) = \mu(x_2^{(\alpha)}) = \ldots = \mu(x_p^{(\alpha)}) = t_\alpha > 0$ and $\sum_{\alpha=1}^{r} t_\alpha p = 1$. In the mapping $\mu \leftrightarrow \mu^*$ we have

$$\mu^* = \sum_{\alpha=1}^{r} t_\alpha(x_1^{(\alpha)} + x_2^{(\alpha)} + \ldots + x_p^{(\alpha)}) .$$

We wish to prove $\mu^2 = \mu$.

Since $L_\alpha x_i^{(\beta)} \subset L_\alpha L_\beta \subset L_\beta$ and L_β is a minimal left ideal of P, we have $L_\alpha x_i^{(\beta)} = L_\beta$ for every β and any i $(1 \leqq i \leqq p)$. Hence

$$t_\alpha(x_1^{(\alpha)} + \ldots + x_p^{(\alpha)})\, t_\beta x_i^{(\beta)} = t_\alpha t_\beta(x_1^{(\beta)} + \ldots + x_p^{(\beta)}) .$$

This implies

$$[\sum_{\alpha=1}^{r} t_\alpha(x_1^{(\alpha)} + \ldots + x_p^{(\alpha)})]\, t_\beta x_i^{(\beta)} = t_\beta(x_1^{(\beta)} + \ldots + x_p^{(\beta)}) \sum_{\alpha=1}^{r} t_\alpha = \\ = \frac{1}{p}\, t_\beta(x_1^{(\beta)} + \ldots + x_p^{(\beta)}) .$$

Further

$$[\sum_{\alpha=1}^{r} t_\alpha(x_1^{(\alpha)} + \ldots + x_p^{(\alpha)})]\, [t_\beta(x_1^{(\beta)} + \ldots + x_p^{(\beta)})] = \frac{1}{p} \cdot p \cdot t_\beta(x_1^{(\beta)} + \ldots + x_p^{(\beta)}) = \\ = t_\beta(x_1^{(\beta)} + \ldots + x_p^{(\beta)}) .$$

Finally

$$[\sum_{\alpha=1}^{r} t_\alpha(x_1^{(\alpha)} + \ldots + x_p^{(\alpha)})] \cdot [\sum_{\beta=1}^{r} t_\beta(x_1^{(\beta)} + \ldots + x_p^{(\beta)})] = \sum_{\beta=1}^{r} t_\beta(x_1^{(\beta)} + \ldots + x_p^{(\beta)}) ,$$

i. e. $\mu^* \mu^* = \mu^*$ and $\mu^2 = \mu$, q. e. d.

Needless to say that an analogous statement can be formulated by replacing minimal left ideals by minimal right ideals.

If μ is an idempotent and $C(\mu) = L_1 \cup L_2 \cup \ldots \cup L_r$ the decomposition of $C(\mu)$ into the union of its minimal left ideals, then μ does not necessarily take the same values in all elements of a fixed chosen ideal L_α. This is shown on the following example.

Example 2,1. Let $S = \{a_1, a_2, a_3, a_4\}$ be the simple semigroup with the following multiplication table:

	a_1	a_2	a_3	a_4
a_1	a_1	a_3	a_3	a_1
a_2	a_4	a_2	a_2	a_4
a_3	a_1	a_3	a_3	a_1
a_4	a_4	a_2	a_2	a_4 .

To get some material for further purposes we shall find all idempotents $\in \mathfrak{M}$. Write $\mu^* = t_1 a_1 + t_2 a_2 + t_3 a_3 + t_4 a_4$, $0 \leqq t_i \leqq 1$, $\sum_{i=1}^{4} t_i = 1$. Elementary calculations show that $\mu^{*2} = \mu^*$ holds if and only if

$$(3) \qquad (t_1 + t_3)(t_1 + t_4) = t_1\,, \quad (t_2 + t_3)(t_2 + t_4) = t_2\,,$$
$$(t_1 + t_3)(t_2 + t_3) = t_3\,, \quad (t_2 + t_4)(t_1 + t_4) = t_4\,.$$

Put $t_1 + t_3 = \xi_1$, $t_2 + t_4 = \xi_2$, $t_1 + t_4 = \eta_1$, $t_2 + t_3 = \eta_2$. We then have $\xi_1 + \xi_2 = \eta_1 + \eta_2 = 1$ and the relations (3) imply $t_1 = \xi_1\eta_1$, $t_2 = \xi_2\eta_2$, $t_3 = \xi_1\eta_2$, $t_4 = \xi_2\eta_1$. Conversely: Choose four real non-negative numbers such that $\xi_1 + \xi_2 = \eta_1 + \eta_2 = 1$. Construct the numbers $t_1 = \xi_1\eta_1$, $t_2 = \xi_2\eta_2$, $t_3 = \xi_1\eta_2$, $t_4 = \xi_2\eta_1$, then the quadruple (t_1, t_2, t_3, t_4) clearly satisfies the relations (3).

Hence: We obtain all idempotents $\mu^* \in \mathfrak{F}(S)$ if in

$$\xi_1(\eta_1 a_1 + \eta_2 a_3) + \xi_2(\eta_1 a_4 + \eta_2 a_2)$$

the numbers $\xi_1, \xi_2, \eta_1, \eta_2$ run through all non-negative real numbers satisfying $\xi_1 + \xi_2 = \eta_1 + \eta_2 = 1$.

Now $S = L_1 \cup L_2 = \{a_1, a_4\} \cup \{a_2, a_3\}$ is the decomposition of S into the union of its minimal left ideals. On $L_1 = \{a_1, a_4\}$ we have $\mu(a_1) = \xi_1\eta_1$, $\mu(a_4) = \xi_2\eta_1$, hence certainly $\mu(a_1) \neq \mu(a_4)$ if $\xi_1 \neq \xi_2$ and $\eta_1 \neq 0$.

Our semigroup is an idempotent semigroup (i.e. every element is an idempotent). Every maximal group is therefore a one point group and the example shows at the same time that an idempotent measure may assume different values on different maximal groups of S.

Our next goal is the proof of the very important assertion that an idempotent measure takes the same values in the points of a fixed group.

We recall first a formula which we shall use for this purpose. Let μ be an idempotent $\in \mathfrak{M}(S)$. Let F be a real-valued function on $S = \{x_1, x_2, \ldots, x_n\}$. We then have

$$\sum_{i=1}^{n} F(x_i)\,\mu(x_i) = \sum_{i=1}^{n} F(x_i)\,[\mu\,\mu(x_i)] = \sum_{i=1}^{n} F(x_i) \sum_{uv=x_i} \mu(u)\,\mu(v) =$$
$$= \sum_{u\in S} \sum_{v\in S} F(uv)\,\mu(u)\,\mu(v)\,.$$

This can be reformulated as follows: If $\mu^* = t_{x_1}x_1 + \ldots + t_{x_n}x_n$ is an idempotent $\in \mathfrak{F}(S)$ and $F(x)$ a real-valued function defined on S, we have

$$\sum_{x_i \in S} F(x_i)\, t_{x_i} = \sum_{x_i \in S} \sum_{y_k \in S} F(x_i y_k)\, t_{x_i} t_{y_k}\,. \tag{4}$$

We now prove:

Lemma 2,3. *Let S be a semigroup, $\mu \in \mathfrak{M}(S)$ an idempotent and $P = C(\mu) = \{x_1, x_2, \ldots, x_q\}$. Denote $\mu^* = t_{x_1}x_1 + \ldots + t_{x_q}x_q$,* where $t_{x_i} > 0$ $(i = 1, 2, \ldots, q)$ and $\sum_{i=1}^{q} t_{x_i} = 1$. *Let L be an arbitrary fixed chosen minimal left ideal of P. Let f denote any real-valued function defined on P. Then $\sum_{k=1}^{q} f(x_k\xi)\, t_{x_k}$ has the same value for every $\xi \in L$.*

Proof. Let e be an idempotent $\in L$. Denote $\varphi(y) = \sum_{k=1}^{q} f(x_k y e)\, t_{x_k}$. For $y \in P$ we have $x_k y e \subset PPL = L$; hence $\varphi(y)$ is a real-valued function defined for every $y \in P$. Put in (4) $F(x_i) = f(x_i y e)$. We have[3])

$$\sum_{x_i \in S} F(x_i)\, t_{x_i} = \sum_{i=1}^{q} f(x_i y e)\, t_{x_i} = \sum_{x_i \in S} \sum_{z_k \in S} F(x_i z_k)\, t_{x_i} t_{z_k} =$$
$$= \sum_{k=1}^{q} \sum_{i=1}^{q} F(z_k x_i)\, t_{x_i} t_{z_k} = \sum_{i=1}^{q} \sum_{k=1}^{q} f(z_k x_i y e)\, t_{x_i} t_{z_k} =$$
$$= \Big\{\sum_{k=1}^{q} f(z_k x_1 y e)\, t_{z_k}\Big\}\, t_{x_1} + \Big\{\sum_{k=1}^{q} f(z_k x_2 y e)\, t_{z_k}\Big\}\, t_{x_2} + \ldots + \Big\{\sum_{k=1}^{q} f(z_k x_q y e)\, t_{z_k}\Big\}\, t_{x_q}\,.$$

Hence

$$\varphi(y) = \varphi(x_1 y)\, t_{x_1} + \varphi(x_2 y)\, t_{x_2} + \ldots + \varphi(x_q y)\, t_{x_q}\,.$$

Suppose that $\varphi(y)$ takes its greatest value in the point $y_0 \in P$. Then

$$\varphi(y_0) = \varphi(x_1 y_0)\, t_{x_1} + \ldots + \varphi(x_q y_0)\, t_{x_q}\,,$$

and, since $t_{x_1} + \ldots + t_{x_q} = 1$,

$$\{\varphi(y_0) - \varphi(x_1 y_0)\}\, t_{x_1} + \{\varphi(y_0) - \varphi(x_2 y_0)\}\, t_{x_2} + \ldots +$$
$$+ \{\varphi(y_0) - \varphi(x_q y_0)\}\, t_{x_q} = 0\,.$$

Since each of the brackets is $\geqq 0$, we have

$$\varphi(y_0) = \varphi(x_1 y_0) = \ldots = \varphi(x_q y_0)\,.$$

This means: $\sum_{k=1}^{q} f(x_k y e)\, t_{x_k}$ takes the same values for $y = y_0$, $y = x_1 y_0, \ldots, y = x_q y_0$, in other words the same value for every $y \in P y_0$. Hence $\sum_{k=1}^{q} f(x_k \xi)\, t_{x_k}$ takes the same value for every $\xi \in P y_0 e$. Now $P y_0 e \subset P y_0 L \subset L$ and since L is a minimal left ideal of P, we have $P y_0 e = L$. This proves Lemma 2,3.

[3]) Hereby the set $\{z_1, \ldots, z_q\}$ is identical with $\{x_1, \ldots, x_q\}$.

Remark 1. The "dual" edition of Lemma 2,3 is as follows. Suppose that the suppositions of Lemma 2,3 are satisfied and let R be a minimal right ideal of P. If f is any fixed chosen real-valued function defined on P, then $\sum_{k=1}^{q} f(\xi x_k)\, t_{x_k}$ assumes the same value for every $\xi \in R$.

Remark 2. Example 2,1 shows that for ξ's chosen from different minimal left ideals $I = \sum_{k=1}^{q} f(x_k \xi)\, t_{x_k}$ can assume different values. In fact, for $\xi \in L_1 = \{a_1, a_4\}$ we have $I = f(a_1)(t_1 + t_3) + f(a_4)(t_2 + t_4) = \xi_1 f(a_1) + \xi_2 f(a_4)$. For $\xi \in L_2 = \{a_2, a_3\}$ we have $I = f(a_3)(t_1 + t_3) + f(a_2)(t_2 + t_4) = \xi_1 f(a_3) + \xi_2 f(a_2)$. By choosing suitably f and ξ_1, ξ_2 these numbers can be made different.

Theorem 2,2. *If $\mu \in \mathfrak{M}(S)$ is an idempotent and G any fixed chosen group-component of $C(\mu)$, then $\mu(x) = \mu(y)$ for every couple $x, y \in G$.*

Proof. Let be $C(\mu) = P = \{x_1, x_2, \ldots, x_q\}$ and $P = \bigcup_{i=1}^{r} L_i = \bigcup_{k=1}^{s} R_k$ (in the meaning introduced above). Write $\mu^* = t_{x_1}x_1 + \ldots + t_{x_q}x_q$, where $t_{x_i} > 0$ $(i = 1, 2, \ldots, q)$ and $\sum_{i=1}^{q} t_{x_i} = 1$. By (4) we have for any real valued function f defined on P (with $\{x_1, \ldots, x_q\} = \{y_1, \ldots, y_q\}$)

$$(5) \qquad \{\sum_{i=1}^{q} f(x_k y_1)\, t_{x_k}\}\, t_{y_1} + \ldots + \{\sum_{k=1}^{q} f(x_k y_q)\, t_{x_k}\}\, t_{y_q} = \sum_{k=1}^{q} f(x_i)\, t_{x_i}\,.$$

Without loss of generality we shall consider the group $G_{11} = R_1 \cap L_1$. Let hereby be $G_{11} = \{x_1, x_2, \ldots, x_m\}$.

Take for $f(x)$ the function $\Phi_{x_1}(x)$ defined by $\Phi_{x_1}(x) = 1$ for $x = x_1$ and $\Phi_{x_1}(x) = 0$ for $x \neq x_1$. By Lemma 2,3 $\sum_{k=1}^{q} f(x_k y)\, t_{x_k} = \sum_{k=1}^{q} \Phi_{x_1}(x_k y)\, t_{x_k}$ assumes for every $y \in L_1$ the same value. If $y \notin L_1$ (hence $y \in L_i$ for a suitably chosen $i \neq 1$), we have $x_k y \in PL_i = L_i$, hence $\Phi_{x_1}(x_k y) = 0$ for every x_k $(k = 1, 2, \ldots, q)$ and therefore $\sum_{k=1}^{q} f(x_k y)\, t_{x_k} = 0$.

The left ideal L_1 can be written in the form $L_1 = \bigcup_{k=1}^{s} [L_1 \cap R_k] = G_{11} \cup G_{21} \cup \ldots \cup G_{s1}$. It contains $v = ms$ different elements. Let us denote them by $y_1, y_2, \ldots, y_v$. Hence $L_1 = \{y_1, y_2, \ldots, y_v\}$ and $\{x_1, x_2, \ldots, x_m\} \subset \{y_1, y_2, \ldots, y_v\}$.

Write in (5) $f(x) = \Phi_{x_1}(x)$. On the left hand side only those brackets can be different from zero in which y is equal to one of the elements $y_1, y_2, \ldots, y_v$. Hereby all these brackets are equal one to the other. On the right hand side there is only one member different from zero, namely $\Phi_{x_1}(x_1)\, t_{x_1} = t_{x_1}$. Hence

$$(6) \qquad \{\sum_{k=1}^{q} \Phi_{x_1}(x_k y)\, t_{x_k}\} \,.\, (t_{y_1} + t_{y_2} + \ldots + t_{y_v}) = t_{x_1}\,, \quad y \in L_1\,.$$

Introducing the functions $\Phi_{x_2}(x), \ldots, \Phi_{x_m}(x)$ defined by

$$\Phi_{x_i}(x) = \left\langle \begin{array}{ll} 1 & \text{for} \quad x = x_i \\ 0 & \text{for} \quad x \neq x_i \end{array} \right. \quad (i = 2, 3, \ldots, m),$$

and repeating the same argument as with $\Phi_{x_1}(x)$, we obtain the following $m - 1$ equations

$$(7) \quad \{\sum_{k=1}^{q} \Phi_{x_i}(x_k y)\, t_{x_k}\}\,(t_{y_1} + t_{y_2} + \ldots + t_{y_v}) = t_{x_i}, \quad y \in L_1 \quad (i = 2, 3, \ldots, m).$$

To prove now that $t_{x_1} = t_{x_2} = \ldots = t_{x_m}$ (i.e. to prove our theorem) it is sufficient to prove

$$\sum_{k=1}^{q} \Phi_{x_1}(x_k y)\, t_{x_k} = \sum_{k=1}^{q} \Phi_{x_2}(x_k y)\, t_{x_k} = \ldots = \sum_{k=1}^{q} \Phi_{x_m}(x_k y)\, t_{x_k}.$$

Denote $I_1 = \sum_{k=1}^{q} \Phi_{x_1}(x_k y)\, t_{x_k}$, $I_2 = \sum_{k=1}^{q} \Phi_{x_2}(x_k y)\, t_{x_k}$. We show $I_1 = I_2$. Since I_1 takes the same value for every $y \in G_{11} = \{x_1, \ldots, x_m\}$ we have in particular $I_1 = \sum_{k=1}^{q} \Phi_{x_1}(x_k x_2) \,.$ $.\, t_{x_k}$. Analogously there holds $I_2 = \sum_{k=1}^{q} \Phi_{x_2}(x_k z)\, t_{x_k}$ for every $z \in G_{11}$. Our assertion will be proved if we can show that $\sum_{k=1}^{q} \Phi_{x_1}(x_k x_2)\, t_{x_k}$ is equal to $\sum_{k=1}^{q} \Phi_{x_2}(x_k z)\, t_{x_k}$ for a suitably chosen $z \in G_{11}$.

Construct in the group G_{11} the element $z = x_2 x_1^{-1} x_2$. We show that

$$\sum_{k=1}^{q} \Phi_{x_1}(x_k x_2)\, t_{x_k} = \sum_{k=1}^{q} \Phi_{x_2}(x_k \,.\, x_2 x_1^{-1} x_2)\, t_{x_k}.$$

To this end it is sufficient to prove that $x_k x_2 = x_1$ holds in P for those and only those x_k for which $x_k(x_2 x_1^{-1} x_2) = x_2$.

a) Let be $x_k x_2 = x_1$. Multiply both sides by $x_1^{-1} x_2 \in G_{11}$. We have $x_k x_2 x_1^{-1} x_2 = x_1 x_1^{-1} x_2$. Now since $x_1 x_1^{-1}$ is the unit element of the group G_{11} the right hand side is equal to x_2. Hence $x_k(x_2 x_1^{-1} x_2) = x_2$.

b) Let be conversely $x_k(x_2 x_1^{-1} x_2) = x_2$. Multiply to the right by the element $(x_1^{-1} x_2)^{-1} \in G_{11}$. We have $x_k x_2 (x_1^{-1} x_2)(x_1^{-1} x_2)^{-1} = x_2 (x_1^{-1} x_2)^{-1}$. Since $(x_1^{-1} x_2) \,.$ $.\, (x_1^{-1} x_2)^{-1}$ is the unit element of the group G_{11} and x_2 is contained in G_{11} we get on the left hand side $x_k x_2$. On the right hand side (since we are dealing with elements of the group G_{11}) we get x_1. Hence $x_k x_2 = x_1$.

This completes the proof of Theorem 2,2.

In what follows we shall consequently use the following notations. If P is a simple subsemigroup of S, then $P = \bigcup_{i=1}^{s} R_i = \bigcup_{k=1}^{r} L_k$ denotes its decomposition into the union of minimal left and right ideals respectively. Further $G_{ik} = R_i L_k$, and $P = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} G_{ik}$ is the decomposition of P into the union of groups. The number of different elements of G_{ik} will be denoted by m. The elements of the group G_{ik} will be denoted by $g_{ik}^{(1)}$, $g_{ik}^{(2)}, \ldots, g_{ik}^{(m)}$.

By $[G_{ik}]$ we shall denote the following element $\in \mathfrak{F}(S)$:

$$[G_{ik}] = \frac{1}{m}(g_{ik}^{(1)} + g_{ik}^{(2)} + \ldots + g_{ik}^{(m)}).$$

The support of $[G_{ik}]$ is G_{ik}. The following relations will be useful:

a) If $g_{jl}^{(\alpha)} \in G_{jl}$, we know that $G_{ik}g_{jl}^{(\alpha)} = G_{il}$. This implies $[G_{ik}]\, g_{jl}^{(\alpha)} = [G_{il}]$ for every i, k, j, l.

b) The relation $G_{ik}G_{jl} = G_{il}$ implies

$$[G_{ik}][G_{jl}] = [G_{ik}] \cdot \frac{1}{m}(g_{jl}^{(1)} + \ldots + g_{jl}^{(m)}) = \frac{1}{m}\{\underbrace{[G_{il}] + \ldots + [G_{il}]}_{m \text{ times}}\}.$$

Hence $[G_{ik}][G_{jl}] = [G_{il}]$.

Let now be ε an idempotent $\in \mathfrak{M}$. Denote $C(\varepsilon) = P$ and write $P = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} G_{ik}$. By Theorem 2,2 ε^* is necessarily of the form

$$\varepsilon^* = \sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik}[G_{ik}] \quad \text{with} \quad \sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik} = 1$$

(t_{ik} being positive numbers). Now ε^* is an idempotent if and only if

$$\sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik}[G_{ik}] \cdot \sum_{j=1}^{s} \sum_{l=1}^{r} t_{jl}[G_{jl}] = \sum_{i=1}^{s} \sum_{l=1}^{r} t_{il}[G_{il}].$$

This implies

$$\sum_{i=1}^{s} \sum_{k=1}^{r} \sum_{j=1}^{s} \sum_{l=1}^{r} t_{ik}t_{jl}[G_{il}] = \sum_{i=1}^{s} \sum_{l=1}^{r} t_{il}[G_{il}]$$

and

$$\sum_{k=1}^{r} \sum_{j=1}^{s} t_{ik}t_{jl} = t_{il},$$

i.e.

(8) $$(\sum_{k=1}^{r} t_{ik}) \cdot (\sum_{j=1}^{s} t_{jl}) = t_{il}.$$

These are rs relations for rs „unknowns" t_{ik}.[4])

Put $\sum_{k=1}^{r} t_{ik} = \xi_i$, $\sum_{j=1}^{s} t_{jl} = \eta_l$ $(i = 1, 2, \ldots, s;\ l = 1, 2, \ldots, r)$. Then $\sum_{i=1}^{s} \xi_i = 1$, $\sum_{l=1}^{r} \eta_l = 1$ and (8) implies $t_{il} = \xi_i \eta_l$.

---

[4]) Also the relation $\sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik} = 1$ is a consequence of (8). For summing through i we have $(\sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik})(\sum_{j=1}^{s} t_{jl}) = \sum_{i=1}^{s} t_{il}$ and since for at least one l $\sum_{i=1}^{s} t_{il} \neq 0$, we have $\sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik} = 1$.

