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ON THE SECOND COVARIANT DERIVATIVE OF A VECTOR FIELD*)

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We shall here introduce a geometrical signification of the operator $\nabla_X \nabla_Y K$, K being a fixed vector field. The proofs of the theorems are routine, and they are omitted.

1. Let B be a differentiable n-dimensional manifold with a linear connection Γ. Let $\tilde{\Gamma}$ be the affine connection canonically associated to Γ. K being a vector field on B, let $\tilde{\Gamma}_K$ be the affine connection associated to Γ and the tensor field ∇K; see [1, p. 74]. Suppose $T = 0$, T being the torsion tensor of the connection $\tilde{\Gamma}$.

Let A^n be the tangent affine space of B at a fixed point $b \in B$, and let $\gamma : (-1, 1) \to B$ be a differentiable curve on B through the point b; suppose, for example, $\gamma(0) = b$. Denote by $\gamma^* : (-1, 1) \to A^n$ (or $\gamma_K^* : (-1, 1) \to A^n$) the development of γ into A^n with respect to $\tilde{\Gamma}$ (or $\tilde{\Gamma}_K$ resp.).

Lemma. *There is a unique affine collineation $C_K : A^n \to A^n$ with the following property: γ being an arbitrary differentiable curve on B through the point b, we have $j_1(\gamma^*)(0) = j_1(C_K \gamma_K^*)(0)$; here, $j_s(F)(p)$ denotes the s-jet of the map F at the point p.*

Consider the tensor L_K of the type (1,2) given by $L_K(X, Y) = \nabla_Y \nabla_X K$. The geometrical significance of this tensor is given by the following

Theorem 1. *Let V_b be a fixed tangent vector of B at b, and let γ be any curve in B through b such that its development γ^* into A^n with respect to $\tilde{\Gamma}$ is tangent to V_b at the point b. Three cases are possible:*

(a) $L_K(V_b, V_b) = 0$, *and we have* $j_2(\gamma^*)(0) = j_2(C_K \gamma_K^*)(0)$.

(b) $L_K(V_b, V_b) = \alpha V_b$, *$\alpha$ a real number* $\neq 0$. *We have* $j_2(\gamma^*)(0) \neq j_2(C_K \gamma_K^*)(0)$, *but there are neighborhoods Ω, $\Omega' \subset (-1, 1)$ of 0 and a map $\mu : \Omega \to \Omega'$; $\mu(0) = 0$, $\mu'(0) \neq 0$; such that*

$$(*) \qquad j_2(\gamma^*|_\Omega) = j_2(C_K \gamma_K^* \mu)(0),$$

$\gamma^*|_\Omega$ being the restriction of γ^* on Ω.

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(c) *The vectors* $L_K(V_b, V_b) = V_b^*$ *and* V_b *are linearly independent. There is no map* μ *such that* $(*)$ *is valid. Let* $A^{n-1} \subset A^n$ *be any hyperplane such that its vector space does not contain the vectors* V_b *and* V_b^*, *and let us denote by* $\pi : A^n \to A^{n-1}$ *the projection of* A^n *onto* A^{n-1} *in the direction of* V_b^*. *We have* $j_2(\pi\gamma^*)(0) = j_2(\pi C_K \gamma_K^*)(0)$.

2. In this section, we present two theorems concerning the possible decomposition of the operator $L_K(X, Y)$.

Let U be a fixed vector field on B. Denote by M_U the set of vector fields K on B with the following property: $K \in M_U$ if and only if the vector fields U and $L_K(X, Y)$ are linearly dependent for any vector fields X, Y on B.

Theorem 2. *Let* $K_1, K_2 \in M_U$. *If* $L_{K_1}(K_2, V) = L_{K_2}(K_1, V)$ *or if* $\nabla_V U$ *and* U *are linearly dependent for each vector field* V *on* B, *then* $[K_1, K_2] \in M_U$.

Further, let T be a fixed tensor field of the type $(1,1)$ on B. Denote by N_T the set of all vector fields K on B with the following property: $K \in N_T$ if and only if the vector fields $T(V)$ and $L_K(V, V)$ are linearly dependent for each vector field V on B.

Theorem 3. *Let* $K_1, K_2 \in N_T$. *If* $L_{K_1}(K_2, V) = L_{K_2}(K_1, V)$ *or* $[(\nabla_U T)(V), T(V)] = 0$ *for any vector fields* U, V *on* B, *then* $[K_1, K_2] \in N_T$.

3. Finally, a result for compact Riemannian manifolds B *based on the well known integral formula*:

Theorem 4. *Let* g *be a Riemannian metric on* B *and* Γ *be the associated connection. Let* K *be a vector field on* B *such that* $L_K(V, V) = f(V, V) K$ *for each vector field* V *on* B, $f(V, W)$ *being a real-valued bilinear function. If* $f(V, V) \geqq 0$ *for each vector field* V *on* B *and* B *is compact, we have* $\nabla K = 0$. *Moreover, if* $f(V, V) = 0$ *implies* $V = 0$ *we have* $K = 0$.

Bibliography

[1] *K. Nomizu:* Lie groups and differential geometry, Publ. of Math. Soc. of Japan, 1956.

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Резюме

ОБ ВТОРОЙ КОВАРИАНТНОЙ ПРОИЗВОДНОЙ ВЕКТОРНОГО ПОЛЯ

АЛОИС ШВЕЦ (Alois Švec), Прага

Дается геометрическое значение оператора $\nabla_Y \nabla_X K$, где K – данное векторное поле.