Consider conversely the element $\varepsilon^* = \sum_{i=1}^{s} \sum_{l=1}^{r} \xi_i \eta_l [G_{il}] \in \mathfrak{F}(S)$, where ξ_i, η_l are positive numbers satisfying $\sum_{i=1}^{s} \xi_i = \sum_{l=1}^{r} \eta_l = 1$. We then have

$$(\varepsilon^*)^2 = \sum_{i=1}^{s} \sum_{l=1}^{r} \xi_i \eta_l [G_{il}] \cdot \sum_{j=1}^{s} \sum_{k=1}^{r} \xi_j \eta_k [G_{jk}] = \sum_{i} \sum_{l} \sum_{j} \sum_{k} \xi_i \eta_l \xi_j \eta_k [G_{ik}] =$$
$$= \left(\sum_{l=1}^{r} \eta_l\right)\left(\sum_{j=1}^{s} \xi_j\right) \cdot \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k [G_{ik}] = \varepsilon^* .$$

We have proved:

Theorem 2,3. *Let P be a simple subsemigroup of a finite semigroup S. Let $P = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} G_{ik}$ be its decomposition into the union of disjoint groups. Then every idempotent $\varepsilon^* \in \mathfrak{F}(S)$ whose support is P is of the form*

$$\varepsilon^* = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k [G_{ik}] , \tag{9}$$

where ξ_i, η_k $(i = 1, \ldots, s;\ k = 1, \ldots, r)$ are positive numbers satisfying $\sum_{i=1}^{s} \xi_i = \sum_{k=1}^{r} \eta_k = 1$.

Conversely if ξ_i, η_k are positive numbers satisfying $\sum_{i=1}^{s} \xi_i = \sum_{k=1}^{r} \eta_k = 1$, then $\sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k [G_{ik}]$ is an idempotent $\in \mathfrak{F}(S)$ whose support is exactly P.

Remark. If in (9) we admit some ξ_i or η_k to be zero, the formula (9) gives again idempotents but the corresponding support is not the whole set P, in fact it is a proper subset of P. Since the support of any idempotent is a simple semigroup, $C(\varepsilon)$ is then also a simple semigroup. The group-components of $C(\varepsilon)$ are identical with some of the groups G_{ik}. (This is being emphasized since P can contain also simple subsemigroups the group-components of which are isomorphic with proper subgroups of the group G_{ik}.)

More precisely we have

Theorem 2,4. *Let P satisfy the suppositions of Theorem 2,3. If the numbers ξ_i, η_k are non-negative real numbers satisfying $\sum_{i=1}^{s} \xi_i = \sum_{k=1}^{r} \eta_k = 1$, then $\varepsilon^* = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \,.\, \eta_k [G_{ik}]$ is an idempotent whose support is either P or such a simple subsemigroup of P whose group-components are isomorphic with G_{ik}.*

Conversely: If Q is a subsemigroup of P, the group-components of which are isomorphic with the maximal groups of P, then every idempotent $\varepsilon^* \in \mathfrak{F}(S)$ having Q for its support is obtained by putting in (9) suitably chosen ξ_i and η_k equal to zero.

Proof. a) Suppose — without loss of generality — that $\xi_{\sigma+1} = \ldots = \xi_s = \eta_{\varrho+1} = \ldots = \eta_r = 0$ $(\sigma < s, \varrho < r)$. The support of the idempotent $\varepsilon^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [G_{ik}]$ is the set

$$Q = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G_{ik} = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} R_i L_k = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} (R_i \cap L_k) = \{R_1 \cup \ldots \cup R_\sigma\} \cap \{L_1 \cup \ldots \cup L_\varrho\}.$$

This is clearly a subsemigroup of P, the group-components of which are isomorphic with the group-components of P.

b) Let conversely Q be a subsemigroup of P having the property mentioned in the formulation of our theorem. Then by Lemma 1,2 Q can be written in the form

$$Q = \{R_{i_1} \cup R_{i_2} \cup \ldots \cup R_{i\sigma}\} \cap \{L_{k_1} \cup L_{k_2} \cup \ldots \cup L_{k\varrho}\}, \tag{10}$$

where $\{i_1, i_2, \ldots, i_\sigma\}$ is a subset of the set of indices $\{1, 2, \ldots, s\}$ and analogously $\{k_1, k_2, \ldots, k_\varrho\} \subset \{1, 2, \ldots, r\}$. By rearranging the relation (10) we have $Q = \bigcup_{\alpha=1}^{\sigma} \bigcup_{\beta=1}^{\varrho} G_{i_\alpha k_\beta}$. Choose positive real numbers $\xi_{i_1}, \ldots, \xi_{i_\sigma}, \eta_{k_1}, \ldots, \eta_{k_\rho}$ such that $\sum_{\alpha=1}^{\sigma} \xi_{i_\alpha} = \sum_{\beta=1}^{\varrho} \eta_{k_\beta} = 1$. Then $\varepsilon^* = \sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\varrho} \xi_{i_\alpha} \eta_{k_\beta} [G_{i_\alpha k_\beta}]$ is an idempotent whose support is Q. But this idempotent can be obtained from the formulae (9) by choosing $\xi_{i_1}, \ldots, \xi_{i_\sigma}, \eta_{k_1}, \ldots, \eta_{k_\rho}$ different from zero while the other ξ_i, η_k are zeros and $\sum_{\alpha=1}^{\sigma} \xi_{i_\alpha} = \sum_{\beta=1}^{\varrho} \eta_{k_\beta} = 1$. This proves our assertion.

3. PRIMITIVE IDEMPOTENTS $\in \mathfrak{M}(S)$ [3a)]

An idempotent π of a semigroup T is said to be primitive if there does not exist an idempotent $\mu \in T$, $\mu \neq \pi$ such that $\pi\mu = \mu\pi = \mu$ holds.

It is known that every idempotent of a compact simple semigroup is primitive (see K. Numakura [16]).

More generally:

Lemma 3,1. *Let T be a compact semigroup and K its kernel. Then those and only those idempotents $\in T$ which are contained in K are primitive idempotents of T.*

[3a)] (Added March 20, 1962.) Some results of this section have been published in the meantime in the paper H. S. Collins, The kernel of a semigroup of measures, Duke Math. J. *28* (1961), 381—392 (September 1961).

(Added in proofs, June 1963.) The actual interest in the subject is reflected by the fact that during the last year further papers concerning this matter have been published. (See H. S. Collins [30], [31], H. S. Collins - R. J. Koch [32], J. S. Pym [33].) A generalization of some results of this paper to the compact case has been announced by the author in the preliminary report [34].

Remark. This lemma is in one or other form known. More detailed information concerning this object can be found in R. J. KOCH [13]. I find it convenient to give a proof, since R. J. Koch is treating the case of a semigroup with zero, where the primitive idempotents are defined in a slightly other form.

Proof. a) Let π be an idempotent $\in T$ contained in K. Since K is a simple semigroup there is no idempotent $\mu \in K$, $\mu \neq \pi$ for which $\pi\mu = \mu\pi = \mu$ holds. At the same time there cannot exist an idempotent $\mu \in T - K$ for which $\pi\mu = \mu$ holds, since $\pi\mu \in \in K\mu \subset K$, a contradiction to $\mu \in T - K$.

b) Let be $\pi_1 \in T - K$. We prove that π_1 cannot be a primitive idempotent of T. Since K is closed in T, $K\pi_1$ is a closed subset of T. At the same time $K\pi_1$ is a subsemigroup since $K\pi_1 \,.\, K\pi_1 \subset KTK\pi_1 \subset K\pi_1$. Analogously $\pi_1 K$ is a closed subsemigroup of T. The intersection $D = K\pi_1 \cap \pi_1 K$ is non-vacuous since $\pi_1 K \,.\, K\pi_1 \subset D$. D is a closed subsemigroup of T. Hence it is compact and it contains therefore an idempotent ε. Since $\varepsilon \in K\pi_1$, $\varepsilon \in \pi_1 K$, there exist elements $s_1, s_2 \in K$ with $\varepsilon = s_1\pi_1 = = \pi_1 s_2$. We have

$$\pi_1\varepsilon = \pi_1(\pi_1 s_2) = \pi_1 s_2 = \varepsilon\,,$$
$$\varepsilon\pi_1 = (s_1\pi_1)\,\pi_1 = s_1\pi_1 = \varepsilon\,.$$

Further we certainly have $\varepsilon \neq \pi_1$, since $\varepsilon \in K$, $\pi_1 \in T - K$. There exists therefore an idempotent $\varepsilon \neq \pi_1$ such that $\pi_1\varepsilon = \varepsilon\pi_1 = \varepsilon$, i.e. π_1 is not a primitive idempotent $\in T$. This proves Lemma 3,1.

Consider the semigroup $\mathfrak{M}(S)$. Since it is compact, it contains a kernel $\mathfrak{N}$ and the idempotents $\in \mathfrak{N}$ (and only these idempotents of $\mathfrak{M}$) are primitive idempotents of $\mathfrak{M}$.

The purpose of this section is to characterize these primitive idempotents. In the papers [19] and [20] I have shown that primitive idempotents play an important role in studying right invariant measures on a semigroup. Every right invariant measure on a semigroup S (if such exists!) is a primitive idempotent $\in \mathfrak{M}$. The converse is not true. The relation between the structure of $\mathfrak{M}$ and the existence of right invariant measures will be studied elsewhere.

Theorem 3,1. *Let S be a semigroup and N its kernel. Let $P \subset N$ be a subsemigroup of N the group-compotents of P being maximal groups of N. Then there exists a primitive idempotent $\in \mathfrak{M}(S)$ the support of which is exactly P.*

More precisely: Every idempotent the support of which is equal to P is a primitive idemponent $\in \mathfrak{M}(S)$.

Proof. Write (in the sense agreed above) $N = \bigcup_{i=1}^{s} R_i = \bigcup_{k=1}^{r} L_k$. Note also that every minimal left ideal of N is at the same time a minimal left ideal of S and that $G_{ik} = = R_i L_k$ are maximal groups of S (and hence maximal groups of N).

By Lemma 1,2 the supposition concerning P implies that P can be written in the form

$$P = \{R_{i_1} \cup R_{i_2} \cup \ldots \cup R_{i_\sigma}\} \cap \{L_{k_1} \cup L_{k_2} \cup \ldots \cup L_{k_\rho}\}\,,$$

where $\{i_1, i_2, \ldots, i_\sigma\} \subset \{1, 2, \ldots, s\}$, $\{k_1, k_2, \ldots, k_\varrho\} \subset \{1, 2, \ldots, r\}$.

Without loss of generality suppose that $\{i_1, \ldots, i_\sigma\} \equiv \{1, 2, \ldots, \sigma\}$, and $\{k_1, k_2, \ldots, k_\varrho\} \equiv \{1, 2, \ldots, \varrho\}$. We then have

$$P = \{\bigcup_{i=1}^{\sigma} R_i\} \cap \{\bigcup_{k=1}^{\varrho} L_k\} \quad \text{and} \quad N = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G_{ik} .$$

Choose positive real numbers $\xi_1, \ldots, \xi_\sigma, \eta_1, \ldots, \eta_\varrho$ such that $\xi_1 + \ldots + \xi_\sigma = \eta_1 + \ldots + \eta_\varrho = 1$ and construct the idempotent π with

$$\pi^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [G_{ik}] .$$

We then have $C(\pi) = P$. (We know that every idempotent π with $C(\pi) = P$ is obtained in this manner.)

Let ε be an idempotent $\in \mathfrak{M}(S)$ for which $\varepsilon = \pi\varepsilon = \varepsilon\pi$ holds. To prove our theorem it is sufficient to show that this relation implies $\pi = \varepsilon$.

$$C(\varepsilon) = C(\pi)\, C(\varepsilon) = P\, C(\varepsilon) \subset \{R_1 \cup \ldots \cup R_\sigma\}\, C(\varepsilon) \subset R_1 \cup \ldots \cup R_\sigma ,$$
$$C(\varepsilon) = C(\varepsilon)\, C(\pi) = C(\varepsilon)\, P \subset C(\varepsilon)\, \{L_1 \cup \ldots \cup L_\varrho\} \subset L_1 \cup \ldots \cup L_\varrho .$$

[Hereby we are using the fact that $L_i(R_k)$ are minimal left (right) ideals of the whole semigroup S.] Hence

$$C(\varepsilon) \subset \{R_1 \cup \ldots \cup R_\sigma\} \cap \{L_1 \cup \ldots \cup L_\varrho\} = P .$$

The relation $C(\varepsilon) = P\, C(\varepsilon) = C(\varepsilon)\, P$ implies $C(\varepsilon) = P\, C(\varepsilon)\, P$. Choose any element $x \in C(\varepsilon) \subset P$. Since P is simple, we have $PxP = P$. Therefore $C(\varepsilon) = P\, C(\varepsilon)\, P \supset PxP = P$, whence $C(\varepsilon) = P$. We proved: If there is an idempotent ε with $\varepsilon = \pi\varepsilon = \varepsilon\pi$, we necessarily have $C(\varepsilon) = P$, hence ε^* can be written in the form

$$\varepsilon^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi'_i \eta'_k [G_{ik}] ,$$

where $\xi'_1, \ldots, \xi'_\sigma, \eta'_1, \ldots, \eta'_\varrho$ are positive numbers satisfying $\xi'_1 + \ldots + \xi'_\sigma = \eta'_1 + \ldots + \eta'_\varrho = 1$.

The relation $\pi\varepsilon = \varepsilon$ implies

$$\sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [G_{ik}] \cdot \sum_{j=1}^{\sigma} \sum_{l=1}^{\varrho} \xi'_j \eta'_l [G_{jl}] = \sum_{i=1}^{\sigma} \sum_{l=1}^{\varrho} \xi'_i \eta'_l [G_{il}] ,$$
$$(\sum_{k=1}^{\varrho} \eta_k)(\sum_{j=1}^{\sigma} \xi'_j) \sum_{i=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_i \eta'_l [G_{il}] = \sum_{i=1}^{\sigma} \sum_{l=1}^{\varrho} \xi'_i \eta'_l [G_{il}] ,$$

hence $\xi_i \eta'_l = \xi'_i \eta'_l$, i.e. $\xi_i = \xi'_i$ $(i = 1, 2, \ldots, \sigma)$. Analogously the relation $\varepsilon\pi = \varepsilon$ implies $\xi'_i \eta_l = \xi'_i \eta'_l$, therefore $\eta_l = \eta'_l$ $(l = 1, 2, \ldots, \varrho)$. Hence $\varepsilon = \pi$, which completes the proof of Theorem 3,1.

Corollary 3,1. *Let S be a semigroup with the kernel N. Then there exists a primitive idempotent $\in \mathfrak{M}(S)$ the support of which is exactly N.*

Lemma 3,2. *Let P be a simple subsemigroup of the semigroup S. Let N be the kernel of S. Then either $P \subset N$ or $P \cap N = \emptyset$.*

Proof. Let be $a \in P \cap N$. Since P is simple, $PaP = P$. Now $PaP \subset PNP \subset N$, hence $P \subset N$, q. e. d.

Theorem 3,2. *If π is a primitive idempotent $\in \mathfrak{M}(S)$, then $C(\pi) \subset N$.*

Proof. Since $C(\pi)$ is a simple semigroup, we have (by Lemma 3,2) either $C(\pi) \subset N$ or $C(\pi) \cap N = \emptyset$. Suppose $C(\pi) \cap N = \emptyset$; we prove that π cannot be a primitive idempotent.

By Corollary 3,1 there is a primitive idempotent v with $C(v) = N$. Consider the measure $\pi v \pi$ (which need not be an idempotent). If $\mathfrak{N}$ is the kernel of $\mathfrak{M}$, the sets $\mathfrak{N}\pi v \pi$ and $\pi v \pi \mathfrak{N}$ are left and right ideals of $\mathfrak{M}$ respectively. These sets are closed and their intersection $\mathfrak{D} = \mathfrak{N}\pi v \pi \cap \pi v \pi \mathfrak{N}$ is non-empty, since $\emptyset \neq \pi v \pi \mathfrak{N} . \mathfrak{N} \pi v \pi \subset \mathfrak{D}$. The set $\mathfrak{D}$ is compact, hence it contains an idempotent ε. This element can be written in both forms $\varepsilon = \sigma_1 \pi v \pi = \pi v \pi \sigma_2$, $\sigma_1, \sigma_2 \in \mathfrak{N}$. Clearly $\varepsilon\pi = \pi\varepsilon = \varepsilon$. Hereby $\varepsilon \neq \pi$, since $C(\varepsilon) = C(\sigma_1)\, C(\pi)\, C(v)\, C(\pi) \subset C(\sigma_1)\, C(\pi)\, N\, C(\pi) \subset N$ (while $C(\pi) \subset S - N$). π is not a primitive idempotent. This contradiction proves Theorem 3,2.

We now prove that the idempotents mentioned in Theorem 3,1 are exactly all primitive idempotents $\in \mathfrak{M}(S)$.

Theorem 3,3. *Let π be a primitive idempotent $\in \mathfrak{M}(S)$. Then $C(\pi)$ is a union of some maximal groups contained in the kernel N.*

Proof. Denote $C(\pi) = P'$. Then P' is a simple subsemigroup contained in N. Suppose that the group-components of P' are not maximal groups of N. We show that π cannot be a primitive idempotent.

Let $P' = L'_1 \cup \ldots \cup L'_\varrho = R'_1 \cup \ldots \cup R'_\sigma$ be the decomposition of P' into the union of minimal left and right ideals of P' respectively and $N = L_1 \cup \ldots \cup L_r = R_1 \cup \cup \ldots \cup R_s$ the corresponding decomposition of N. By Lemma 1,1 to every L'_i there is an L_j $(1 \leqq j \leqq r)$ such that $L'_i = P' \cap L_j$. Analogously for minimal right ideals R'_i. Without loss of generality let be $L'_i = P' \cap L_i$ $(i = 1, 2, \ldots, \varrho)$ and $R'_i = R_i \cap P'$ $(i = 1, 2, \ldots, \sigma)$.

Consider the semigroup

$$P = (R_1 \cup R_2 \cup \ldots \cup R_\sigma) \cap (L_1 \cup L_2 \cup \ldots \cup L_\varrho) .$$

Denoting $G_{ik} = R_i L_k$, $R'_i L'_k = G'_{ik}$ we have $P' = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G'_{ik}$, $P = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G_{ik}$, and according to the supposition we have $G'_{ik} \subset G_{ik}$ and $G'_{ik} \neq G_{ik}$.

By Theorem 2,3 π^* can be written in the form

$$\pi^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [G'_{ik}] , \quad 0 < \xi_i \leqq 1 , \quad 0 < \eta_k \leqq 1 , \quad \sum_{i=1}^{\sigma} \xi_i = \sum_{k=1}^{\varrho} \eta_k = 1 .$$

To prove that π is not a primitive idempotent $\in \mathfrak{M}$ it is sufficient to find an idempotent ε such that $\pi \neq \varepsilon$ and $\pi\varepsilon = \varepsilon\pi = \varepsilon$.

Construct the measure ε with $\varepsilon^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [G_{ik}]$. Then ε is an idempotent and $\varepsilon \neq \pi$, since $C(\varepsilon) = P \supsetneqq C(\pi) = P'$.

If $G_{ik} = \{g_{ik}^{(1)}, \ldots, g_{ik}^{(m)}\}$, $G'_{ik} = \{g'^{(1)}_{ik}, \ldots, g'^{(m')}_{ik}\}$, $m' < m$, m'/m, denote

$$[G_{ik}] = \frac{1}{m}(g_{ik}^{(1)} + \ldots + g_{ik}^{(m)}), \quad [G'_{ik}] = \frac{1}{m'}(g'^{(1)}_{ik} + \ldots + g'^{(m')}_{ik}).$$

For $g'_{jl} \in G'_{jl} \subset G_{jl}$ we have $[G_{ik}]\, g'_{jl} = [G_{il}]$ and $g'_{jl}[G_{ik}] = [G_{jk}]$. Therefore

$$[G_{ik}]\,[G'_{jl}] = [G_{ik}] \cdot \frac{1}{m'}(g'^{(1)}_{jl} + \ldots + g'^{(m')}_{jl}) = \frac{1}{m'}\, \{\underbrace{[G_{il}] + \ldots + [G_{il}]}_{m'\text{-times}}\},$$

i.e. $[G_{ik}]\,[G'_{jl}] = [G_{il}]$ and analogously $[G'_{jl}]\,[G_{ik}] = [G_{jk}]$. Now we have

$$\varepsilon^*\pi^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [G_{ik}] \cdot \sum_{j=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_j \eta_l [G'_{jl}] = (\sum_{k=1}^{\varrho} \eta_k)(\sum_{j=1}^{\sigma} \xi_j) \sum_{i=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_i \eta_l [G_{il}] = \varepsilon^*.$$

Analogously $\pi^*\varepsilon^* = \varepsilon^*$. Hence π is not a primitive idempotent. This proves Theorem 3,3.

Summarily we have proved:

Theorem 3,4. *Let S be a finite semigroup, $N = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} G_{ik}$ the decomposition of its kernel into the union of maximal groups. An idempotent $\pi \in \mathfrak{M}(S)$ is primitive if and only if $C(\pi)$ is the union of some maximal groups G_{ik} of N.*

For every primitive idempotent π we have $\pi^* = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k [G_{ik}]$, where ξ_i, η_k are non-negative numbers with $\sum_{i=1}^{s} \xi_i = \sum_{k=1}^{r} \eta_k = 1$. Conversely: If the non-negative numbers ξ_i, η_k are such that $\sum_{i=1}^{s} \xi_i = \sum_{k=1}^{r} \eta_k = 1$, then $\sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k [G_{ik}]$ is a primitive idempotent $\in \mathfrak{F}(S)$.

We are now able to clarify the structure of the kernel $\mathfrak{N}$. Since $\mathfrak{N}$ is a compact simple semigroup, $\mathfrak{N}$ is a union of isomorphic groups. By Lemma 3,1 $\mathfrak{N}$ contains those and only those idempotents $\in \mathfrak{M}(S)$ which are primitive idempotents $\in \mathfrak{M}$. If $\mathfrak{G}(\pi)$ is the maximal group having π as its unit element we may write $\mathfrak{N} = \bigcup_{\pi \in \mathfrak{N}} \mathfrak{G}(\pi)$ the sum running through all primitive idempotents $\in \mathfrak{M}$. We show that $\mathfrak{G}(\pi)$ reduces to a single element (i.e. $\mathfrak{G}(\pi) = \pi$).

Theorem 3,5. *The kernel $\mathfrak{N}$ of the semigroup $\mathfrak{M}(S)$ is identical with the set of all primitive idempotents $\in \mathfrak{M}(S)$.*

Proof. It is known (see f.i. [16]) that $\mathfrak{G}(\pi)$ is given by the formula $\mathfrak{G}(\pi) = \pi\mathfrak{N}\pi$. If μ is any element $\in \mathfrak{N}$ it is contained in some group $\mathfrak{G}(\pi')$, hence $\mu\pi' = \mu$. This implies $C(\mu) = C(\mu)\, C(\pi') \subset C(\mu)\, N \subset N$, hence the support of any $\mu \in \mathfrak{N}$ is contained in N. With the same notations as above for any $\mu \in \mathfrak{N}$ the corresponding μ^* may therefore be written in the form

$$\mu^* = \sum_{\alpha=1}^{m} \sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik}^{(\alpha)} g_{ik}^{(\alpha)}, \quad \text{where} \quad 0 \leqq t_{ik}^{(\alpha)} \leqq 1 \quad \text{and} \quad \sum_{\alpha=1}^{m} \sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik}^{(\alpha)} = 1 .$$

If the primitive idempotent π^* is written in the form of Theorem 3,4, we have

$$\pi^*\mu^*\pi^* = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k [G_{ik}] \cdot \sum_{\alpha=1}^{m} \sum_{j=1}^{s} \sum_{l=1}^{r} t_{jl}^{(\alpha)} g_{jl}^{(\alpha)} \cdot \sum_{u=1}^{s} \sum_{v=1}^{r} \xi_u \eta_v [G_{uv}] =$$

$$= (\sum_{k=1}^{r} \eta_r) \cdot (\sum_{\alpha=1}^{m} \sum_{j=1}^{s} \sum_{l=1}^{r} t_{jl}^{(\alpha)}) (\sum_{u=1}^{s} \xi_u) \cdot \sum_{i=1}^{s} \sum_{v=1}^{r} \xi_i \eta_v [G_{iv}] = 1 \,.\, 1 \,.\, 1 \,.\, \pi^* .$$

Hence $\pi\mu\pi = \pi$ for every $\mu \in \mathfrak{N}$, i.e. $\mathfrak{G}(\pi) = \pi\mathfrak{N}\pi = \pi$, q.e.d.

Our next goal is to describe a natural isomorphic representation of the kernel $\mathfrak{N}$.

Every element $\pi \in \mathfrak{N}$, $\pi^* = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k [G_{ik}]$ is uniquely determined by the ordered set of $s + r$ non-negative numbers $\{\xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_r\}$ satisfying the conditions $\sum_{i=1}^{s} \xi_i = \sum_{k=1}^{r} \eta_k = 1$.

If

$$(\pi')^* = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi'_i \eta'_k [G_{ik}] \to \{\xi'_1, \ldots, \xi'_s, \eta'_1, \ldots, \eta'_r\} ,$$

$$(\pi'')^* = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi''_i \eta''_k [G_{ik}] \to \{\xi''_1, \ldots, \xi''_s, \eta''_1, \ldots, \eta''_r\} .$$

Then

$$(\pi'\pi'')^* = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi'_i \eta'_k [G_{ik}] \cdot \sum_{j=1}^{s} \sum_{l=1}^{r} \xi''_j \eta''_k [G_{jl}] =$$

$$= \sum_{i=1}^{s} \sum_{l=1}^{r} \xi'_i \eta''_l [G_{il}] \to \{\xi'_1, \ldots, \xi'_s, \eta''_1, \ldots, \eta''_r\} .$$

Therefore we can state:

Theorem 3,6. *Let S be a finite semigroup with the kernel N containing s minimal right and r minimal left ideals respectively. Let $\mathfrak{T}$ be the set of all $(s + r)$-tuples of non-negative real numbers $\{\xi_1, \ldots, \xi_s, \eta_1, \ldots, \eta_r\}$ satisfying the conditions $\xi_1 + \ldots + \xi_s = \eta_1 + \ldots + \eta_r = 1$. Define in $\mathfrak{T}$ a multiplication $\odot$ by the relation*

$$\{\xi'_1, \ldots, \xi'_s, \eta'_1, \ldots, \eta'_r\} \odot \{\xi''_1, \ldots, \xi''_s, \eta''_1, \ldots, \eta''_r\} = \{\xi'_1, \ldots, \xi'_s, \eta''_1, \ldots, \eta''_r\} .$$

Then $\mathfrak{T}$ is isomorphic with the kernel $\mathfrak{N}$ of the semigroup $\mathfrak{M}(S)$.

Remark. Let π_0 be a fixed chosen element $\in \mathfrak{N}$, $\pi_0^* = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i^{(0)} \eta_k^{(0)} [G_{ik}]$. The minimal left ideal of $\mathfrak{N}$ generated by π_0 can be written in the form $\mathfrak{L}_{\pi_0} = \mathfrak{N}\pi_0$. If $\pi \in \mathfrak{N}$, we have

$$\pi^*\pi_0^* = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k [G_{ik}] \cdot \sum_{j=1}^{s} \sum_{l=1}^{r} \xi_j^{(0)} \eta_l^{(0)} [G_{jl}] = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k^{(0)} [G_{ik}] .$$

Analogously for the minimal right ideal generated by π_0 we have $\mathfrak{R}_{\pi_0} = \pi_0\mathfrak{N}$ and for any element $\pi_0\pi \in \mathfrak{R}_{\pi_0}$ we have $\pi_0^*\pi^* = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i^{(0)}\eta_k[G_{ik}]$. Moreover $\mathfrak{N} = \mathfrak{L}_{\pi_0}\mathfrak{R}_{\pi_0}$ and $\mathfrak{R}_{\pi_0}\mathfrak{L}_{\pi_0} = \{\pi_0\}$.

If

$$\pi_0 \to \{\xi_1^{(0)}, \ldots, \xi_s^{(0)}, \eta_1^{(0)}, \ldots, \eta_r^{(0)}\},$$

then

$$\mathfrak{L}_{\pi_0} \to \{\xi_1, \ldots, \xi_s, \eta_1^{(0)}, \ldots, \eta_r^{(0)}\}$$

(where $\xi_1, \ldots, \xi_s$ run through all non-negative numbers with $\sum_{i=1}^{s} \xi_i = 1$), and

$$\mathfrak{R}_{\pi_0} \to \{\xi_1^{(0)}, \ldots, \xi_s^{(0)}, \eta_1, \ldots, \eta_r\}$$

(η_k running through all non-negative numbers satisfying $\sum_{k=1}^{r} \eta_k = 1$).

We know that all minimal left ideals of $\mathfrak{N}$ are isomorphic. Taking account of the multiplication introduced in Theorem 3,6 we obtain:

Every minimal left ideal of $\mathfrak{N}$ is isomorphic with the set of all s-tuples $\{\xi_1, \ldots, \xi_s\}$ the multiplication being defined by

$$\{\xi_1', \xi_2', \ldots, \xi_s'\} \odot \{\xi_1'', \xi_2'', \ldots, \xi_s''\} = \{\xi_1', \xi_2', \ldots, \xi_s'\}.$$

Appendix. Put the question whether there exists a semigroup S such that $\mathfrak{M}(S)$ is a simple semigroup.

In this case we necessarily have $\mathfrak{M}(S) = \mathfrak{N}$ and every $\mu \in \mathfrak{M}(S)$ is an idempotent. Also every subset $A \subset S$ must be a semigroup. Let N be the kernel of S. a) If $S \neq N$, then there is a μ such that $C(\mu) \not\subset N$. Therefore $\mu \notin \mathfrak{N}$ (see Theorem 3,2). Hence $\mathfrak{M}(S) \neq \mathfrak{N}$. b) Suppose $S = N$. Since every subset of S must be a semigroup, every element $\in S$ is an idempotent. We may write $S = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} g_{ik}$ each g_{ik} being an idempotent.

Let be $s > 1$. We assert that then we necessarily have $r = 1$. Suppose, for an indirect proof, $r > 1$ and consider the subset $\{g_{21}, g_{12}\}$. Since this must be a semigroup $g_{21} \cdot g_{12}$ would be either g_{21} or g_{12}. But this is certainly false since $g_{21}g_{12} = g_{22}$. Therefore $r = 1$ and S is of the form $S = \bigcup_{i=1}^{s} g_{s1}$. Since $g_{i1}g_{k1} = g_{i1}$, every element of this semigroup is a right unit of S. Analogously if $s = 1$ the semigroup S is of the form $S = \bigcup_{k=1}^{r} g_{ik}$ and for any two elements $\in S$ we have $g_{1k}g_{1l} = g_{1l}$, so that every element $\in S$ is a left unit of S.

Conversely: If S is a semigroup of one of the two types just described, it is easily seen that every $\mu \in \mathfrak{M}(S)$ is an idempotent $\in \mathfrak{M}(S)$ and since the support of every μ contains whole groups $\in S$ it is a primitive idempotent $\in \mathfrak{M}(S)$, hence $\mathfrak{M}(S) = \mathfrak{N}$.

We have proved:

Theorem 3,7. *Let S be a finite semigroup. $\mathfrak{M}(S)$ is a simple semigroup if and only if S belongs to one of the following two classes of semigroups:*

a) $S = \{e_1, e_2, \ldots, e_r\}$, *where* $e_i e_k = e_k$ *for every couple* i, k;

b) $S = \{e_1, e_2, \ldots, e_s\}$, *where* $e_i e_k = e_i$ *for every couple* i, k.

4. A FURTHER LEMMA

Let S be a finite semigroup, $\mu \in \mathfrak{M}(S)$ and $C(\mu) = A \subset S$. The closure of the sequence $\mu, \mu^2, \mu^3, \ldots$ contains an idempotent ε. What can be said about the set $C(\varepsilon)$?

If P_1 is the subsemigroup generated by A, we have $C(\mu) = A \subset P_1$ and $C(\mu^k) \subset P_1$ holds for every integer $k > 0$.

Lemma 4,1. *Suppose that $\mu \in \mathfrak{M}$ belongs to the idempotent ε and $C(\mu) = A$. If P_1 is the subsemigroup generated by A, then $C(\varepsilon) \subset P_1$.*

Proof. Suppose for an indirect proof that $\mathfrak{D} = C(\varepsilon) - P_1 \neq \emptyset$, and $P_1 \cap C(\varepsilon) = \{x_1, \ldots, x_u\}$, $\mathfrak{D} = \{y_1, \ldots, y_v\}$. Then ε^* can be written in the form $\varepsilon^* = t_1 x_1 + \ldots + t_u x_u + t'_1 y_1 + \ldots + t'_v y_v$, where $\sum_{i=1}^{u} t_i + \sum_{k=1}^{v} t'_k = 1$ and $\min(t'_1, \ldots, t'_v) = \delta > 0$. Consider the neighbourhood $o(\varepsilon^*)$ of ε^* containing all elements $t_1 x_1 + \ldots + t_u x_u + \xi_1 y_1 + \ldots + \xi_v y_v$ with $\frac{1}{2}\delta < \xi_1 \leqq 1, \ldots, \frac{1}{2}\delta < \xi_v \leqq 1$ and $\sum_{i=1}^{v} \xi_i = 1 - \sum_{i=1}^{u} t_i$. Since $\varepsilon \in \overline{\{\mu, \mu^2, \ldots\}}$, there is an integer $k > 0$ such that $\mu^{*k} \in o(\varepsilon^*)$. This is impossible since for every integer $k > 0$ the coefficients of $y_1, \ldots, y_v$ in μ^{*k} are zeros. This contradiction proves our assertion.

Remark 1. It follows from our proof that also for any $\nu \in \overline{\{\mu, \mu^2, \mu^3, \ldots\}}$ we have $C(\nu) \subset P_1$.

Remark 2. Even in the case that A itself is a semigroup and $C(\mu^k) = A$ for every k we cannot conclude $C(\varepsilon) = A$ but merely $C(\varepsilon) \subset A$. This can be shown on the simplest two-element semigroup $S = \{z, a\}$ with $a^2 = a$, $z^2 = az = za = z$. Put $\mu^* = t_1 z + t_2 a$, where $0 < t_2 < 1$. We have

$$(\mu^*)^k = (t_1 z + t_2 a)^k = t_1^k z + \binom{k}{1} t_1^{k-1} t_2 z + \ldots + \binom{k}{1} t_1 t_2^{k-1} z + t_2^k a =$$
$$= z(t_1 + t_2)^k - z t_2^k + t_2^k a = (1 - t_2^k)\, z + t_2^k a\,.$$

Clearly $\varepsilon^* = z$. Hence $C(\mu^k) = S = \{z, a\}$ for every $k \geqq 1$, while $C(\varepsilon) = z$.

Remark 3. We shall show later (see Theorem 7,5 below) that $C(\varepsilon)$ is contained in the minimal two-sided ideal of P_1.

In what follows we shall primarily be interested in the converse question: If μ belongs to a given ε what can be said about the set $C(\mu)$.

We shall first treat the case of a simple semigroup without zero, then the case of a simple smigroup with zero and finally the general case. It will turn out that this proceeding is a natural one.

5. THE CASE OF A SIMPLE SEMIGROUP (WITHOUT ZERO)

In this section we shall study a simple semigroup S. The following notations will consequently be used throughout the whole section: $S = \bigcup_{i=1}^{s} R_i = \bigcup_{k=1}^{r} L_k$ denotes the decomposition of S into the union of its minimal right and left ideals respectively. Further $S = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} G_{ik}$ is the decomposition of S into its group-components.

If ε is an idempotent $\in \mathfrak{M}(S)$, then $C(\varepsilon) = H$ is a subsemigroup of S which is itself simple. $H = R'_1 \cup \ldots \cup R'_\sigma = L'_1 \cup \ldots \cup L'_\varrho$ will denote the decomposition of H into the union of its minimal right and left ideals respectively and $H = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G'_{ik}$, $G'_{ik} = R'_i L'_k$, the decomposition of H into the union of group-components.

Without loss of generality we may suppose that R_i is the minimal right ideal of S containing R'_i ($i = 1, \ldots, \sigma$) and L_k the minimal left ideal of S containing L'_k ($k = 1, \ldots, \varrho$).

Denote

$$H_1 = (R_1 \cup R_2 \cup \ldots \cup R_\sigma) \cap (L_1 \cup L_2 \cup \ldots \cup L_\varrho).$$

Then H_1 is the largest simple subsemigroup of S containing the same idempotents as H. It has been proved in [18] that H_1 can be decomposed modulo (H, H) into the union of pairwise disjoint classes of the form

$$H_1 = H \cup HaH \cup HbH \cup \ldots, \tag{11}$$

where $a, b, \ldots$ are suitably chosen elements $\in H_1$. Note – in particular – that $HaH = H$ if and only if $a \in H$.

5,1. We first prove:

Theorem 5,1. *Let S be a (finite) simple semigroup. Suppose that $\mu \in \mathfrak{M}(S)$ belongs to the idempotent ε and μ is a regular element $\in \mathfrak{M}(S)$. Put $C(\varepsilon) = H$. Let H_1 denote the largest subsemigroup of S containing the same idempotents as H. Then $C(\mu)$ is identical with a unique class of the decomposition* (11).

Proof. Since μ is regular, we have $\mu = \mu\varepsilon = \varepsilon\mu$. Hence $C(\mu) = H\, C(\mu) = C(\mu)\, H$ and $C(\mu) = H\, C(\mu)\, H$. With the notations introduced above we have

$$C(\mu) = C(\mu)\, H = C(\mu)\, \{L'_1 \cup \ldots \cup L'_\varrho\} \subset C(\mu)\, \{L_1 \cup \ldots \cup L_\varrho\} \subset L_1 \cup \ldots \cup L_\varrho,$$

$$C(\mu) = H\, C(\mu) = \{R'_1 \cup \ldots \cup R'_\sigma\}\, C(\mu) \subset \{R_1 \cup \ldots \cup R_\sigma\}\, C(\mu) \subset R_1 \cup \ldots \cup R_\sigma.$$

Hence
$$C(\mu) \subset \{R_1 \cup \ldots \cup R_\sigma\} \cap \{L_1 \cup \ldots \cup L_\varrho\} = H_1 .$$
If $C(\mu) = \{x_1, x_2, \ldots, x_v\}$, then
$$C(\mu) = x_1 H \cup x_2 H \cup \ldots \cup x_v H = H x_1 \cup H x_2 \cup \ldots \cup H x_v$$
and
$$(12) \qquad C(\mu) = H x_1 H \cup H x_2 H \cup \ldots \cup H x_v H .$$
Our theorem will be proved if we can show that $H x_1 H = H x_2 H = \ldots = H x_v H$. Obviously it is sufficient to show that $H x_1 H = H x_2 H$.

Since μ is a regular element $\in \mathfrak{M}(S)$, there exists a regular element $\mu' \in \mathfrak{M}(S)$ such that $\mu'\mu = \varepsilon$, hence $C(\mu')\,C(\mu) = H$. Also the element μ' satisfies $C(\mu') \subset H_1$ and if $C(\mu') = \{y_1, y_2, \ldots, y_u\}$ we have
$$C(\mu') = y_1 H \cup y_2 H \cup \ldots \cup y_u H = H y_1 \cup H y_2 \cup \ldots \cup H y_u .$$
Since
$$\{H y_1 \cup \ldots \cup H y_u\} . \{x_1 H \cup \ldots \cup x_v H\} = H ,$$
we have $H y_i x_k H \subset H$, i.e. $H y_i x_k H = H$, hence $y_i x_k \in H$ for every couple i, k $(1 \leqq i \leqq u,\ 1 \leqq k \leqq v)$.

Denote $y_1 x_1 = h' \in H$. The element y_1 is contained in some minimal left ideal L_i, hence $y_1 \in L_i \cap H_1$. Analogously there is a right ideal R_k such that $x_1 \in R_k \cap H_1$. Let e_{ki} be the idempotent $\in R_k \cap L_i$, $e_{ki} \in H \subset H_1$. The set $R_k \cap H_1$ is a minimal right ideal of H_1. Hence $x_1(R_k \cap H_1) = R_k \cap H_1$. There exists therefore an element $x_1' \in R_k \cap H_1$ such that $x_1 x_1' = e_{ki}$. Hereby e_{ki} is a right unit for L_i. Therefore the relation $y_1 x_1 = h'$ implies successively $y_1 x_1 x_1' = h' x_1'$, $y_1 e_{ki} = h' x_1'$, $y_1 = h' x_1'$. Hence $H y_1 = H h' x_1' \subset H x_1'$. Since two classes in the decomposition of H_1 modulo H are either disjoint or identical (see [18], Theorem 3,1), we have $H y_1 = H x_1'$.

Consider now the relation $H y_1 x_2 H = H$. It implies $H x_1' x_2 H = H$, hence $x_1' x_2 = h'' \in H$. Multiplying to the left by x_1 we get $x_1 x_1' x_2 = x_1 h''$, $e_{ki} x_2 = x_1 h''$. Further we have $e_{ki} x_2 H = x_1 h'' H \subset x_1 H$, hence $e_{ki} x_2 H = x_1 H$, $H x_1 H = H e_{ki} x_2 H \subset H x_2 H$, therefore $H x_1 H = H x_2 H$, q.e.d.

The following theorem is a consequence of Theorem 5,1:

Theorem 5,2. *Let the suppositions of Theorem 5,1 be satisfied. Let μ be any (non-necessarily regular) element $\in \mathfrak{M}(S)$ belonging to the idempotent ε. Then $C(\mu)$ is contained in a unique class $H x_0 H$ of the decomposition* (11).

Proof. The suppositions imply that $\varepsilon\mu = \mu\varepsilon$ is a regular element $\in \mathfrak{M}(S)$. Hence, by Theorem 5,1, $C(\mu\varepsilon) = H x_0 H$ with a suitably chosen element $x_0 \in H_1$. This is equivalent to $C(\mu)\,H = H x_0 H$. Analogously we have $H\,C(\mu) = H x_0 H$.

We now prove that $C(\mu) \subset H_1$. Let be $b \in C(\mu)$. If b were contained in a minimal right ideal of S and different from $R_1, \ldots, R_\sigma$, say in the right ideal $R_{\sigma+1}$, we would have

$bH \subset bS \subset R_{\sigma+1}S \subset R_{\sigma+1}$. At the same time we have $bH \subset Hx_0H \subset H_1$. This is impossible since $H_1 \cap R_{\sigma+1} = \emptyset$. Analogously we prove that $b \in \{L_1 \cup \ldots \cup L_\varrho\}$, hence $b \in \{R_1 \cup \ldots \cup R_\sigma\} \cap \{L_1 \cup \ldots \cup L_\varrho\} = H_1$. Therefore $C(\mu) \subset H_1$.

Let now be $b \in C(\mu)$. For every $b \in H_1$ we have $b \in HbH$, hence $b \in H\ C(\mu)\ H$, i.e. $b \in (Hx_0H) . H = Hx_0H$. Therefore $C(\mu) \subset Hx_0H$, q.e.d.

Example 5,1. The following example will be useful also later on. Consider the semigroup $S = \{a_1, a_2, a_3, a_4\}$ with the multiplication table

	a_1	a_2	a_3	a_4
a_1	a_1	a_2	a_3	a_4
a_2	a_2	a_1	a_4	a_3
a_3	a_1	a_2	a_3	a_4
a_4	a_2	a_1	a_4	a_3

In our notation we have $G_{11} = \{a_1, a_2\}$, $G_{12} = \{a_3, a_4\}$ and $S = G_{11} \cup G_{12}$. The idempotents $\in \mathfrak{M}(S)$ are of the form: a) Either $t_1a_1 + t_3a_3$ with $t_1 + t_3 = 1$, or b) $t_1 . \frac{1}{2}(a_1 + a_2) + t_3 . \frac{1}{2}(a_3 + a_4)$ with $t_1 + t_3 = 1$. (These later are primitive idempotents $\in \mathfrak{M}(S)$.)

Choose f.i. the idempotent $\varepsilon^* = ta_1 + (1 - t)\,a_3$, $0 < t < 1$, with t fixed. Then $H = C(\varepsilon) = \{a_1, a_3\}$, $H_1 = S$ and the decomposition (11) takes the form $S = H_1 = H \cup Ha_2H = \{a_1, a_3\} \cup \{a_2, a_4\}$. If μ is a regular element $\in \mathfrak{M}(S)$, we have by Theorem 5,1 either $C(\mu) = \{a_1, a_3\}$ or $C(\mu) = \{a_2, a_4\}$. We shall show that both cases may take place.

Let us find more generally all elements $\mu \in \mathfrak{M}(S)$ belonging to the idempotent ε. By Theorem 5,2 we have either $C(\mu) \subset \{a_1, a_3\}$ or $C(\mu) \subset \{a_2, a_4\}$. Write $\mu_1^* = t_1a_1 + t_3a_3$, $t_1 + t_3 = 1$. Now $(\mu_1^*)^k = t_1a_1 + t_3a_3 = \mu_1$ for every integer $k > 0$; therefore μ_1 belongs to ε if and only if $\mu_1 = \varepsilon$, i.e. $t_1 = t$, $t_3 = 1 - t$. Write $\mu_2^* = t_2a_2 + t_4a_4$, $t_2 + t_4 = 1$. An elementary calculation shows that $(\mu_2^*)^2 = t_2a_1 + t_4a_3$. Since $t_2a_1 + t_4a_3$ is an idempotent, we then necessarily have $t_2a_1 + t_4a_3 = ta_1 + (1 - t)\,a_3$, i. e. $t_2 = t$, $t_4 = 1 - t$, hence $\mu_2^* = ta_2 + (1 - t)\,a_4$. This element μ_2^* is regular since besides of $\mu_2^{*2} = \varepsilon^*$ we can verify that $\mu_2^*\varepsilon^* = \varepsilon^*\mu_2^* = \mu_2$. Thus we have proved: All elements belonging to ε^* are regular; in fact they form the two-element group $\{\varepsilon^*, \mu_2 = ta_2 + (1 - t)\,a_4\}$.

All maximal subgroups of $\mathfrak{M}(S)$ are either the two-element groups just considered, or one-point groups (namely the primitive idempotents). If μ is any element $\in \mathfrak{M}(S)$ for which neither $C(\mu) \subset \{a_1, a_3\}$, $C(\mu) \subset \{a_2, a_4\}$, then μ belongs to a primitive idempotent ε_0 and, moreover, it can be proved that $\lim_{n=\infty} \mu^n$ exists and $= \varepsilon_0$ (see Theorem 8,3 below).

We show on this example that – in general – it is not true that for a regular μ the set $C(\mu)$ can be written as a (simple) left class of the form $C(\mu) = Ha$. For, $\mu_2^* = ta_2 + (1 - t)\,a_4$, $0 < t < 1$, is a regular element belonging to the idempotent $\varepsilon^* = ta_1 + (1 - t)\,a_3$, but $\{a_2, a_4\}$ cannot be written in the form $Hx_0 = \{a_1, a_3\}\,x_0$

with an $x_0 \in S$. (Hence, in contrary to the case when S is a group, to get reasonable results we are compelled to deal with double coset decompositions and not merely with simple left or right coset decompositions.)

Theorem 5,1 and 5,2 imply also the following

Corollary 5,1. *Let S be a simple semigroup and ε the point mass at the idempotent e_{ik}. The set of all elements $\in \mathfrak{M}(S)$ belonging to ε forms a group isomorphic with G_{ik} (the maximal group of S containing e_{ik} as its unit element).*

Proof. In our case $H = C(\varepsilon) = e_{ik}$. The set H_1 is the group G_{ik}. The decomposition (11) is the trivial decomposition of H_1: $H_1 = G_{ik} = \{g_{ik}^{(1)}\} \cup \{g_{ik}^{(2)}\} \cup \ldots \cup \{g_{ik}^{(m)}\}$. Theorem 5,1 implies that for regular elements $\mu \in \mathfrak{M}(S)$ we have necessarily $C(\mu) = g_{ik}^{(l)}$, $1 \leqq l \leqq m$. Conversely every $\mu \in \mathfrak{M}(S)$ with $\mu^* = g_{ik}^{(l)}$ is regular and belongs to ε, with $\varepsilon^* = e_{ik}$. Further Theorem 5,2 implies that also for any $\mu^* \in \mathfrak{F}(S)$ belonging to e_{ik} (a priori non-necessarily regular) $C(\mu)$ is a one point set. In other words: Every μ belonging to such an idempotent is regular. This proves our assertion.

Remark. Suppose on the other hand that (ε being an idempotent) $C(\varepsilon) = H$ is a semigroup which is maximal in the sense that there does not exist a larger simple subsemigroup of S having the same idempotents as H. Then the decomposition (11) reduces to the trivial decomposition $H_1 = H$. In this case there is a unique regular element belonging to ε (namely ε itself). Now the idempotents having supports of the form described and contained in the kernel $\mathfrak{N}$ are primitive idempotents (see Theorem 3,4). Our result gives in the case of a simple semigroup a new proof of the assertion that the maximal group $\in \mathfrak{M}(S)$ belonging to a primitive idempotent $\in \mathfrak{M}(S)$ is a one point group.

5,2. In what follows we need the following elementary group considerations.

Let G be a finite group containing m elements and $G' = \{g_1, g_2, \ldots, g_{m_1}\}$ a subgroup of G containing m_1 elements. Consider the double coset decomposition $G = G' \cup G'aG' \cup G'bG' \cup \ldots$ We ask: How many different elements are contained in a class $G'aG'$.

The classes $G'ag_1, G'ag_2, \ldots, G'ag_{m_1}$ are either disjoint or identical. If $G'ag_i = G'ag_j$, then $ag_ig_j^{-1}a^{-1} \in G'$, $g_ig_j^{-1} \in a^{-1}G'a$, i.e. $g_ig_j^{-1} \in D = G' \cap a^{-1}G'a$. Conversely let be $k \in D = G' \cap a^{-1}G'a$ and $k = a^{-1}g'a$, then $G'ak = G'aa^{-1}g'a = G'a$. Let us put $G' = Dk_1 \cup Dk_2 \cup Dk_3 \cup \ldots \cup Dk_p$ with disjoint summands (k_1 being the unit element of G'). Clearly $p \mid m_1$. Let be $x \in Dk_i$, $y \in Dk_i$, $x = a^{-1}g'_xak_i$, $y = a^{-1}g'_yak_i$. Then $G'ax = G'aa^{-1}g'_xak_i = G'ak_i$ and similarly $G'ay = G'ak_i$. For $x \in Dk_i$, $y \in Dk_j$, $i \neq j$, we have $G'ax = G'ak_i$, $G'ay = G'ak_j$ and since $G'ak_i \neq G'ak_j$, we have finally $G'ax \neq G'ay$. Hence among the classes $G'ag_1, G'ag_2, \ldots, G'ag_{m_1}$ there are exactly p different classes. Each of them occurs exactly m_1/p times. We have proved: If we consider formally all m_1^2 products xay, $x \in G'$, $y \in G'$ we get m_1p different elements each of which occurs m_1/p times.

The following is the consequence of this fact. Denote $T_a = G'aG'$ in the set-theoretical sense (i.e. we take different elements just one times). Let be $T_a = \{\tau_1, \tau_2, \ldots, \tau_{m_1 p}\}$. Denote $[T_a] = 1/(m_1 p) . (\tau_1 + \tau_2 + \ldots + \tau_{m_1 p})$.

We then have

Lemma 5,1. $[T_a] = [G'aG'] = [G']a[G']$.

Proof. By definition

$$[G']a[G'] = \frac{1}{m_1^2}(g_1 + \ldots + g_{m_1})\, a\, (g_1 + g_2 + \ldots + g_{m_1}) .$$

As we have just seen among the summands on the right hand side there are exactly $m_1 p$ different elements (namely $\tau_1, \tau_2, \ldots, \tau_{m_1 p}$); each of them occurs m_1/p times. Hence

$$[G']a[G'] = \frac{m_1/p\,(\tau_1 + \tau_2 + \ldots + \tau_{m_1 p})}{m_1^2} = \frac{1}{m_1 p}(\tau_1 + \tau_2 + \ldots + \tau_{m_1 p}) = [T_a] ,$$

q.e.d.

5,3. We shall now try to identify the regular elements μ belonging to a given idempotent ε.

Let be $\varepsilon^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [G'_{ik}]$ and $H = C(\varepsilon) = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G'_{ik}$. Let further $H_1 = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G_{ik}$ be the maximal subsemigroup of S containing the same idempotents as H. Since μ is regular we have $\mu = \varepsilon\mu = \mu\varepsilon$ and $C(\mu) = HaH$ with a suitably chosen $a \in C(\mu) \subset H_1$.

By Theorem 3,2 of the paper [18] we have $HaH \cap G_{ik} = G'_{ik}aG'_{ik}$. Denote $G'_{ik}aG'_{ik} = T_{ik} \subset G_{ik}$; then $C(\mu) = HaH \cap H_1 = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} T_{ik}$. The set T_{ik} is one of the classes of the decomposition of G_{ik} modulo (G'_{ik}, G'_{ik}). Since all groups G_{ik} and G'_{ik} respectively are isomorphic, every T_{ik} ($i = 1, \ldots, \sigma$; $k = 1, \ldots, \varrho$) contains the same number of elements. Denote $T_{ik} = \{a_{ik}^{(1)}, \ldots, a_{ik}^{(p)}\}$. The element μ^* can be written in the form

$$\mu^* = \sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\varrho} \sum_{\gamma=1}^{p} t_{\alpha\beta}^{(\gamma)} a_{\alpha\beta}^{(\gamma)} \quad \text{with} \quad \sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\varrho} \sum_{\gamma=1}^{p} t_{\alpha\beta}^{(\gamma)} = 1 .$$

The relation $\mu = \varepsilon\mu\varepsilon$ implies

$$\mu^* = \sum_{k=1}^{\sigma} \sum_{i=1}^{\varrho} \xi_i \eta_k [G'_{ik}] \cdot \sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\varrho} \sum_{\gamma=1}^{p} t_{\alpha\beta}^{(\gamma)} a_{\alpha\beta}^{(\gamma)} \cdot \sum_{j=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_j \eta_l [G'_{jl}] . \tag{13}$$

Let us first consider $[G'_{ik}]a_{\alpha\beta}^{(\gamma)}[G'_{jl}]$. If $e_{\alpha\beta}$ is the unit element of $G_{\alpha\beta}$, we have $G'_{ik}a_{\alpha\beta}^{(\gamma)}G'_{jl} = G'_{ik}e_{\alpha\beta}a_{\alpha\beta}^{(\gamma)}e_{\alpha\beta}G'_{jl} = G'_{i\beta}a_{\alpha\beta}^{(\gamma)}G'_{\alpha l}$. Since $G'_{ik}a_{\alpha\beta}^{(\gamma)}G'_{jl}$ does not depend on k and j, we have in particular $G'_{ik}a_{\alpha\beta}^{(\gamma)}G'_{jl} = G'_{il}a_{\alpha\beta}^{(\gamma)}G'_{il}$.

We next show: If b is any element $\in HaH = C(\mu)$, we have $G'_{il}bG'_{il} = G'_{il}aG'_{il}$. If $b \in HaH$, we may write $b = x_{\gamma\delta}ay_{\kappa\lambda}$, $x_{\gamma\delta} \in G'_{\gamma\delta}$, $y_{\kappa\lambda} \in G'_{\kappa\lambda}$ with suitably chosen $\gamma, \delta, \kappa, \lambda$. Hence $G'_{il}bG'_{il} = G'_{il}x_{\gamma\delta}ay_{\kappa\lambda}G'_{il} = G'_{i\delta}aG'_{\kappa l}$. Analogously as above we prove that the product $G'_{i\delta}aG'_{\kappa l}$ is independent of δ and κ, and hence, in particular, it is equal to $G'_{il}aG'_{il}$, i.e. $G'_{il}bG'_{il} = G'_{il}aG'_{il}$.

Since $a_{\alpha\beta}^{(\gamma)} \in HaH = C(\mu)$, we have $G'_{il}a_{\alpha\beta}^{(\gamma)}G'_{il} = G'_{il}aG'_{il} = T_{il}$ and by Lemma 5,1 $[T_{il}] = [G'_{il}a_{\alpha\beta}^{(\gamma)}G'_{il}] = [G'_{il}]a_{\alpha\beta}^{(\gamma)}[G'_{il}]$ for every α, β, γ. Therefore $[G'_{ik}a_{\alpha\beta}^{(\gamma)}G'_{jl}] = [G'_{il}a_{\alpha\beta}^{(\gamma)}G'_{il}] = [T_{il}]$, i.e. $[G'_{ik}]a_{\alpha\beta}^{(\gamma)}[G'_{jl}] = [T_{il}]$ for every $\alpha, \beta, \gamma, k, j$.

The relation (13) implies

$$\mu^* = (\sum_{k=1}^{\varrho} \eta_k)(\sum_{j=1}^{\sigma} \xi_j)(\sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\varrho} \sum_{\gamma=1}^{p} t_{\alpha\beta}^{(\gamma)}) \cdot \sum_{i=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_i \eta_l [T_{il}] ,$$

i.e.

$$\mu^* = \sum_{i=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_i \eta_l [T_{il}] .$$

We have proved the following interesting theorem:

Theorem 5,3. *Let μ be a regular element $\in \mathfrak{M}(S)$ belonging to ε with $\varepsilon^* = \sum_{i=1}^{\varrho} \sum_{k=1}^{\sigma} \xi_i \eta_k [G'_{ik}]$. Let G_{ik} be the maximal group containing the group G'_{ik}. Denote $G_{ik} \cap C(\mu) = T_{ik}$. We then have $\mu^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [T_{ik}]$.*

Let ε be an idempotent, $C(\varepsilon) = H$, and consider again the decomposition

$$H_1 = H \cup HaH \cup \ldots \cup HwH . \tag{14}$$

Theorem 5,3 implies that a class $HaH = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} T_{ik}$ is the support of at most one regular measure belonging to ε. Theorem 5,1 asserts conversely that the support of every regular measure belonging to ε is one class of the decomposition (14). Since (14) contains only a finite number of classes we get

Corollary 5,2. *Let ε be a fixed chosen idempotent $\in \mathfrak{M}(S)$. The set of regular elements belonging to ε forms a finite group.*

If S is any semigroup and Γ a subset of $\mathfrak{M}(S)$, we shall denote $\bigcup_{\mu \in \Gamma} C(\mu)$ by $C(\Gamma)$ and refer to it as the support of Γ. If Γ is a semigroup, $C(\Gamma)$ is clearly a subsemigroup of S. [For $\mu_1, \mu_2 \in \Gamma$ implies $\mu_1\mu_2 \in \Gamma$ and $C(\mu_1)\, C(\mu_2) = C(\mu_1\mu_2) \subset C(\Gamma)$.] We may also prove that $C(\bar{\Gamma}) = C(\Gamma)$.

If S is finite simple and $\mathfrak{G}(\varepsilon)$ the group of regular elements belonging to ε, then $C(\mathfrak{G}(\varepsilon))$ is a subsemigroup of S and since any subsemigroup of S is a simple semigroup, we have:

Corollary 5,3. *If S is finite simple, then $C(\mathfrak{G}(\varepsilon))$ is a simple subsemigroup of S.*

Clearly $C(\mathfrak{G}(\varepsilon)) \subset H_1$.

5,4. The question arises whether each of the classes in the decomposition (14) is a support of some regular measure belonging to ε. The answer to this question is negative. This can be shown on the simplest case of a non-commutative group, i.e.

the symetric group of three elements. Let $S = \{a_1, a_2, \ldots, a_6\}$ be the group with the multiplication table

	a_1	a_2	a_3	a_4	a_5	a_6
a_1	a_1	a_2	a_3	a_4	a_5	a_6
a_2	a_2	a_3	a_1	a_5	a_6	a_4
a_3	a_3	a_1	a_2	a_6	a_4	a_5
a_4	a_4	a_6	a_5	a_1	a_3	a_2
a_5	a_5	a_4	a_6	a_2	a_1	a_3
a_6	a_6	a_5	a_4	a_3	a_2	a_1 .

Choose f. i. the idempotent $\varepsilon^* = \frac{1}{2}(a_1 + a_4)$. Then $H = \{a_1, a_4\}$ and the decomposition (14) takes the form

$$S = H \cup Ha_2H = \{a_1, a_4\} \cup \{a_2, a_3, a_5, a_6\} .$$

If Ha_2H were the support of a regular measure μ, then by Theorem 5,3 μ^* would have the form $\mu^* = \frac{1}{4}(a_2 + a_3 + a_5 + a_6)$. Since there exist at most two regular measures belonging to ε (namely ε and μ) and the set of the regular measures (belonging to ε) is a group, there would hold necessarily $\mu^2 = \varepsilon$. But this is not true, since a simple calculation shows that $\mu^{*2} = \frac{1}{2}(\varepsilon^* + \mu^*)$ and $\varepsilon^* = \frac{1}{2}(\varepsilon^* + \mu^*)$ implies $\varepsilon^* = \mu^*$, contrary to the supposition. Hence Ha_2H is not the support of a regular measure belonging to ε.[5])

Now the natural problem arises: We have to find the classes in the decomposition (14) that are supports of some regular element $\in \mathfrak{M}$ (belonging to ε).

If S is a group Б. М. Клосс ([12]) has proved that HaH is the support of a regular measure if and only if $HaH = Ha = aH$, i.e. Ha is a two-sided class of the decomposition of the group S modulo the subgroup H. In other words Ha is contained in the normalizer of the subgroup H in the group S. Example 5,1 shows that in the case of a semigroup (which is not a group) this condition is not necessary. For in this example $Ha_2H = \{a_2, a_4\}$ is the support of a regular measure μ_2 with $\mu_2^* = ta_2 + (1 - t)\, a_4$, though $Ha_2 = \{a_2\}$ and $a_2H = \{a_2, a_4\}$, hence $Ha_2 \neq a_2H$.

To clarify the situation we shall give further necessary conditions that a regular μ with $\mu^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [T_{ik}]$ must satisfy in order to belong to the idempotent ε with $\varepsilon^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [G'_{ik}]$.

If μ is regular, there exists a regular measure $\mu^{(0)}$ such that $\mu\mu^{(0)} = \mu^{(0)}\mu = \varepsilon$. Denote $C(\mu^{(0)}) = HbH$. We have $HbH \cap G_{jl} = G'_{jl}bG'_{jl}$. Denote further $T_{jl}^{(0)} =$

[5]) The group belonging to ε contains a single element, namely ε itself. It is also possible to show (f.i. by direct calculations) that μ^* belongs to the idempotent $\varepsilon_0^* = \frac{1}{6}(a_1 + a_2 + a_3 + a_4 + a_5 + a_6)$. μ is, of course, not a regular element $\in \mathfrak{M}(S)$. For, since it belongs to ε_0 and $C(\varepsilon_0) = S$, ε_0 is clearly a primitive idempotent $\in \mathfrak{M}(S)$ (see Theorem 3,4). ε_0 itself is the unique regular element $\in \mathfrak{M}(S)$ belonging to ε_0 (see Theorem 3,5).

$= G'_{jl} b G'_{jl} \subset G_{jl}$. By Theorem 5,3 we have $(\mu^{(0)})^* = \sum_{j=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_j \eta_l [T_{jl}^{(0)}]$ and the relation $\mu\mu^{(0)} = \varepsilon$ implies

$$(15) \qquad \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [T_{ik}] \cdot \sum_{j=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_j \eta_l [T_{jl}^{(0)}] = \sum_{i=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_i \eta_l [G'_{il}] .$$

We have

$$[T_{ik}] [T_{jl}^{(0)}] = [G'_{ik}] a [G'_{ik}] [G'_{jl}] b [G'_{jl}] = [G'_{ik}] a [G'_{il}] b [G'_{jl}] .$$

By the proof of Theorem 5,3 we have further $[G'_{ik}] a [G'_{il}] = [G'_{il}] a [G'_{il}]$, hence $[T_{ik}] [T_{jl}^{(0)}] = [G'_{il}] a [G'_{il}] b [G'_{jl}]$, and since also $[G'_{il}] b [G'_{jl}] = [G'_{il}] b [G'_{il}]$, we have finally $[T_{ik}] [T_{jl}^{(0)}] = [G'_{il}] a [G'_{il}] b [G'_{il}]$.

The relation (15) implies

$$\Big(\sum_{k=1}^{\varrho} \eta_k\Big) \Big(\sum_{i=1}^{\sigma} \xi_j\Big) \sum_{i=1}^{\sigma} \sum_{l=1}^{\varrho} [G'_{il}] a [G'_{il}] b [G'_{il}] \xi_i \eta_l = \sum_{i=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_i \eta_l [G'_{il}] ,$$

i.e. [5a])

$$[G'_{il}] a [G'_{il}] b [G'_{il}] = [G'_{il}] ,$$
$$[G'_{il}] a [G'_{il}] \,.\, [G'_{il}] b [G'_{il}] = [G'_{il}] ,$$
$$[T_{il}] [T_{il}^{(0)}] = [G'_{il}] ,$$

for every couple i, l. Analogously $\mu^{(0)}\mu = \varepsilon$ implies $[T_{il}^{(0)}] [T_{il}] = [G'_{il}]$ for every couple i, l.

Introducing (in the set-theoretical sense) $T_{il} = G'_{il} a G'_{il}$, $T_{il}^{(0)} = G'_{il} b G'_{il}$ we have also

$$(16) \qquad T_{il} T_{il}^{(0)} = G'_{il} \quad \text{and} \quad T_{il}^{(0)} T_{il} = G'_{il} .$$

The expression $T_{il} = G'_{il} a G'_{il}$ shows that T_{il} can be written as a union of disjoint left classes in the form

$$T_{il} = a_1 G'_{il} \cup a_2 G'_{il} \cup \dots \quad (a_1, a_2, \dots \in G_{il}) .$$

Analogously we may write

$$T_{il}^{(0)} = G'_{il} b_1 \cup G'_{il} b_2 \cup \dots \quad (b_1, b_2, \dots \in G_{il}) .$$

We now prove that T_{il} contains a unique left class of the decomposition of G_{il} modulo G'_{il}. We prove it indirectly. Suppose that this is not the case. The relation (16) implies

$$\{G'_{il} b_1 \cup G'_{il} b_2 \cup \dots\} \,.\, \{a_1 G'_{il} \cup a_2 G'_{il} \cup \dots\} = G'_{il} ,$$

i.e. $G'_{il} b_1 a_1 G'_{il} \subset G'_{il}$, $G'_{il} b_1 a_2 G'_{il} \subset G'_{il}$. This implies $b_1 a_1 = g_{il} \in G'_{il}$, $b_1 = g_{il} a_1^{-1}$, i.e. $G'_{il} b_1 = G'_{il} g_{il} a_1^{-1} = G'_{il} a_1^{-1}$. But then $G'_{il} b_1 a_2 G'_{il} = G'_{il} a_1^{-1} a_2 G'_{il} \subset G'_{il}$ implies $a_1^{-1} a_2 = g_{il}^{(0)} \in G'_{il}$, $a_2 = a_1 g_{il}^{(0)}$ and $a_2 G'_{il} = a_1 g_{il}^{(0)} G'_{il} = a_1 G'_{il}$, which is a contradiction.

[5a]) Hereby we use the fact that $G'_{il}(a G'_{il} b)\, G'_{il} \subset G_{il}(S)\, G_{il} \subset R_i S L_l = R_i L_l = G_{il}$.

Hence T_{il} is really of the form $T_{il} = a_1 G'_{il}$. Analogously we prove that T_{il} is a unique right class of the decomposition of G_{il} modulo G'_{il}, i. e. $T_{il} = G'_{il}\bar{a}_1$, $\bar{a}_1 \in G_{il}$. We have $T_{il} = a_1 G'_{il} = G'_{il}\bar{a}_1$. Now $a_1 \in a_1 G'_{il}$, hence $a_1 = \bar{g}_{il}\bar{a}_1$ with $\bar{g}_{il} \in G'_{il}$. Therefore $a_1 G'_{il} = G'_{il}(\bar{g}_{il})^{-1} a_1 = G'_{il}a_1$. Moreover $(G'_{il}a_1) G'_{il} = (a_1 G'_{il}) G'_{il} = a_1 G'_{il} = G'_{il}a_1$. The class $a_1 G'_{il}$ is a two-sided class. We have proved

Theorem 5,4. *If μ is a regular element $\in \mathfrak{M}(S)$ belonging to the idempotent ε, $C(\varepsilon) = H = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G'_{ik}$, G_{ik} the maximal group containing G'_{ik} and $C(\mu) = HaH$, then $G_{il} \cap HaH = T_{il}$ is exactly one two-sided class in the decomposition of the group G_{il} modulo the subgroup G'_{il} (for all i, l).*

Remark. Let $\mathfrak{G}(\varepsilon)$ be the set of all regular elements belonging to ε. Denote $C(\mathfrak{G}(\varepsilon)) = H_0$. Clearly $H \subset H_0 \subset H_1$. Theorem 5,4 implies that (for every couple i, k) G'_{ik} is a normal subgroup of $H_0 \cap G_{ik} = G_{ik}^{(0)}$.

Now we are able to find an answer to our question and to prove a theorem analogous to that of Б. М. Клосс.

Recall a somewhat other characterisation of a regular element $\in \mathfrak{M}(S)$. If ε is an idempotent $\in \mathfrak{M}(S)$, then μ is a regular element belonging to ε if and only if: 1) $\mu\varepsilon = \varepsilon\mu = \mu$, 2) there is a $\mu_0 \in \mathfrak{M}(S)$ such that $\mu\mu_0 = \mu_0\mu = \varepsilon$, and 3) $\mu_0\varepsilon = \varepsilon\mu_0 = \mu_0$.[6])

We first prove

Lemma 5,2. *Let ε be an idempotent $\in \mathfrak{M}(S)$ with $\varepsilon^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i\eta_k[G'_{ik}]$ and $C(\varepsilon) = H$. Let HbH be any class of the decomposition* (14). *Denote $G_{ik} \cap HbH = V_{ik}$ and construct the measure v with $v^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i\eta_k[V_{ik}]$. Then $\varepsilon v = v\varepsilon = v$ holds.*

Proof. Note first that Theorem 3,2 of the paper [18] implies that $V_{ik} = G'_{ik}bG'_{ik}$. Since by Lemma 5,1 $[V_{ik}] = [G'_{ik}]b[G'_{ik}]$, we have

$$\begin{aligned}\varepsilon^* v^* &= \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i\eta_k[G'_{ik}] \cdot \sum_{j=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_j\eta_l[V_{jl}] = \\ &= \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \sum_{j=1}^{\sigma} \sum_{l=1}^{\varrho} [G'_{ik}]\,[G'_{jl}]b[G'_{jl}]\;\xi_i\eta_k\xi_j\eta_l = \\ &= \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \sum_{j=1}^{\sigma} \sum_{l=1}^{\varrho} [G'_{il}]b[G'_{jl}]\;\xi_i\eta_k\xi_j\eta_l\,.\end{aligned}$$

According to the proof of Theorem 5,3 we have $[G'_{il}]b[G'_{jl}] = [V_{il}]$. Hence

$$\varepsilon^* v^* = \Big(\sum_{k=1}^{\varrho} \eta_k\Big)\Big(\sum_{j=1}^{\sigma} \xi_j\Big) \sum_{i=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_i\eta_l[V_{il}] = v^*\,.$$

We prove analogously $v^*\varepsilon^* = v^*$. Therefore $\varepsilon v = v\varepsilon = v$, q. e. d.

---

[6]) For then the semigroup generated by μ, μ_0, ε is a group with ε as unit element, hence μ is contained in a subgroup of $\mathfrak{M}(S)$.

Theorem 5,5. *Let* $\varepsilon \in \mathfrak{M}(S)$ *be an idempotent with* $\varepsilon^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [G'_{ik}]$, $C(\varepsilon) = H = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G'_{ik}$ *and* $H_1 = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G_{ik}$ *the largest subsemigroup of S containing the same idempotents as H. Consider the decomposition*

$$H_1 = H \cup HaH \cup HbH \cup \dots$$

Let HaH be such a class that for every couple i, k the set $G_{ik} \cap HaH = V_{ik}$ *is exactly one two-sided class in the decomposition of the group* G_{ik} *modulo* G'_{ik}. *Then the measure* ν *with* $\nu^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [V_{ik}]$ *is a regular element* $\in \mathfrak{M}(S)$ *belonging to* ε.

Remark. Theorem 5,5 implies that $G_{ik}^{(0)} = H_0 \cap G_{ik} = C(\mathfrak{G}(\varepsilon)) \cap G_{ik}$ is exactly the whole normalizer of the group G'_{ik} in G_{ik}, so that the harmony of our result is really a complete one.

Proof. With respect to Lemma 5,2 it is sufficient to prove that there is a $\nu_0 \in \mathfrak{M}(S)$ such that $\nu\nu_0 = \nu_0\nu = \varepsilon$ and $\nu_0\varepsilon = \varepsilon\nu_0 = \nu_0$.

The element a is contained in a group, say $G_{\alpha\beta}$, $G_{\alpha\beta} \subset H_1$. Denote by $\bar{a}$ the element $\in G_{\alpha\beta}$ such that $a\bar{a} = \bar{a}a = e_{\alpha\beta}$ ($e_{\alpha\beta}$ the unit element of the group $G_{\alpha\beta}$). Denote $H\bar{a}H \cap G_{ik} = V_{ik}^{(0)}$ and construct the element $\nu_0^* = \sum_{j=1}^{\sigma} \sum_{i=1}^{\varrho} \xi_j \eta_l [V_{jl}^{(0)}]$. Since $V_{jl}^{(0)} = G'_{jl}\bar{a}G'_{jl}$ and $[V_{jl}^{(0)}] = [G'_{jl}]\bar{a}[G'_{jl}]$, we have

$$\begin{aligned}[V_{ik}]\,[V_{jl}^{(0)}] &= [G'_{ik}]a[G'_{ik}]\,[G'_{jl}]\bar{a}[G'_{jl}] = [G'_{ik}]a[G'_{il}]\bar{a}[G'_{jl}] = \\ &= [G'_{ik}]\,(ae_{\alpha\beta}[G'_{il}]e_{\alpha\beta}\bar{a})\,[G'_{jl}] = [G'_{ik}]\,(a[G'_{\alpha\beta}]\,\bar{a}])\,[G'_{jl}]\,.\end{aligned}$$

Now, with respect to our (very essential) supposition, we have

$$a[G'_{\alpha\beta}]\,\bar{a} = [G'_{\alpha\beta}]\,a\bar{a} = [G'_{\alpha\beta}]\,e_{\alpha\beta} = [G'_{\alpha\beta}]\,.$$

Hence

$$[V_{ik}]\,[V_{jl}^{(0)}] = [G'_{ik}]\,[G'_{\alpha\beta}]\,[G'_{jl}] = [G'_{il}]\,.$$

Therefore

$$\nu^*\nu_0^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i\eta_k[V_{ik}] \cdot \sum_{j=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_j\eta_l[V_{jl}^{(0)}] = (\sum_{j=1}^{\sigma} \xi_j)(\sum_{k=1}^{\varrho} \eta_k) \sum_{i=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_i\eta_l[G'_{il}] = \varepsilon^*\,.$$

Analogously we prove $\nu_0^*\nu^* = \varepsilon^*$.

Further we have

$$\begin{aligned}\nu_0^*\varepsilon^* &= \sum_{j=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_j\eta_l[V_{jl}^{(0)}] \cdot \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i\eta_k[G'_{ik}] = \sum_{j,l,i,k} \xi_j\eta_l\xi_i\eta_k[G'_{jl}]\bar{a}[G'_{jl}]\,[G'_{ik}] = \\ &= \sum_{jlik} \xi_j\eta_l\xi_i\eta_k[G'_{jl}]\bar{a}[G'_{jk}]\,.\end{aligned}$$

Now $[G'_{jl}]\,\bar{a} = [G'_{jl}]\,e_{\alpha\beta}\bar{a} = [G'_{j\beta}]\,\bar{a}$ shows that $[G'_{jl}]\,\bar{a}$ is independent of l, hence – in particular – equal to $[G'_{jk}]\,\bar{a}$. Therefore $[G'_{jl}]\bar{a}[G'_{jk}] = [G'_{jk}]\bar{a}[G'_{jk}] = [V_{jk}^{(0)}]$. This implies $\nu_0^*\varepsilon^* = \nu_0^*$. Analogously $\varepsilon^*\nu_0^* = \nu_0^*$, which completes the proof of Theorem 5,5.

6. THE CASE OF A SIMPLE SEMIGROUP WITH ZERO

A semigroup S with zero z is called to be simple if it does not contain any two-sided ideal different from z and S itself. In this section we shall study such semigroups using hereby essentially the results of section 5.

Let S be a simple semigroup with zero z and ε an idempotent $\in \mathfrak{M}(S)$ with $\varepsilon^* \neq z$. Denote again $H = C(\varepsilon)$. Since H is a simple semigroup without zero [or a one point group[7])] we have either $H = \{z\}$ or H does not contain z. (We are using hereby the results of the paper [18].) The first case being excluded by supposition, H does not contain z.

Let $H = \bigcup_{i=1}^{\sigma} R'_i = \bigcup_{k=1}^{\varrho} L'_k$ be the decomposition of the semigroup H into its minimal right and left ideals respectively. In the paper [18] (Lemma 2,2) we have proved that there exist minimal left and right ideals of S respectively such that

$$H_1 = \{R_1 \cup \ldots \cup R_\sigma\} \cap \{L_1 \cup \ldots \cup L_\varrho\} - \{z\}$$

is the (unique) maximal simple subsemigroup of S containing the same idempotents as H. (Hereby $R_\alpha \cap H_1$ is a minimal right ideal of H_1 and $L_\beta \cap H_1$ is a minimal left ideal of H_1.)

By Lemma 1,3 we decompose H_1 modulo (H, H) into the union of disjoint classes

$$H_1 = H \cup Hx_1H \cup Hx_2H \cup \ldots,$$

where $x_1, x_2, \ldots$ are suitably chosen elements $\in H_1$.

6,1. Let now be μ a regular element $\in \mathfrak{M}(S)$ belonging to the idempotent ε, with $\varepsilon^* \neq z$. Then $\mu = \mu\varepsilon = \varepsilon\mu$ and $C(\mu) = C(\mu)\,H = H\,C(\mu) = H\,C(\mu)\,H$. Analogously as in the proof of Theorem 5,1 we have $C(\mu) = H\,C(\mu) = \{R'_1 \cup \ldots \cup R'_\sigma\}\,C(\mu) \subset \{R_1 \cup \ldots \cup R_\sigma\}\,C(\mu) \subset \{R_1 \cup \ldots \cup R_\sigma\}$, and by the same argument $C(\mu) \subset \{L_1 \cup \ldots \cup L_\varrho\}$. Hence $C(\mu) \subset \{R_1 \cup \ldots \cup R_\sigma\} \cap \{L_1 \cup \ldots \cup L_\varrho\}$.

We prove that $C(\mu)$ cannot contain z, hence $C(\mu) \subset H_1$. Suppose that S contains v elements, $S = \{z, g_2, g_3, \ldots, g_v\}$. Write $\mu^* = t_1 z + t_2 g_2 + \ldots + t_v g_v$ with $\sum_{i=1}^{v} t_i = 1$. Since μ is regular, there is an element μ' with $(\mu')^* = t'_1 z + t'_2 g_2 + \ldots + t'_v g_v$, $\sum_{i=1}^{v} t'_i = 1$, such that

$$(t_1 z + t_2 g_2 + \ldots + t_v g_v)(t'_1 z + t'_2 g_2 + \ldots + t'_v g_v) = \varepsilon^* .$$

If there were $z \in C(\mu)$, i. e. $t_1 > 0$, the coefficient of z in the product on the left hand side would be a positive number $\geqq$ than $t_1(t'_1 + t'_2 + \ldots + t'_v) = t_1$. Thus we would have $z \in C(\mu\mu') = C(\varepsilon) = H$ contrary to the assumption. This proves $C(\mu) \subset H_1$.

[7]) To avoid confusions we shall mean by a semigroup with zero a semigroup containing at least two elements and so we shall not consider the case of a one point group separately.

Now we may consider μ as a regular measure defined on the simple subsemigroup H_1 with $C(\mu) = H \subset H_1$ and we may apply Theorem 5,1 and Theorem 5,3. We thus obtain the following result:

Theorem 6,1. *Let S be a finite simple semigroup with zero z. Let $\varepsilon^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \,.$ $. \eta_k[G'_{ik}]$ be an idempotent $\in \mathfrak{M}(S)$, $\varepsilon^* \neq z$. Let $C(\varepsilon) = H = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G'_{ik}$ and μ be a regular element $\in \mathfrak{M}(S)$ belonging to ε. Denote by $H_1 = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G_{ik}$ the largest subsemigroup of S containing the same idempotents as H. Then $C(\mu) = Hx_0H$ for a suitably chosen $x_0 \in H_1$ and $\mu^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k[T_{ik}]$, where $T_{ik} = G_{ik} \cap C(\mu) =$ $= G'_{ik}x_0G'_{ik}$.*

Analogously as in Theorem 5,2 we may prove

Theorem 6,2. *Let the suppositions of Theorem 6,1 be satisfied. If μ is any (non-necessarily regular) element belonging to the idempotent ε, we have $C(\mu) \subset Hx_0H$ with a suitably chosen $x_0 \in H_1$.*

The following Corollary is analogous to Corollaries 5,1 and 5,2:

Corollary 6,1. *Suppose that the suppositions of Theorem 6,1 are satisfied. Then the set of regular elements belonging to ε is a finite group. If moreover ε is a point mass at the idempotent $e \in S$, $e \neq z$, then the set of all elements $\in \mathfrak{M}(S)$ belonging to ε forms a group which is isomorphic with the maximal group $G(e) \subset S$ belonging to e.*

6,2. The natural question arises to find the elements which belong to ε with $\varepsilon^* = z$. This question is solved by

Theorem 6,3. *Let S be a finite simple semigroup with zero z. The element $\mu \in$ $\in \mathfrak{M}(S)$ belongs to ε with $\varepsilon^* = z$ if and only if the subsemigroup P_1 generated by $C(\mu)$ contains z.*

Proof. 1. Let be $A = C(\mu)$. Consider the sequence of sets

$$A, A^2, A^3, \ldots, A^{k-1}, A^k, \ldots \tag{17}$$

Since S is finite, the sequence (17) contains only a finite number of different members. Let k be the first exponent such that $A^k = A^l$ for some $l < k$. We then have $A^{k+1} =$ $= A^{l+1}$, $A^{k+2} = A^{l+2}$, ... The set $P_1 = A \cup A^2 \cup \ldots \cup A^k$ is a semigroup since

$$(A \cup A^2 \cup \ldots \cup A^k)(A \cup A^2 \cup \ldots \cup A^k) \subset A \cup A^2 \cup \ldots \cup A^k .$$

Moreover P_1 is the least semigroup containing A since every semigroup containing A necessarily contains P_1.

Suppose that $z \in P_1$. Then there is an integer h $(1 \leqq h \leqq k)$ such that $z \in A^h$, i.e. $z \in [C(\mu)]^h = C(\mu^h)$. If $P_1 = \{z, g_2, \ldots, g_u\}$, the element $(\mu^h)^*$ can be written in the form

$$(\mu^h)^* = t_1 z + t_2 g_2 + \ldots + t_u g_u, \quad t_1 > 0, \quad \sum_{i=1}^{u} t_i = 1.$$

For any integer $j > 0$ put

$$(\mu^j)^* = t_1^{(j)} z + t_2^{(j)} g_2 + \ldots + t_u^{(j)} g_u, \quad \sum_{i=1}^{u} t_u^{(j)} = 1.$$

We have

$$(\mu^{h+j})^* = (t_1 z + t_2 g_2 + \ldots + t_u g_u)(t_1^{(j)} z + t_2^{(j)} g_2 + \ldots + t_u^{(j)} g_u)$$

and the coefficient of z is $\geqq t_1(t_1^{(j)} + \ldots + t_u^{(j)}) = t_1$ independently of the numbers $t_i^{(j)}$. This means that in every member of the sequence

(18) $$(\mu^h)^*, (\mu^{h+1})^*, (\mu^{h+2})^*, \ldots$$

the coefficient of z is $\geqq t_1$. This implies that every element of the closure of the sequence (18) and also the idempotent ε^* to which μ^* belongs has the coefficient of z greater or equal $t_1 > 0$. Therefore μ belongs to such an idempotent ε for which $\varepsilon^* = \bar{t}_0 z + \bar{t}_2 g_2 + \ldots + \bar{t}_u g_u$ and $\bar{t}_0 \geqq t_1$. Now since $C(\varepsilon)$ is a simple semigroup without zero and since such a semigroup if it contains more than one element cannot contain z, we have necessarily $C(\varepsilon) = z$. This proves our assertion.

2. Suppose conversely that P_1 does not contain z. By [18] (Theorem 2,1) P_1 is a simple semigroup without zero. If μ belongs to the idempotent ε we have $H = C(\varepsilon) \subset P_1$. Hence $\varepsilon^* \neq z$. This completes the proof of Theorem 6,3.

Remark. Theorem 6,3 can be considered as a special case of a more general theorem which will be proved below (see Theorem 7,5).

7. THE GENERAL CASE

Now we shall treat the case of a general finite semigroup. We are led to a decomposition of S into the union of some disjoint subsets which turn out to be the known Green's F-classes. (See J. A. Green [8], or R. H. Bruck [1] and E. C. Ляпин [15].)

We are starting from a simple subsemigroup of S and obtain a somewhat modified treatment convenient for our purposes.

7,1. Let H be a simple subsemigroup of S. The least two-sided ideal of S containing H is clearly the set $J_H = H \cup SH \cup HS \cup SHS$. Since $H = H^2 \subset SH$, $HS \subset H \,.\, HS \subset SHS$, and $SH = SHH \subset SHS$, we have clearly $J_H = SHS$.

If $h \in H$, we have $HhH = H$, hence $J_H = SHhHS \subset ShS$. On the other hand we have, of course, $ShS \subset SHS$. Hence $J_H = ShS$ for every $h \in H$.

Notation. In what follows, for brevity, we shall denote the principal ideal of S generated by x, i.e. the set $x \cup Sx \cup xS \cup SxS$, by the symbol $\langle x \rangle$.

Definition. If H is a simple subsemigroup of S, then by H^* we shall denote the set $H^* = \{x \mid x \in S, \langle x \rangle = SHS\}$.

Clearly $H \subset H^*$ and $H^* \subset SHS = J_H$. (In the sense of J. A. Green ([8]) the set H^* is the F-class containing H.)

Every simple subsemigroup H of S is contained in some maximal simple subsemigroup $H^{(0)} \subset S$. But we shall show below (see Example 7,2) that such a maximal simple subsemigroup $H^{(0)}$ is — in general — not uniquely determined.

Lemma 7,1. *If H and H' are simple subsemigroups of S and $H \subset H'$, then $H' \subset H^*$.*

Proof. For any $h \in H \subset H'$ we have $H'hH' = H'$, hence $SH'S = SH'hH'S \subset \subset ShS = SHS$; conversely $H \subset H'$ implies $SHS \subset SH'S$, hence $SH'S = SHS$.

For $h' \in H'$ we have $SH'S = SH'h'H'S \subset Sh'S$, on the other hand $Sh'S \subset SH'S$, hence $SH'S = Sh'S$. Further we have

$$SH'S \subset h' \cup Sh' \cup h'S \cup SH'S = \\ = \langle h' \rangle \subset H' \cup SH' \cup H'S \cup SH'S \subset H'H'H' \cup SH'H' \cup H'H'S \cup \\ \cup SH'S \subset SH'S,$$

hence $\langle h' \rangle = SHS$, i.e. $h' \in H^*$; therefore $H' \subset H^*$.

Lemma 7.2, *If H', H'' are two simple subsemigroups of S, then either $(H')^* \cap \cap (H'')^* = \emptyset$ or $(H')^* = (H'')^*$.*

Proof. Let be $(H')^* \cap (H'')^* \neq \emptyset$. Then there is a $w \in S$ such that $\langle w \rangle = SH'S$ and at the same time $\langle w \rangle = SH''S$. Hence $SH'S = SH''S$. This implies that for every y for which $\langle y \rangle = SH'S$ we have also $\langle y \rangle = SH''S$ and conversely. Hence $(H')^* = = (H'')^*$.

In particular we have

Corollary 7,1. If H', H'' are two simple subsemigroups of S for which $H' \cap H'' \neq \emptyset$, then $(H')^* = (H'')^*$.

The following two examples (both being simple semigroups with zero) serve to clarify somewhat the possible situations.

Example 7,1. *Let $S = \{z, a_1, a_2, a_3, a_4\}$ be a semigroup with the multiplication table:*

	z	a_1	a_2	a_3	a_4
z	z	z	z	z	z
a_1	z	a_1	z	a_3	z
a_2	z	z	a_2	z	a_4
a_3	z	z	a_3	z	a_1
a_4	z	a_4	z	z	z .

This semigroup contains three disjoint maximal simple subsemigroups. These are $H' = \{a_1\}$, $H'' = \{a_2\}$, $H''' = \{z\}$. We have $(H')^* = \{a_1, a_2, a_3, a_4\} = (H'')^*$, $(H''')^* = \{z\}$.

Example 7,2. *Let* $S = \{z, a_1, a_2, a_3, a_4\}$ be the semigroup with the multiplication table

	z	a_1	a_2	a_3	a_4
z	z	z	z	z	z
a_1	z	z	a_1	z	a_3
a_2	z	z	a_2	z	a_4
a_3	z	a_1	a_1	a_3	a_3
a_4	z	a_2	a_2	a_4	a_4

This semigroup contains again three maximal simple subsemigroups. These are $H' = \{a_2, a_4\}$, $H'' = \{a_3, a_4\}$, $H''' = \{z\}$. We have again $(H')^* = (H'')^* = \{a_1, a_2, a_3, a_4\}$, $(H''')^* = \{z\}$. Note that in this example there are two maximal simple subsemigroups H', H'' which are not disjoint, since $H' \cap H'' = \{a_4\} \neq \emptyset$. This example gives at the same time a negative answer to a question raised by R. Croisot (see [5], p. 369, footnote 7).

Let H, J_H and H^* have the sense introduced above. Consider the set $K_H = J_H - H^*$ (i. e. the set of elements $\in J_H$ which do not generate the ideal J_H). It is known – and easy to prove – that K_H is a two-sided ideal of S.[8]) Consider now the difference semigroup J_H/K_H in the sense of D. Rees ([17]). The elements of this semigroup are the elements $\in J_H - K_H = H^*$ together with an adjoint zero element O_{H^*}. In the set $H_0^* = H^* \cup \{O_{H^*}\}$ the product is defined as follows: If $x, y \in H^*$ and $xy \in H^*$ the product denotes the same element as in S; if $xy \in K_H$ we define $xy = O_{H^*}$; further $x \,.\, O_{H^*} = O_{H^*} \,.\, x = O_{H^*}$ for every $x \in H_0^*$. It is known (see J. A. Green [8]) that H_0^* is a simple semigroup with zero for which $(H_0^*)^2 = H_0^*$. If H^* itself is a semigroup, H^* is a simple semigroup without zero. For every $a \neq O_{H^*}$, $a \in H_0^*$, we have $H_0^* a H_0^* = H_0^*$.

Summarily we have

Lemma 7,3. *Let H be a simple subsemigroup of S. Then there exists a uniquely determined subset H^* of S with the following properties*: a) $H^* \supset H$; b) *H^* contains every maximal simple subsemigroup H' for which $H' \supset H$ holds*; c) *H^* is either a simple semigroup without zero or by adjoining a zero element O_{H^*} and defining a natural multiplication the set $H^* \cup \{O_{H^*}\}$ becomes a simple semigroup with zero. For two simple subsemigroups H', H'' we have either $(H')^* = (H'')^*$ or $(H')^* \cap (H'')^* = \emptyset$.*

[8]) To prove this we show first that $y \in K_H$ implies $Sy \in K_H$. Suppose indirectly that this is not the case and that there is an $x \in Sy \cap H^*$ (i. e. x generates J_H). Write $x = uy$, $u \in S$. We then have $J_H = \langle x \rangle = uy \cup Suy \cup uyS \cup SuyS \subset Sy \cup Sy \cup SyS \cup SyS \subset \langle y \rangle \subset J_H \cup SJ_H \cup J_HS \cup SJ_HS \subset J_H$. Hence $\langle y \rangle = J_H$, which contradicts $y \in K_H$. Therefore $Sy \in K_H$ and analogously $yS \in K_H$, which proves that K_H is a two-sided ideal of S.

7,2. We return to the study of $\mathfrak{M}(S)$ and we prove

Theorem 7,1. *Let S be a finite semigroup and ε an idempotent $\in \mathfrak{M}(S)$. Denote $C(\varepsilon) = H$. If μ belongs to the idempotent ε and μ is a regular element $\in \mathfrak{M}(S)$, then $C(\mu) \subset H^*$.*

Proof. a) If μ is regular, we have $\varepsilon\mu = \mu\varepsilon = \mu$, hence $C(\mu) = C(\mu)\,H = H\,C(\mu) = H\,C(\mu)\,H$.

b) We first show that $S\,C(\mu)\,S = SHS$. We have $C(\mu) = C(\mu)\,H \subset SH = SH\,.\,H \subset SHS$, hence $S\,C(\mu)\,S \subset SHS$. Since μ is regular, there is a regular element $\mu_0 \in \mathfrak{M}(S)$ such that $\mu\mu_0 = \varepsilon$. Therefore $C(\mu)\,C(\mu_0) = H$. Now we have $SHS = S\,C(\mu)\,.\,C(\mu_0)\,S \subset S\,C(\mu)\,S$. This implies $SHS = S\,C(\mu)\,S$.

c) We next prove that for every $c \in C(\mu)$ we have $ScS = SHS$. If c is any element $c \in C(\mu)$ the relation $C(\mu)\,C(\mu_0) = H$ implies the existence of an element $c_0 \in C(\mu_0)$ such that $cc_0 = h \in H$. Further $C(\mu) = H\,C(\mu)$ implies the existence of two elements $h_1 \in H$, $c_1 \in C(\mu)$ such that $c = h_1c_1$. We now have $SHS = ShS = Scc_0S = Sc(c_0S) \subset ScS = Sh_1c_1S = Sh_1(c_1S) \subset Sh_1S = SHS$, whence $ScS = SHS$.

d) Finally we have

$$c \in C(\mu) = H\,C(\mu)\,H \subset S\,C(\mu)\,S = SHS = ScS\,,$$
$$cS \in C(\mu)\,S = H\,C(\mu)\,S \subset S\,C(\mu)\,S = ScS\,,$$
$$Sc \in S\,C(\mu) = S\,C(\mu)\,H \subset S\,C(\mu)\,S = ScS\,.$$

Hence for every $c \in C(\mu)$ we have $\langle c \rangle = c \cup Sc \cup cS \cup ScS = ScS = SHS$. Therefore every $c \in C(\mu)$ is contained in H^*. This proves our theorem.

Remark. Roughly to say the foregoing theorem locates, so to speak, $C(\mu)$ for a regular μ. $C(\mu)$ does not lie "far away" from the set $C(\varepsilon) = H$. It remains in H^* which itself is uniquely determined by H.

The location of the set $C(\mu)$ is also more precisely described by the following essentially sharper

Theorem 7,2. *Let S be a finite semigroup, $\varepsilon^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i\eta_k[G'_{ik}]$ an idempotent $\in \mathfrak{F}(S)$, and $H = C(\varepsilon) = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G'_{ik}$. Let $H_1 = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G_{ik}$ be the largest simple subsemigroup of S containing the same idempotents as H. Suppose that μ belongs to ε and μ is a regular element $\in \mathfrak{M}(S)$. We then have:*

a) *$C(\mu) = Hx_0H$, where x_0 is a suitably chosen element $\in H_1$,*

b) *$\mu^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i\eta_k[T_{ik}]$, where $T_{ik} = C(\mu) \cap G_{ik} = G'_{ik}x_0G'_{ik}$.*

Proof. Let J_H, K_H and H^* have the same meaning as introduced above. Consider the difference semigroup $H_0^* = J_H/K_H$. The elements of this semigroup are the elements $\in H^*$ together with the zero adjoint O_{H^*}.

The semigroup H_0^* is a simple semigroup with zero. We have $H \subset H^* \subset H_0^*$. Since $C(\mu) \subset H^*$, we may consider μ as a regular measure defined on H_0^* and belonging to the idempotent ε. Since $C(\varepsilon) = H$, and H does not contain O_{H^*}, we can apply Theorem 6,1. Let H_1 be the greatest subsemigroup of S containing the same idempotents as H. We know that $H_1 \subset H^*$. By Theorem 6,1 we have $C(\mu) = Hx_0H$ with a suitably chosen element $x_0 \in H_1$ and we get the explicit expression of μ^* in the form given above. This proves Theorem 7,2.

Analogously as in Corollaries 5,1 and 6,1 we get a Corollary which (in contradistinction to the case of simple semigroups) is dealing only with regular elements $\in \mathfrak{M}(S)$.

Corollary 7,2. *Let S be a finite semigroup and ε an idempotent $\in \mathfrak{M}(S)$. Then the set of all regular elements $\in \mathfrak{M}(S)$ belonging to ε is a finite group. If ε is a point mass at the idempotent $e \in S$, then the group of regular elements $\in \mathfrak{M}(S)$ belonging to ε is isomorphic with the maximal group $G(e)$. If ε is such that $C(\varepsilon)$ is a maximal group of S, then the set of regular elements belonging to ε reduces to a single element (namely ε itself).*

7,3. Now we shall try to locate $C(\mu)$ for a non-necessarily regular $\mu \in \mathfrak{M}(S)$. The result we obtain is of course not so simple as the result formulated in the corresponding Theorems 5,2 and 6.2.

Let S be a semigroup and $x \in S$. Denote $F_x = \{y \mid y \in S, \langle y \rangle = \langle x \rangle\}$. Two sets F_u, F_v are either identical or disjoint. The semigroup S can be written as a union of pairwise disjoint classes $S = \bigcup_{x \in S} F_x$. These are the F-classes of J. A. Green ([8]). Each of our sets H^* is an F-class, but there may exist also F-classes without idempotents, i.e. F-classes that do not contain simple subsemigroups.

Introduce in the set of F-classes a partial ordering by the statement $F_x \leqq F_y$ if $\langle x \rangle \subset \langle y \rangle$. Every F-class F_x consists of the generators of a certain principal ideal J, whereby J itself is the union of all F-classes F' with $F' \leqq F_x$.

Consider two classes F_u, F_v generating the principal ideals J_1, J_2 and let u, v be any elements $\in J_1$ and $\in J_2$ respectively. What can be said about the product uv? Clearly $uv \in F_uF_v \subset J_uJ_v \subset J_u \cap J_v$. If uv is contained in the class F_w, we have necessarily $F_w \cap [J_u \cap J_v] \neq \emptyset$. Now

$$J_u = F_u \cup \{\text{union of classes } F_\alpha \cup F_\beta \cup \ldots \text{ which are } \leqq F_u\},$$
$$J_v = F_v \cup \{\text{union of classes } F_{\alpha'} \cup F_{\beta'} \cup \ldots \text{ which are } \leqq F_v\}.$$

If F_w has a non-empty intersection with some class it is identical with it. Therefore $F_w = F_\alpha = F_{\alpha'}$ for suitably chosen α and α'. Hence certainly $F_w \leqq F_u$, $F_w \leqq F_v$. We have proved:

Lemma 7,4. *If $u \in F_u$, $v \in F_v$, then uv is contained in such a class F_w that $F_w \leqq F_u$, $F_w \leqq F_v$.*

Remark. In general it is not true that the product F_uF_v is contained in some class F_w. Simple examples show that this product can be scattered throughout several F-classes.

Theorem 7,3. *Let $\mu \in \mathfrak{M}(S)$ belong to the idempotent ε. Denote $C(\varepsilon) = H$. Then $C(\mu) \subset \bigcup_{F \geqq H^*} F$, where the union runs through all F-classes which are $\geqq$ as the F-class H^*.*

Proof. $\mu\varepsilon$ is a regular element $\in \mathfrak{M}(S)$. By Theorem 7,1 we have $C(\mu\varepsilon) \subset H^*$, i.e. $C(\mu) H \subset H^*$. Let be $c \in C(\mu)$ and $h \in H$. Since $ch \in H^*$, we have by Lemma 7,4 $F_c \geqq H^*$. Hence $C(\mu) \subset \bigcup_{F_c \geqq H^*} F_c$, q.e.d.

7,4. The relation between μ and ε is also clarified by Theorems 7,4 and 7,5 formulated below.

Let Γ be any group having ε as its unit element, $\Gamma \subset \mathfrak{G}(\varepsilon)$ (the maximal group of $\mathfrak{M}$ belonging to ε). By Theorem 7,2 we have $C(\mu) \subset H_1$ for any $\mu \in \Gamma$. Now $C(\Gamma) = \bigcup_{\mu \in \Gamma} C(\mu)$ is a semigroup (contained in H_1), hence a simple subsemigroup of H_1. This implies the following slight generalization of Corollary 5,3:

Lemma 7,5. *If Γ is any subgroup of $\mathfrak{M}(S)$, then $C(\Gamma)$ is a simple subsemigroup of S.*

We next prove

Theorem 7,4. *Let S be finite and Γ the maximal group contained in $\overline{\{\mu, \mu^2, \mu^3, \ldots\}}$. Let further be P the subsemigroup of S generated by $C(\mu)$. If J is the minimal two-sided ideal of P, we have $C(\Gamma) = J$.*

Proof. Analogously as in Theorem 6,3 we may write $P = C(\mu) \cup C(\mu^2) \cup \ldots \cup C(\mu^k)$, where k is an integer. Denote $C(\Gamma) = K \subset P$, and suppose that μ belongs to $\varepsilon \in \Gamma$.

If $x \in P$, then $x \in C(\mu^l)$ with an $l \leqq k$. We have $C(\varepsilon)\, x\, C(\varepsilon) \subset C(\varepsilon)\, C(\mu^l)\, C(\varepsilon) = C(\varepsilon\mu^l\varepsilon)$ and since $\varepsilon\mu^l\varepsilon \in \Gamma$, we have $C(\varepsilon)\, x\, C(\varepsilon) \subset K$. This implies $PxP \cap K \neq \emptyset$ for every $x \in P$. Now the set $J = \bigcap_{x \in P} PxP \neq \emptyset$ is the minimal two-sided ideal of P. If $y \in J \subset P$, we have $PyP \subset PJP \subset J$, hence $J = PyP$. Since $y \in P$, we have also $J \cap K = PyP \cap K \neq \emptyset$ and since K is a simple subsemigroup of P, we have (by Lemma 3,2) $K \subset J$.

Let again be $x \in P$ and $x \in C(\mu^l)$, with an integer l. Let further ν be *any* element $\in \Gamma$. We then have $x\, C(\nu) \subset C(\mu^l)\, C(\nu) = C(\mu^l \nu)$ and since $\mu^l\nu \in \Gamma$, $x\, C(\nu) \subset K$. This implies $x \bigcup_{\nu \in \Gamma} C(\nu) \subset K$, i.e. $xK \subset K$. Analogously $Kx \subset K$ for every $x \in P$. Hence K is a two-sided ideal of P. Since $K \subset J$ and J is minimal, we have $K = J$, which completes the proof of our theorem.

In particular, we have $C(\varepsilon) \subset C(\Gamma) \subset J = K$, hence we can strengthen Lemma 4,1 as follows:

Theorem 7,5. *If S is finite, μ belongs to ε, and P is the subsemigroup generated by $C(\mu)$, then $C(\varepsilon)$ is contained in the minimal two-sided ideal of P.*

Let μ belong to the idempotent ε. Consider the sequence

$$\mu, \mu^2, \mu^3, \ldots \tag{19}$$

One of the fundamental questions is as follows: Under what conditions does the sequence (19) converge?

If (19) converges we have necessarily $\lim_{n=\infty} \mu^n = \varepsilon$.

It is known and easy to prove that $\lim_{n=\infty} \mu^n$ exists if and only if $\mu\varepsilon = \varepsilon\mu = \varepsilon$. In other words this is the case if and only if the maximal group contained in the closure of $\{\mu, \mu^2, \mu^3, \ldots\}$ is a one point group. (See Б. М. Клосс [12], also E. Hewitt, H. S. Zuckerman [9].) An alternative answer to this question is given by:

Theorem 8,1. *Let $\mu \in \mathfrak{M}(S)$ belong to the idempotent ε. Denote $C(\varepsilon) = H$. Then $\lim_{n=\infty} \mu^n$ exists if and only if $H\,C(\mu)\,H = H$.*

Proof. 1. The condition is necessary. If $\lim_{n=\infty} \mu^n$ exists, we have $\varepsilon\mu = \varepsilon$, i.e. $H\,C(\mu) = H$, whence $H\,C(\mu)\,H = H$.

2. The condition is sufficient. Write H in the form $H = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G'_{ik}$ and $\varepsilon^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [G'_{ik}]$. Let further be $G'_{ik} = \{g_{ik}^{(1)}, g_{ik}^{(2)}, \ldots, g_{ik}^{(m)}\}$. Consider the measure $\varepsilon\mu\varepsilon \in \mathfrak{M}(S)$. Since $H\,C(\mu)\,H = H$ we may write $(\varepsilon\mu\varepsilon)^* = \sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\varrho} \sum_{\gamma=1}^{m} t_{\alpha\beta}^{(\gamma)} g_{\alpha\beta}^{(m)}$, where $\sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\varrho} \sum_{\gamma=1}^{m} t_{\alpha\beta}^{(\gamma)} = 1$.

Further we have

$$\varepsilon^* g_{jl}^{(\gamma)} \varepsilon^* = \sum_{i}^{\sigma} \sum_{k}^{\varrho} \xi_i \eta_k [G'_{ik}] \cdot g_{jl}^{(\gamma)} \cdot \sum_{u=1}^{\sigma} \sum_{v=1}^{\varrho} \xi_u \eta_v [G'_{uv}] \,.$$

Since

$$[G'_{ik}] g_{jl}^{(\gamma)} [G'_{uv}] = [G'_{il}]\,[G'_{uv}] = [G'_{iv}]\,,$$

we have

$$\varepsilon^* g_{jl}^{(\gamma)} \varepsilon^* = \Big(\sum_{k=1}^{\varrho} \eta_k\Big)\Big(\sum_{u=1}^{\sigma} \xi_u\Big) \sum_{i=1}^{\sigma} \sum_{v=1}^{\varrho} \xi_i \eta_v [G'_{iv}] = \varepsilon^* \,.$$

Therefore

$$(\varepsilon^* \mu^* \varepsilon^*) = \varepsilon^* (\varepsilon\mu\varepsilon^*)^* \,\varepsilon^* = \varepsilon^* \sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\varrho} \sum_{\gamma=1}^{m} t_{\alpha\beta}^{(\gamma)} g_{\alpha\beta}^{(m)} \varepsilon^* = \Big(\sum_{\alpha=1}^{\sigma} \sum_{\beta=1}^{\varrho} \sum_{\gamma=1}^{m} t_{\alpha\beta}^{(\gamma)}\Big)\, \varepsilon^* = \varepsilon^* \,.$$

The relation $(\varepsilon\mu\varepsilon)^* = \varepsilon^*$ implies $(\varepsilon\mu)^2 = \varepsilon\mu$. Since $\varepsilon\mu$ is an idempotent and at the same time an element belonging to ε, we have $\varepsilon\mu = \varepsilon$. Analogously $\mu\varepsilon = \varepsilon$. This proves our theorem.

For simple semigroups without zero this condition takes a much simpler form:

Theorem 8,2. *Let S be a simple semigroup without zero, $\mu \in \mathfrak{M}(S)$, μ belongs to ε and $C(\varepsilon) = H$. Then $\lim_{n=\infty} \mu^n$ exists if and only if $C(\mu) \subset H$.*

Proof. In the foregoing theorem we have proved that $\lim_{n=\infty} \mu^n$ exists if and only if $H\, C(\mu)\, H = H$.

a) Suppose that $H\, C(\mu)\, H = H$ is satisfied. By Theorem 5,2 $C(\mu)$ is necessarily contained in H_1 (the largest simple subsemigroup of S containing the same idempotents as H). Now for $x \in H_1$ we have $HxH = H$ if and only if $x \in H$, hence $C(\mu) \subset H$.

b) Suppose conversely that $C(\mu) \subset H$. We then have $H\, C(\mu)\, H \subset H$. Now, since H is a simple semigroup without zero, its two-sided ideal $H\, C(\mu)\, H$ is identical with H, hence $H\, C(\mu)\, H = H$. This proves our assertion.

Other formulations of Theorems 8,1 and 8,2 can be obtained by means of Theorem 7,4.

Theorem 8,3. *Suppose that the suppositions of Theorem 8,1 are satisfied. Then $\lim_{n=\infty} \mu^n$ exists if and only if H is the minimal two-sided ideal of the semigroup P generated by $C(\mu)$.*

Proof. $\lim_{n=\infty} \mu^n$ exists if and only if the group Γ (considered in Theorem 7,4) reduces to $\Gamma = \{\varepsilon\}$.

If $\Gamma = \{\varepsilon\}$, then (by Theorem 7,4) $C(\Gamma) = C(\varepsilon) = H$ is the minimal two-sided ideal of P. If conversely $C(\varepsilon) = H$ is the minimal two-sided ideal of P, then by Theorem 7,2 and 7,4 Γ contains a single element, i.e. ε.

Theorem 8,4. *Suppose that the suppositions of Theorem 8,2 are satisfied. Then $\lim_{n=\infty} \mu^n$ exists if and only if H is identical with the subsemigroup generated by $C(\mu)$.*

Proof. This follows from Theorem 8,3 since P being a simple semigroup cannot contain a proper subsemigroup $H \neq P$.

Remark. Theorems 8,3 and 8,4 are related to the results of K. Urbanik [29], who considered the case of compact groups.

We mention two special results which can easily be proved directly. Suppose that S is a simple semigroup with zero z. If μ belongs to the point mass at z, we have [with respect to $H = C(\varepsilon) = z$] $z\, C(\mu)\, z = z$, i.e. $H\, C(\mu)\, H = H$. Hence:

Corollary 8,1. *If S is a simple semigroup with zero z and μ belongs to ε with $\varepsilon^* = z$, then $\lim_{n=\infty} \mu^n$ exists (and is equal to ε).*

More generally:

Theorem 8,5. *If S is any semigroup, $\mu \in \mathfrak{M}(S)$, and μ belongs to a primitive idempotent ε, then $\lim_{n=\infty} \mu^n$ exists (and is equal to ε).*

Proof. By Theorem 3,5 the maximal group belonging to a primitive idempotent ε is a one point group, hence $\varepsilon\mu = \mu\varepsilon = \varepsilon$, q.e.d.

The following can be considered as an extension of the Weyl's equidistribution theorem:

Theorem 8,6. *Let S be finite and $\mu \in \mathfrak{M}(S)$. Denote $\sigma_n^* = 1/n \,.\, (\mu^* + \mu^{*2} + \ldots + \mu^{*n})$. Then $\lim_{n=\infty} \sigma_n$ exists and is equal to an idempotent $\sigma \in \mathfrak{M}(S)$. If P is the subsemigroup of S generated by $C(\mu)$ and J the minimal two-sided ideal of P, then $C(\sigma) = J$.*

Proof. Denote by $\mathfrak{H}_\mu$ the closed convex hull of the subsemigroup $\{\mu^*, \mu^{*2}, \mu^{*3}, \ldots\} \subset \mathfrak{F}(S)$. Clearly $C(\mathfrak{H}_\mu) = P$. (See the proof of Theorem 7,4.) Let σ^* be any cluster point of the sequence $\sigma_1^*, \sigma_2^*, \sigma_3^*, \ldots$ Clearly $\sigma^* \in \mathfrak{H}_\mu$. Since $\mu^*\sigma_n^* - \sigma_n^* = 1/n \,.\, (\mu^{*n+1} - \mu^*)$, it is easily seen that $\mu^*\sigma^* = \sigma^*$. Since this relation implies $\sigma^* = \mu^*\sigma^* = \mu^{*2}\sigma^* = \mu^{*3}\sigma^* = \ldots$, we have also $\sigma^* = (t_1\mu^* + t_2\mu^{*2} + t_3\mu^{*3} + \ldots)\,\sigma^*$ for any $t_i \geqq 0$ with $\sum_i t_i = 1$, consequently (with respect to the continuity) $\sigma^* = \lambda^*\sigma^*$ for every $\lambda^* \in \mathfrak{H}_\mu$. This means that $\mathfrak{H}_\mu$ being considered as an abelian subsemigroup of $\mathfrak{F}(S)$ contains σ^* as its zero element. Since any semigroup contains at most one zero element, there is a unique cluster point of the sequence $\sigma_1^*, \sigma_2^*, \sigma_3^*, \ldots$ and $\lim_{n=\infty} \sigma_n^* = \sigma^*$ follows by compactness. Moreover σ^* is an idempotent and at the same time the minimal (two-sided) ideal of $\mathfrak{H}_\mu$.

The relation $\sigma^*\mathfrak{H}_\mu = \mathfrak{H}_\mu\sigma^* = \sigma^*$ implies $C(\sigma)\,C(\mathfrak{H}_\mu) = C(\mathfrak{H}_\mu)\,C(\sigma) = C(\sigma)$, i.e. $P\,C(\sigma) = C(\sigma)\,P = C(\sigma)$. Since $C(\sigma) \subset P$, $C(\sigma)$ is a two-sided ideal of P. We have $J\,C(\sigma) \subset J \cap C(\sigma)$, hence $J \cap C(\sigma) \neq \emptyset$, and since $C(\sigma)$ is a simple semigroup, we have by Lemma 3,2 $C(\sigma) \subset J$. Since finally J is the minimal two-sided ideal of P, we have $C(\sigma) = J$. This completes the proof of Theorem 8,6.

9. THE CASE OF SEMIGROUPS ADMITTING RELATIVE INVERSES

The fact that the support of an idempotent $\varepsilon \in \mathfrak{M}(S)$ is a simple semigroup has led us to consider the relations between the simple subsemigroups of S. We now proceed in this direction.

Recall first the fact we have used several times, namely that any subgroup of any semigroup S is contained in a maximal subgroup of S and the maximal subgroups are pairwise disjoint.

Analogously it is possible to prove (see R. Croisot [5]) that every left simple subsemigroup of S is contained in a uniquely determined maximal left simple subsemigroup and two maximal left simple subsemigroups are disjoint.

Also every simple subsemigroup of a finite semigroup S is contained in a maximal simple subsemigroup of S, but Example 7,2 shows that two maximal simple subsemigroups may have a non-empty intersection (which itself is simple).

However, there exist important classes of semigroups in which two maximal simple subsemigroups are disjoint. To such classes belong f.i. the semigroups which can be written as a union of (disjoint) groups ("semigroups admitting relative inverses").

9,1. We adopt the terminology introduced by A. H. Clifford in [2]. Let be S any (non-necessarily finite) semigroup which can be written as a union of disjoint subsemigroups $S = \bigcup_{\alpha \in \Lambda} S_\alpha$. Suppose that $S_\alpha S_\beta$ and $S_\beta S_\alpha$ $(\alpha, \beta \in \Lambda)$ are contained in the same semigroup S_γ, $\gamma \in \Lambda$. Then we shall say that S is a semilattice of semigroups $\{S_\alpha \mid \alpha \in \Lambda\}$.

A. H. Clifford proved (see [2]):

A semigroup is a union of groups if and only if it is a semilattice of completely simple semigroups.

R. Croisot (see [5]) generalized this result as follows:

A semigroup S is a union of simple semigroups if and only if for every two-sided ideal J of S we have: $x^2 \in J$ if and only it $x \in J$.

Further it is proved in the same paper:

If S is the union of simple semigroups $S = \bigcup_{\alpha \in \Lambda} S_\alpha$, then S is a semilattice of simple semigroups.

If S is a union of simple semigroups then every simple subsemigroup of S is contained in a maximal simple subsemigroup and any two maximal simple subsemigroups are disjoint.

Remark 1. If S is an idempotent semigroup, S is a union of one point groups. The decomposition of such a semigroup into a semilattice of simple semigroups has also been described by E. Hewitt and H. S. Zuckerman ([10]) and D. McLean ([14]).

Remark 2. If S is commutative every simple subsemigroup of S is a group and the maximal simple subsemigroups are disjoint. Again S is a union of simple semigroups if and only if S is a union of disjoint groups. Hereby S is a semilattice of groups. Measures on such finite semigroups have been studied in great detail by E. Hewitt and H. S. Zuckerman in the paper [9].

9,2. In what follows we shall apply our results to a finite semigroup S which is a union of groups. We write in the above sense $S = \bigcup_{\alpha \in \Lambda} S_\alpha$, so that each S_α is a maximal simple subsemigroup of S.

We introduce a partial ordering into the set of indices Λ by writting $\alpha \leqq \beta$ if $S_\alpha S_\beta \subset S_\alpha$. The set of indices forms then clearly a semilattice.

If ε is an idempotent $\in \mathfrak{M}(S)$, $H = C(\varepsilon)$ is a simple subsemigroup of S and it is contained in a maximal simple subsemigroup S_α. In this case the set S_α is identical with the set H^* introduced in section 7.

Theorem 7,3 and 7,1 imply the following

Theorem 9,1. *Let S be finite and let $S = \bigcup_{\gamma \in \Lambda} S_\gamma$ be a semilattice of simple semigroups. Let $\varepsilon \in \mathfrak{M}(S)$ be an idempotent for which $C(\varepsilon) = H \subset S_\alpha$. If μ belongs to ε, we have $C(\mu) \subset \bigcup_{\gamma \geqq \alpha} S_\gamma$. If μ is regular, we have moreover $C(\mu) \subset S_\alpha$.*

For a regular μ we may use Theorem 7,2 and we have

Theorem 9,2. *Let the suppositions of Theorem 9,1 be satisfied. If μ is a regular element $\in \mathfrak{M}(S)$ belonging to ε, then $C(\mu) = Hx_0H$ with a suitably chosen $x_0 \in S_\alpha$.*

Theorem 9,1 can be strengthned as follows:

Theorem 9,3. *Let S be finite and $S = \bigcup_{\gamma \in \Lambda} S_\gamma$ be a semilattice of simple semigroups. Let be $\mu \in \mathfrak{M}(S)$ and suppose that $C(\mu)$ has a non-empty intersection with the subsemigroups $S_\alpha, S_\beta, \ldots, S_\tau$. Let τ_0 be the least element of the semilattice generated by $\alpha, \beta, \ldots, \tau$. Then μ belongs to an idempotent ε for which $C(\varepsilon) = H \subset S_{\tau_0}$ holds.*

Proof. Let be $C(\varepsilon) = H \subset S_\delta$, $\delta \in \Lambda$. The element $\varepsilon\mu$ is a regular element belonging to ε, hence $C(\varepsilon\mu) \subset S_\delta$, i.e. $H\,C(\mu) \subset S_\delta$. By supposition $C(\mu) \cap S_\alpha = D_\alpha \neq \emptyset$, $\ldots$, $C(\mu) \cap S_\tau = D_\tau \neq \emptyset$ and $C(\mu) = D_\alpha \cup \ldots \cup D_\tau$, hence $H(D_\alpha \cup \ldots \cup D_\tau) \subset S_\delta$. Since every element $\in S_\delta S_\alpha$ is contained in the same semigroup S_ξ and $HD_\alpha \subset S_\delta$, we have necessarily $S_\xi = S_\delta$, hence $S_\delta S_\alpha \subset S_\delta$, i.e. $\delta \leqq \alpha$. Analogously $\delta \leqq \beta, \ldots, \delta \leqq \tau$. This implies $\delta \leqq \tau_0$. By Lemma 4,1 $C(\varepsilon) \subset P_1$, where P_1 is the least semigroup containing $C(\mu) = D_\alpha \cup \ldots \cup D_\tau$. Consider the semilattice $J \subset \Lambda$ generated by the elements $\alpha, \beta, \ldots, \tau \in \Lambda$. τ_0 is the least element of J. Since $\bigcup_{\xi \in J} S_\xi$ is a semigroup, we certainly have $P_1 \subset \bigcup_{\xi \in J} S_\xi$. Since $C(\varepsilon) = H \subset P_1 \subset \bigcup_{\nu \in J} S_\nu$ and $C(\varepsilon) \subset S_\delta$, we have $S_\delta \subset \bigcup_{\nu \in J} S_\nu$, hence $\delta \in J$ and $\delta \geqq \tau_0$, therefore $\delta = \tau_0$, q.e.d.

10. THE CASE OF IDEMPOTENT SIMPLE SEMIGROUPS

In this section we return to a more detailed study of simple semigroups without zero supposing moreover that each element $\in S$ is idempotent.

With the same notations as in section 5 we may write $S = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} \{g_{ik}\}$, where every g_{ik} is an idempotent. By Theorem 2,4 for every $\varepsilon \in \mathfrak{M}(S)$ we have $\varepsilon^* = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k g_{ik}$, where $0 \leqq \xi_i \leqq 1$, $0 \leqq \eta_k \leqq 1$, $\sum_{i=1}^{s} \xi_i = \sum_{k=1}^{r} \eta_k = 1$ and every idempotent $\in \mathfrak{M}(S)$ is obtained in this manner. By Theorem 3,4 every idempotent $\in \mathfrak{M}(S)$ is a primitive idempotent $\in \mathfrak{M}(S)$. By Theorem 3,5 the kernel $\mathfrak{N}$ of $\mathfrak{M}(S)$ is identical with the set of all idempotents $\in \mathfrak{M}(S)$ and the structure of $\mathfrak{N}$ is described by Theorem 3,6.

10,1. The following theorem gives further information about the structure of $\mathfrak{M}(S)$.

Theorem 10,1. *Let S be an idempotent simple semigroup. Then*

a) *For any* $\mu \in \mathfrak{M}(S)$ *the element* μ^2 *is an idempotent (hence* $\mathfrak{M}(S)$ *is a torsion semigroup).*

b) *The product of any two idempotents* $\in \mathfrak{M}(S)$ *is an idempotent.*

c) *If* μ' *belongs to* ε', μ'' *belongs to* ε'', *then* $\mu'\mu'' = \varepsilon'\varepsilon''$.

d) *The set of all elements belonging to* ε *is a subsemigroup* $\mathfrak{K}(\varepsilon)$.

Proof. a) Let μ be any element with $\mu^* = \sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik} g_{ik}$ and $\sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik} = 1$, $t_{ik} \geqq 0$.

We have

$$(\mu^2)^* = \sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik} g_{ik} \cdot \sum_{j=1}^{s} \sum_{l=1}^{r} t_{jl} g_{jl} = \sum_{i=1}^{s} \sum_{k=1}^{r} \sum_{j=1}^{s} \sum_{l=1}^{r} t_{ik} t_{lj} g_{il},$$

$$(\mu^3)^* = \sum_{i=1}^{s} \sum_{k=1}^{r} \sum_{j=1}^{s} \sum_{l=1}^{r} t_{ik} t_{jl} g_{il} \cdot \sum_{u=1}^{s} \sum_{v=1}^{r} t_{uv} g_{uv} = \sum_{ikjluv} t_{ik} t_{jl} t_{uv} g_{iv} =$$

$$= \Big(\sum_{j=1}^{s} \sum_{l=1}^{r} t_{jl}\Big) \cdot \sum_{i=1}^{s} \sum_{k=1}^{r} \sum_{u=1}^{s} \sum_{v=1}^{r} t_{ik} t_{uv} g_{iv} = (\mu^2)^* .$$

The relation $\mu^3 = \mu^2$ implies $\mu^4 = \mu^2$, hence μ^2 is an idempotent.

b) Since $\varepsilon' \in \mathfrak{N}$, $\varepsilon'' \in \mathfrak{N}$, we have also $\varepsilon'\varepsilon'' \in \mathfrak{N}$ and since every element $\in \mathfrak{N}$ is an idempotent, $\varepsilon'\varepsilon''$ is an idempotent.

c) Let be

$$(\mu')^* = \sum_{i=1}^{s} \sum_{k=1}^{r} t'_{ik} g_{ik}, \quad (\mu'')^* = \sum_{i=1}^{s} \sum_{k=1}^{r} t''_{ik} g_{ik}, \quad \sum_{ik} t'_{ik} = \sum_{ik} t''_{ik} = 1, \quad t'_{ik} \geqq 0, \quad t''_{ik} \geqq 0 .$$

By supposition

$$(\mu')^{*2} = \sum_{i=1}^{s} \sum_{k=1}^{r} \sum_{j=1}^{s} \sum_{l=1}^{r} t'_{ik} t'_{jl} g_{il} = \varepsilon' ,$$

$$(\mu'')^{*2} = \sum_{\alpha=1}^{s} \sum_{\beta=1}^{r} \sum_{\gamma=1}^{s} \sum_{\delta=1}^{r} t''_{\alpha\beta} t''_{\gamma\delta} g_{\alpha\delta} = \varepsilon'' .$$

On the other hand we have

$$(\varepsilon'\varepsilon'')^* = \sum_{\substack{ikjl \\ \alpha\beta\gamma\delta}} t'_{ik} t'_{jl} t''_{\alpha\beta} t''_{\gamma\delta} g_{i\delta} = \Big(\sum_{jl} t'_{\gamma l}\Big) \Big(\sum_{\alpha\beta} t''_{\alpha\beta}\Big) \sum_{\substack{ik \\ \gamma\delta}} t'_{ik} t''_{\gamma\delta} g_{i\delta} =$$

$$= \Big(\sum_{ik} t'_{ik} g_{ik}\Big) \Big(\sum_{\gamma\delta} t''_{\gamma\delta} g_{\gamma\delta}\Big) = (\mu'\mu'')^* ,$$

which proves our assertion.

d) This follows directly from c).

Remark. $\mathfrak{M}(S)$ can be written as a union of disjoint subsemigroups $\mathfrak{M} = \bigcup_{\varepsilon} \mathfrak{K}(\varepsilon)$, where ε runs through all idempotents $\in \mathfrak{M}(S)$. Every semigroup $\mathfrak{K}(\varepsilon)$ contains a unique idempotent and for any two elements $\nu, \mu \in \mathfrak{K}(\varepsilon)$ we have $\mu\nu = \varepsilon$. Further $\mathfrak{K}(\varepsilon_1) \cdot \mathfrak{K}(\varepsilon_2) = \varepsilon_1 \varepsilon_2 \in \mathfrak{K}(\varepsilon_1 \varepsilon_2)$ and also – since $\mathfrak{M}^2 = [\bigcup_{\varepsilon} \mathfrak{K}(\varepsilon)]^2 \subset \mathfrak{N}$ and $\mathfrak{N}^2 = \mathfrak{N}$ – we have $[\mathfrak{M}(S)]^2 = \mathfrak{N}$.[9])

While the structure of $\mathfrak{N}$ is sufficiently described by Theorem 3,6, the decomposition $\mathfrak{M} = \bigcup_{\varepsilon} \mathfrak{K}(\varepsilon)$ does not enable a sufficiently clear insight into the structure of $\mathfrak{M}$. This is due also to the fact that the sets $\mathfrak{K}(\varepsilon)$ need not be isomorphic. This will be shown on the Example 10,1 below.

10,2. We shall try to find all elements $\in \mathfrak{M}(S)$ belonging to the idempotent ε with $\varepsilon^* = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k g_{ik}$, $0 \leqq \xi_i, \eta_k \leqq 1$, $\sum_{i=1}^{s} \xi_i = \sum_{k=1}^{r} \eta_k = 1$.

Write $\mu^* = \sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik} g_{ik}$, $\sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik} = 1$, $t_{ik} \geqq 0$. Since the square of any element is an idempotent, there must hold:

$$\sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik} g_{ik} \sum_{j=1}^{s} \sum_{l=1}^{r} t_{jl} g_{jl} = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k g_{ik} ,$$

$$\sum_{ikjl} t_{ik} t_{jl} g_{il} = \sum_{il} \xi_i \eta_l g_{il} ,$$

$$\sum_{k=1}^{r} \sum_{j=1}^{s} t_{ik} t_{jl} = \xi_i \eta_l \quad (1 \leqq i \leqq s,\ 1 \leqq l \leqq r) .$$

The summation through all i gives

$$\Big(\sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik}\Big) \sum_{j=1}^{s} t_{jl} = \eta_l \sum_{i=1}^{s} \xi_i ,$$

i.e.

$$\sum_{j=1}^{s} t_{jl} = \eta_l \quad (l = 1, 2, \dots, r) . \tag{20}$$

Analogously by summing through l we get

$$\sum_{k=1}^{r} t_{ik} = \xi_i \quad (i = 1, 2, \dots, s) . \tag{21}$$

The relations (20) and (21) give necessary conditions for the validity of $\mu^2 = \varepsilon$. Conversely, if t_{ik} satisfy (20) and (21) (and $0 \leqq t_{ik} \leqq 1$) it is easily seen that $\mu^2 = \varepsilon$ holds. We have proved:

Theorem 10,2. *The measure μ with $\mu^* = \sum_{i=1}^{s} \sum_{k=1}^{r} t_{ik} g_{ik}$ belongs to the idempotent ε with $\varepsilon^* = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k g_{ik}$ if and only if the (linear) conditions* (20) *and* (21) *hold.*

[9]) The semigroup $\mathfrak{M}(S)$ is a totally non-commutative semigroup in the sense introduced in the paper [21].

Remark. With fixed chosen $\xi_1, \dots, \xi_s, \eta_1, \dots, \eta_r$ the relations (20) and (21) give $r + s$ equations for rs "unknowns" t_{ik}:

$$(22)\qquad \begin{aligned} t_{11} + t_{12} + \dots + t_{1r} &= \xi_1 ,\\ &\vdots\\ t_{s1} + t_{s2} + \dots + t_{sr} &= \xi_s ,\\ t_{11} + t_{21} + \dots + t_{s1} &= \eta_1 ,\\ &\vdots\\ t_{1r} + t_{2r} + \dots + t_{sr} &= \eta_r .\end{aligned}$$

These equations are not independent since f.i. the sum of the first s equations is identical with the sum of the remaining r equations. But if we drop f.i. the last equation the linear forms (in t_{ik}) on the left hand sides of the remaining equations are linearly independent. For if there existed real numbers $\lambda_1, \dots, \lambda_s, \lambda_{s+1}, \dots, \lambda_{s+r-1}$ such that

$$\sum_{\alpha=1}^{s} \lambda_\alpha(t_{\alpha 1} + \dots + t_{\alpha r}) + \sum_{\beta=1}^{r-1} \lambda_{s+\beta}(t_{1\beta} + \dots + t_{s\beta}) = 0$$

we would first have $\lambda_1 = \lambda_2 = \dots = \lambda_s = 0$, since $t_{1r}, t_{2r}, \dots, t_{sr}$ do not occur in the remaining forms. Next a relation of the form

$$\lambda_{s+1}(t_{11} + \dots + t_{s1}) + \dots + \lambda_{s+r-1}(t_{1,r-1} + \dots + t_{s,r-1}) = 0$$

is possible only if $\lambda_{s+1} = \dots = \lambda_{s+r-1} = 0$ since every variable t_{ik} occurs exactly once. Hence the rank of the system (22) is $r + s - 1$. a) If $r = 1$, the system (22) is of the form $t_{11} = \xi_1, \dots, t_{s1} = \xi_s$, $t_{11} + t_{12} + \dots + t_{s1} = \eta_1$, and it has a unique solution for $t_{11}, \dots, t_{s1}$. Analogously for the case $s = 1$. b) Hence we may suppose that $s > 1, r > 1$. Then $rs - (r + s - 1) = (r - 1)(s - 1) > 0$. Now we may calculate from (22) some $r + s - 1$ "unknowns" t_{ik} as polynomial of the remaining quantities t_{ik}. Unfortunately this does not yet solve our problem. To find all measures μ belonging to ε it is necessary to find those solutions for which $0 \leqq t_{ik} \leqq 1$. One such solution always exists. It is namely sufficient to put $t_{ik} = \xi_i\eta_k$. We then have $0 \leqq \leqq t_{ik} = \xi_i\eta_k \leqq 1$ and the equations (22) are clearly satisfied. (Of course, from our point of view, this solution is trivial, since it leads to $\mu = \varepsilon$.)

10,3. Example 10,1. Let $S = \{g_{11}, g_{12}, g_{21}, g_{22}\}$ be the semigroup with the multiplication $g_{ik}g_{jl} = g_{il}$. This semigroup is isomorphic with the semigroup of Example 2,1. If a measure μ with $\mu^* = \sum_{i=1}^{2} \sum_{k=1}^{2} t_{ik}g_{ik}$ belongs to ε with $\varepsilon^* = \sum_{i=1}^{2} \sum_{k=1}^{2} \xi_i\eta_k g_{ik}$, there must hold

$$t_{11} + t_{12} = \xi_1 , \quad t_{21} + t_{22} = \xi_2 , \quad t_{11} + t_{21} = \eta_1 , \quad t_{12} + t_{22} = \eta_2 .$$

The "general" solution of this system is given by

$$\begin{pmatrix} t_{11}, & t_{12} \\ t_{21}, & t_{22} \end{pmatrix} = \begin{pmatrix} t_{11} & , \xi_1 - t_{11} \\ \eta_1 - t_{11}, & \xi_2 - \eta_1 + t_{11} \end{pmatrix}.$$

The condition $0 \leqq t_{ik} \leqq 1$ is satisfied if and only if t_{11} is such that $\max(0, \eta_1 - \xi_2) \leqq \leqq t_{11} \leqq \min(\xi_1, \eta_1)$. The corresponding measure μ is of the form

$$\mu^* = t_{11}g_{11} + (\xi_1 - t_{11})\,g_{12} + (\eta_1 - t_{11})\,g_{21} + (\xi_2 - \eta_1 + t_{11})\,g_{22}\,.$$

To prove our statement concerning the semigroup $\mathfrak{K}(\varepsilon)$ mentioned above choose first f.i. $\varepsilon_1^* = g_{11}$ (i.e. $\xi_1 = \eta_1 = 1, \xi_2 = \eta_2 = 0$). We then have necessarily $t_{11} = 1$, $t_{12} = t_{21} = t_{22} = 0$. Hence the semigroup $\mathfrak{K}(\varepsilon_1)$ corresponding to ε_1 is the one point group $\{g_{11}\}$. Choose next f.i. $\varepsilon_2^* = \frac{1}{4}(g_{11} + g_{12} + g_{21} + g_{22})$, i.e. $\xi_1 = \eta_1 = \xi_2 = = \xi_2 = \frac{1}{2}$. The measures μ belonging to this idempotent are given by the formula

$$\mu^* = t_{11}(g_{11} + g_{22}) + (\tfrac{1}{2} - t_{11})\,(g_{12} + g_{21})\,,$$

where t_{11} runs through all real numbers satisfying $0 \leqq t_{11} \leqq \frac{1}{2}$. Hence $\mathfrak{K}(\varepsilon_2)$ is an infinite semigroup which is clearly not isomorphic with $\mathfrak{K}(\varepsilon_1)$.

10,4. It is possible to find also a – rather trivial – representation of $\mathfrak{M}(S)$ by means of some "matrices" of real numbers. Consider the set of all $s \times r$ "matrices" $\mathbf{M}$ of the form

$$\mathbf{M} = \begin{pmatrix} t_{11}, & \dots, & t_{1r} \\ \vdots & & \\ t_{s1}, & \dots, & t_{sr} \end{pmatrix}, \tag{23}$$

whose elements are non-negative numbers. For two $s \times r$ "matrices" $\mathbf{M}_1 = (t'_{ik})$ and $\mathbf{M}_2 = (t''_{ik})$ we define the product by

$$\mathbf{M}_1 \odot \mathbf{M}_2 = (t_{ik})\,, \tag{24}$$

where

$$t_{ik} = (\sum_{l=1}^{r} t'_{il})\,(\sum_{j=1}^{s} t''_{jk})\,. \tag{25}$$

This is an associative operation, so that the set of these "matrices" forms a semigroup. If we restrict ourselves to the matrices of the form (23) in which $\sum_{ik} t_{ik} = 1$, we get again a semigroup, since the relations $\sum_{il} t'_{il} = 1$, $\sum_{jk} t''_{jk} = 1$ in (25) imply $\sum_{i,k} t_{ik} = 1$.

Let now be $S = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} \{g_{ik}\}$. To every μ with $\mu = \sum_{ik} t_{ik}g_{ik}$, $0 \leqq t_{ik} \leqq 1$ let us assign the "matrix"

$$\mu \to \mathbf{M}(\mu) = \begin{pmatrix} t_{11}, & \dots, & t_{1r} \\ \vdots & & \\ t_{s1}, & \dots, & t_{sr} \end{pmatrix}.$$

If $(\mu')^* = \sum_{ik} t'_{ik}g_{ik}$, $(\mu'')^* = \sum_{jl} t''_{jl}g_{jl}$, we have $(\mu'\mu'')^* = \sum_{il} t_{il}g_{il}$.[10]) Hence if $\mu' \to \to \mathbf{M}(\mu')$, $\mu'' \to \mathbf{M}(\mu'')$, we have clearly $\mu'\mu'' \to \mathbf{M}(\mu'\mu'') = \mathbf{M}(\mu') \odot \mathbf{M}(\mu'')$. Since this correspondence is a one-to-one we have proved:

[10]) Explicitly: $(\mu'\mu'')^* = \sum_i \sum_k t'_{ik}\, g_{ik} \cdot \sum_j \sum_l t''_{jl} g_{jl} = \sum_{il} (\sum_k t'_{ik})\,(\sum_j t''_{jl})\, g_{il} = \sum_{il} t_{il} g_{il}$.

If $\mathfrak{T}$ *is the set of all "matrices" of the form* (23) *in which* $\sum_{ik} t_{ik} = 1, 0 \leqq t_{ik} \leqq 1,$ *and we define in* $\mathfrak{T}$ *a multiplication by the relations* (24) *and* (25), *then* $\mathfrak{M}(S)$ *is isomorphic to* $\mathfrak{T}$.

11. MAXIMAL IDEMPOTENTS OF $\mathfrak{M}(S)$

As a counterpart to the investigations of section 3 we prove some results concerning maximal idempotents.

Definition. An idempotent ε of any semigroup T is said to be a maximal idempotent of T if $\varepsilon = \varepsilon\mu = \mu\varepsilon$ and μ an idempotent implies $\mu = \varepsilon$.

The following theorem holds for any finite semigroup.

Theorem 11,1. *Let* S *be a finite semigroup and* ε *a maximal idempotent* $\in \mathfrak{M}(S)$. *Then* $C(\varepsilon)$ *does not contain a proper subsemigroup having the same idempotents as* $C(\varepsilon)$.

Proof. Let be $C(\varepsilon) = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G'_{ik}$ the decomposition of $C(\varepsilon)$ into the union of groups and $\varepsilon^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i\eta_k[G'_{ik}]$. Suppose for an indirect proof that there is a subsemigroup $P \subsetneqq C(\varepsilon)$, where P contains the same idempotents as $C(\varepsilon)$, hence $P = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G''_{ik}$, where $G''_{ik} \subsetneqq G'_{ik}$. Construct the measure μ, $\mu^* = \sum_{j=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_j\eta_l[G''_{jl}]$ (with the same ξ_j, η_l). Clearly $\varepsilon \neq \mu$. Further

$$\varepsilon^*\mu^* = \sum_{k=1}^{\varrho} \eta_k \sum_{j=1}^{\sigma} \xi_j \sum_{i=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_i\eta_l[G'_{ik}]\,[G''_{jl}] = \sum_{i=1}^{\sigma} \sum_{l=1}^{\varrho} \xi_i\eta_l[G'_{il}] = \varepsilon^*$$

and analogously $\mu^*\varepsilon^* = \varepsilon^*$. Hence ε is not a maximal idempotent, contrary to the assumption.

In the case of a simple semigroup without zero the converse statement is also true:

Theorem 11,2. *Let* S *be a simple semigroup,* $H \subset S$ *and suppose that* H *does not contain a proper subsemigroup containing the same idempotents as* H. *Let* ε *be an idempotent with* $C(\varepsilon) = H$. *Then* ε *is a maximal idempotent* $\in \mathfrak{M}(S)$.

Proof. Write in our usual notations $S = \bigcup_{i=1}^{s} R_i = \bigcup_{k=1}^{r} L_k = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} G_{ik}$. Suppose that μ is an idempotent for which $\varepsilon = \mu\varepsilon = \varepsilon\mu$ holds.

$C(\mu)$ is a simple semigroup. Without loss of generality let $R_1, R_2, \ldots, R_\sigma$ ($L_1, L_2, \ldots L_\varrho$) be exactly all minimal right (left) ideals of S having a non-empty intersection with $C(\mu)$. We then have

$$C(\mu) \subset \{R_1 \cup \ldots \cup R_\sigma\} \cap \{L_1 \cup \ldots \cup L_\varrho\} = H_1 .$$

By Theorem 1,2 of the paper [18] $C(\mu)$ contains exactly all idempotents $\in H_1$. The relations

$$C(\varepsilon) = C(\varepsilon)\, C(\mu) \subset C(\varepsilon)\, \{L_1 \cup \ldots \cup L_\varrho\} \subset L_1 \cup \ldots \cup L_\varrho$$

and

$$C(\varepsilon) = C(\mu)\, C(\varepsilon) \subset \{R_1 \cup \ldots \cup R_\sigma\}\, C(\varepsilon) \subset R_1 \cup \ldots \cup R_\sigma$$

imply

$$C(\varepsilon) \subset \{R_1 \cup \ldots \cup R_\sigma\} \cap \{L_1 \cup \ldots \cup L_\sigma\} = H_1 .$$

Since $C(\varepsilon)$ is a simple semigroup, it necessarily contains an idempotent. Since further all idempotents $\in H_1$ are contained in $C(\mu)$ we have $C(\mu) \cap C(\varepsilon) \neq \emptyset$. Let be $a \in C(\varepsilon) \cap C(\mu)$. Then

$$C(\varepsilon) = C(\varepsilon)\, C(\mu) = C(\mu)\, C(\varepsilon)\, C(\mu) \supset C(\mu)\,.\, a\,.\, C(\mu) = C(\mu)\,.$$

Hence $C(\varepsilon) \supset C(\mu)$. Therefore $C(\varepsilon)$ contains all idempotents $\in H_1$. Now since $C(\varepsilon) = H$ does not contain a proper subsemigroup containing the same idempotents as H, we have necessarily $C(\varepsilon) = C(\mu)$.

If $C(\varepsilon) = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G'_{ik}$, we may write $\varepsilon^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [G'_{ik}]$, $\mu^* = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi'_i \eta'_k [G'_{ik}]$, where $\xi_i, \eta_k, \xi'_i, \eta'_k$ are positive numbers satisfying $\sum_i \xi_i = \sum_k \eta_k = \sum_i \xi'_i = \sum_k \eta'_k = 1$. By supposition we have

$$\sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [G'_{ik}] = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [G'_{ik}] \cdot \sum_{j=1}^{\sigma} \sum_{l=1}^{\varrho} \xi'_j \eta'_l [G'_{jl}] =$$
$$= \sum_{j=1}^{\sigma} \sum_{l=1}^{\varrho} \xi'_j \eta'_l [G'_{jl}] \cdot \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [G'_{ik}]\,.$$

Hence

$$\sum_i \sum_k \xi_i \eta_k [G'_{ik}] = \sum_i \sum_k \xi_i \eta'_k [G'_{ik}] = \sum_i \sum_k \xi'_i \eta_k [G'_{ik}]\,.$$

Therefore $\xi_i \eta_k = \xi_i \eta'_k = \xi'_i \eta_k$, i. e. $\xi_i = \xi'_i$, $\eta_k = \eta'_k$. The relation $\varepsilon = \mu\varepsilon = \varepsilon\mu$ is satisfied if and only if $\mu = \varepsilon$, hence ε is a maximal idempotent $\in \mathfrak{M}(S)$, q. e. d.

Theorems 11,1 and 11,2 imply:

Corollary 11,1. *Let S be a simple semigroup. An idempotent $\varepsilon \in \mathfrak{M}(S)$ is a maximal idempotent $\in \mathfrak{M}(S)$ if and only if $C(\varepsilon)$ does not contain a proper subsemigroup containing the same idempotents as $C(\varepsilon)$.*

In particular:

Corollary 11,2. *Let S be a simple semigroup in which the product of two idempotents is an idempotent. Then ε is a maximal idempotent $\in \mathfrak{M}(S)$ if and only if $C(\varepsilon)$ is an idempotent semigroup.*

Example 11,1. We show on a simple example that Theorem 11,2 and Corollary 11,1 need not hold if S is not simple. Consider the semigroup $S = \{a_0, a_1, a_2\}$ with the multiplication table

	a_0	a_1	a_2
a_0	a_0	a_0	a_0
a_1	a_0	a_1	a_2
a_2	a_0	a_2	a_1

The one point group $\{a_0\}$ does not contain a proper subsemigroup, while ε with $\varepsilon^* = a_0$ is not a maximal idempotent $\in \mathfrak{M}(S)$ since we have $\varepsilon^* = \varepsilon^* a_1 = a_1 \varepsilon^*$.

By an analogous argument as in Theorem 11,2 (and using Theorem 2,2 of [18]) we may prove also

Theorem 11,3. *Let S be a simple semigroup with zero z in which $S^2 \neq z$. Let further be $H \subset S$ a subsemigroup not containing the zero element and H not containing a proper subsemigroup with the same idempotents as H. If ε is any idempotent $\in \mathfrak{M}(S)$ for which $C(\varepsilon) = H$ holds, then ε is a maximal idempotent $\in \mathfrak{M}(S)$.*

Remark. It follows from Theorem 11,1 that if S is a simple semigroup, ε a point mass at the idempotent $e_{ik} \in S$, then ε is a maximal idempotent $\in \mathfrak{M}(S)$. From Corollary 5,1 we know that the set of all elements belonging to this idempotent is a group. The question arises whether also in general the set of all elements belonging to any other maximal idempotent $\in \mathfrak{M}(S)$ is a group. Example 10,1 shows that this is not the case. For the idempotent ε_2 considered in this example is a maximal idempotent, but $\mathfrak{K}(\varepsilon_2)$ is clearly not a group, since ε_2 is at the same time a primitive idempotent and the set of regular elements belonging to ε_2 reduces to $\{\varepsilon_2\}$.

Let S be again a finite semigroup. Decompose S into the union of F-classes (see section 7,3). Denote (in accordance with 7,1) by H^*-classes those F-classes that contain idempotents and define $H_x^* \leqq H_y^*$ if $\langle x \rangle \subset \langle y \rangle$. A H_x^*-class is said to be a maximal H^*-class if there is no H^*-class that is strictly greater than H_x^*.

Theorem 11,4. *Let H^* be a maximal H^*-class of S and H a simple subsemigroup of S contained in H^*. Suppose that H does not contain a proper subsemigroup containing the same idempotents as H. If ε is an idempotent with $C(\varepsilon) = H$, then ε is a maximal idempotent $\in \mathfrak{M}(S)$.*

Proof. Let μ be any idempotent $\in \mathfrak{M}(S)$ with $\mu\varepsilon = \varepsilon\mu = \varepsilon$. We have to show that $\mu = \varepsilon$.

The set $C(\mu) = H_0$ is a simple subsemigroup contained in a H^*-class, say H_0^*. We first show that $H_0^* = H^*$. By supposition we have $HH_0 = H_0 H = H$. If $u_0 \in H_0 \subset H_0^*$, $u \in H \subset H^*$, Lemma 7,4 implies that the product $u_0 u$ is contained in such a F-class F_w that $F_w \leqq H_0^*$. But $u_0 u \in H \subset H^*$, hence $F_w = H^*$ and therefore $H^* \leqq H_0^*$. Since H^* is a maximal H^*-class we have $H^* = H_0^*$. Hence $C(\mu) = H_0 \subset H^*$.

Consider now the class H^*, adjoin a zero element O_{H^*}, and define in $S_1 = H^* \cup O_{H^*}$ a natural multiplication (see Lemma 7,3). Then S_1 is a simple semigroup with zero and $S_1^2 \neq O_{H^*}$ (since H^* contains idempotents). Now ε and μ may be considered as measures defined on S_1 satisfying $\mu\varepsilon = \varepsilon\mu = \varepsilon$. Since H considered as a subsemigroup of S_1 satisfies the suppositions of Theorem 11,3, the relation $\mu\varepsilon = \varepsilon\mu = \varepsilon$ implies $\mu = \varepsilon$. Hence ε is a maximal idempotent of S. This proves our theorem.

I am unable to prove the converse of Theorem 11,4 namely that the support of a maximal idempotent $\varepsilon \in \mathfrak{M}(S)$ is contained in a maximal H^*-class. It is possible that in general this *conjecture* does not hold but at this writing I cannot find an example to prove this.

However in the commutative case the conjecture holds.

If S is commutative, every H^*-class is a maximal group of S. The idempotents ε satisfying the condition mentioned in Theorem 11,1 are point masses at some idempotents of S. An idempotent e contained in a maximal H^*-class is a maximal idempotent of S. For if e were not a maximal idempotent $\in S$ there would exist an idempotent f with $ef = e$. We have $Sf = (Se)f = S(ef) = Se$. f is not contained in Se, since otherwise we would have $ef = f$, hence $e = f$. Therefore $Sf \supsetneqq Se$, a contradiction with the maximality of the H^*-class containing e. Conversely if e is an idempotent which is contained in a non-maximal H^*-class, say H_0^*, then there is a H^*-class, say H_{00}^*, such that $H_0^* \leqq H_{00}^*$. If f is the idempotent $\in H_{00}^*$, we have $Se \subsetneqq Sf$. Now f is clearly a unit element for every element $\in Se$, in particular $ef = e$. Hence e is not a maximal idempotent $\in S$. Summarily: The idempotents contained in maximal H^*-classes are exactly the maximal idempotents $\in S$.

Let now ε be a maximal idempotent $\in \mathfrak{M}(S)$. By Theorem 11,1 we have necessarily $\varepsilon^* = e$, where e is an idempotent $\in S$. If e were not a maximal idempotent $\in S$ there would exist an idempotent $f, f \neq e$, such that $ef = e$. If μ is the idempotent $\in \mathfrak{M}(S)$ with $\mu^* = f$ we would have $\varepsilon\mu = \varepsilon$, $\mu^* \neq e$, hence ε is not a maximal idempotent $\in \mathfrak{M}(S)$, contrary to the assumption.

This implies:

Theorem 11,5. *If S is commutative, the maximal idempotents $\in \mathfrak{M}(S)$ are the point masses at the maximal idempotents of S.*

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Резюме

О ВЕРОЯТНОСТНЫХ РАСПРЕДЕЛЕНИЯХ НА НЕКОММУТАТИВНЫХ ПОЛУГРУППАХ

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На протяжении всей работы S — конечная полугруппа. Мерой μ называем неотрицательную аддитивную множественную функцию, определенную на подмножествах S, для которой $\mu(S) = 1$. Носителем меры μ называется множество $C(\mu) = \{x \in S / \mu(x) \neq 0\}$. Обозначим символом $\mathfrak{M}(S)$ множество всех мер полугруппы S.

Пусть $\nu_1, \nu_2 \in \mathfrak{M}(S)$. Произведением $\nu_1 \nu_2$ будем называть меру, определенную соотношением $\nu_1 \nu_2(x) = \sum_{uv = x} \nu_1(u)\, \nu_2(v)$ $(x \in S)$.

Если $\mathfrak{A}(S)$ — полугрупповая алгебра полугруппы $S = \{x_1, \ldots, x_n\}$ над полем действительных чисел и $\mathfrak{F}(S) \subset \mathfrak{A}(S)$ — подмножество всех элементов вида $\sum_{i=1}^{n} t_i x_i$, где $t_i \geqq 0$, $\sum_{i=1}^{n} t_i = 1$, то $\nu(x) \in \mathfrak{M}(S) \leftrightarrow \sum_{i=1}^{n} \nu(x_i)\, x_i \in \mathfrak{F}(S)$ — изоморфизм. В дальнейшем отождествим $\mathfrak{M}(S)$ и $\mathfrak{F}(S)$ и будем меры писать в виде $\mu = \sum_{i=1}^{n} t_i x_i$. Кроме того, будем считать, что S вложено в $\mathfrak{M}(S)$ (как множество точечных масс).

Если ввести в $\mathfrak{M}(S)$ произведение мер и топологию естественным образом, то $\mathfrak{M}(S)$ превращается в бикомпактную (хаусдорфову) полугруппу. Целью работы является изучение строения полугруппы $\mathfrak{M}(S)$.

В разделе 1 приводятся вспомогательные результаты, касающиеся бикомпактных полугрупп, которые необходимы в дальнейшем.

В разделе 2 получены все идемпотенты $\in \mathfrak{M}(S)$. Если $\mu = \mu^2$, то $S(\mu)$ — вполне простая полугруппа. Пусть $C(\mu) = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} G_{ik}$ — разложение $C(\mu)$ в (дизъюнктные изоморфные) групповые компоненты. Тогда для $x, y \in G_{ik}$ имеет место $\mu(x) = \mu(y)$. Пусть, наоборот, $P = \bigcup_{i=1}^{s} \bigcup_{k=1}^{r} G_{ik}$ — любая вполне простая подполугруппа полугруппы S. Если $G_{ik} = \{g_{ik}^{(1)}, \ldots, g_{ik}^{(m)}\}$, то обозначим $[G_{ik}] = (1/m) \cdot (g_{ik}^{(1)} + \ldots + g_{ik}^{(m)})$ (мера Хаара группы G_{ik}). Пусть ξ_i, η_k — положительные числа, для которых $\sum_{i=1}^{s} \xi_i = \sum_{k=1}^{r} \eta_k = 1$. Тогда $\varepsilon = \sum_{i=1}^{s} \sum_{k=1}^{r} \xi_i \eta_k [G_{ik}]$ — идемпотент $\in \mathfrak{M}(S)$, и каждый идемпотент $\in \mathfrak{M}(S)$, носителем которого является P, получается таким образом. (В частности, любая вполне простая подполугруппа является носителем некоторой идемпотентной меры).

В разделе 3 характеризуются примитивные идемпотенты $\in \mathfrak{M}(S)$. Оказывается, что совокупность всех примитивных идемпотентов есть ядро $\mathfrak{N}$ полугруппы $\mathfrak{M}(S)$. Получено одно изоморфное представление ядра $\mathfrak{N}$.

Скажем, что $\mu \in \mathfrak{M}(S)$ принадлежит к идемпотенту ε, если ε — (единственный) идемпотент, лежащий в замыкании последовательности $\{\mu, \mu^2, \mu^3, \ldots\}$. Элемент $\mu \in \mathfrak{M}(S)$ называется регулярным, если он лежит в некоторой подгруппе из $\mathfrak{M}(S)$.

Пусть μ — регулярный элемент, принадлежащий к идемпотенту $\varepsilon = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} . \xi_i \eta_k [G'_{ik}]$. Пусть $H_1 = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G_{ik}$ — (единственная) максимальная вполне простая полугруппа, имеющая те же идемпотенты как $H = \bigcup_{i=1}^{\sigma} \bigcup_{k=1}^{\varrho} G'_{ik}$.

Рассмотрим разложение в дизьюнктные классы: $H_1 = \bigcup_{a \in H_1} HaH$ (*). Оказывается, что $C(\mu) = HaH$ с некоторым удобно выбранным $a \in H_1$. Обозначим $C(\mu) \cap G_{ik} = HaH \cap G_{ik} = G'_{ik} a G'_{ik} = T_{ik}$. Далее, если $T_{ik} = \{t_{ik}^{(1)}, \ldots, t_{ik}^{(p)}\}$, пусть $[T_{ik}] = (1/p)(t_{ik}^{(1)} + \ldots + t_{ik}^{(p)})$. Оказывается, что для данных ε и $C(\mu)$, μ однозначно определено, а именно $\mu = \sum_{i=1}^{\sigma} \sum_{k=1}^{\varrho} \xi_i \eta_k [T_{ik}]$. Этот результат доказывается в разделе 5 для вполне простых полугрупп (без нуля) и обобщается в разделах 6 и 7.

Доказаны тоже необходимые и достаточные условия для того, чтобы класс HbH из разложения (*) был носителем некоторой регулярной меры μ. В разделах 5—7 доказаны тоже теоремы, касающиеся ненеобходимо регулярных мер. В разделе 7 разбирается, кроме того, вопрос о нахождении всех вполне простых подполугрупп в общем случае.

В разделе 8 доказаны некоторые необходимые и достаточные условия для существования $\lim_{n=\infty} \mu^n$. Доказана тоже следующая теорема: Обозначим $\sigma_n = (1/n)(\mu + \mu^2 + \ldots + \mu^n)$. Тогда $\lim_{n=\infty} \sigma_n$ существует и равняется некоторому идемпотенту $\sigma \in \mathfrak{M}(S)$. Если P — полугруппа, порожденная множеством $C(\mu)$, и I — минимальный двусторонний идеал из P, то $C(\sigma) = I$.

В разделах 9 и 10 исследуется $\mathfrak{M}(S)$ для некоторых специальных типов полугрупп.

В разделе 11 исследуются максимальные идемпотентны $\in \mathfrak{M}(S)$.

Обобщение некоторых результатов на случай бикомпактного S будет предметом другой работы.