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NON-HOLONOMIC CONNECTIONS ON VECTOR BUNDLES, II

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4. HIGHER ORDER CONNECTIONS AND PSEUDO-CONNECTIONS ON VECTOR BUNDLES

In this paragraph we shall deal with sequences of pseudo-connections or connections of subsequent orders. These sequences may be finite or infinite. All the results are formulated in such a way that they apply to infinite sequences, nevertheless it will be always evident how to modify these formulations to obtain the corresponding results for finite sequences. These evident formulations will not be given explicitly.

Definition 4.1. Let $q \geqq 1$ be an arbitrary integer. *A non-holonomic* (or *semi-holonomic*) *pseudo-conection of q-th order* on the vector bundle E is a bundle isomorphism

(4.1) $$NH^q : \tilde{S}^q(E) \to \tilde{T}^q(E),$$

(or accordingly

(4.2) $$SH^q : \bar{S}^q(E) \to \bar{T}^q(E)).$$

One could define quite similarly the holonomic pseudo-connections on E. However we shall not need it in the next as we are not concerned with the holonomic case in this paper.

We begin with the study of the relation between non-holonomic pseudo-connections and (first order) pseudo-connections on the non-holonomic jet and tensor prolongations.

Suppose we are given a sequence $\{\tilde{H}^q_S\}$ $(q \geqq 1)$ of pseudo-connections

(4.3) $$\tilde{H}^q_S : \tilde{S}^q(E) = S^1(\tilde{S}^{q-1}(E)) \to T^1(\tilde{S}^{q-1}(E))$$

on $\tilde{S}^{q-1}(E)$ $(q = 1, 2, \ldots)$, or a sequence $\{\tilde{H}^q_T\}$ $(q \geqq 1)$ of pseudo-connections

(4.4) $$\tilde{H}^q_T : S^1(\tilde{T}^{q-1}(E)) \to T^1(\tilde{T}^{q-1}(E)) = \tilde{T}^q(E),$$

and consider the sequence of "diagram sequences"

$$(4.5)\qquad \tilde{S}^q(E) \xrightarrow{\tilde{H}^q_S} \dots \longrightarrow \tilde{T}^k(\tilde{S}^{q-k}(E)) \xrightarrow{\tilde{T}^k(\tilde{H}^{q-k}_S)} \tilde{T}^{k+1}(\tilde{S}^{q-k-1}(E)) \to \dots$$
$$\dots \xrightarrow{\tilde{T}^{q-1}(\tilde{H}^1_S)} \tilde{T}^q(E)\,; \quad q = 1, 2, \dots,$$

or

$$(4.6)\qquad \tilde{S}^q(E) \xrightarrow{\tilde{S}^{q-1}(\tilde{H}^1_T)} \dots \longrightarrow \tilde{S}^{k+1}(\tilde{T}^{q-k-1}(E)) \xrightarrow{\tilde{S}^k(\tilde{H}^{q-k}_T)} \tilde{S}^k(\tilde{T}^{q-k}(E)) \to \dots$$
$$\dots \xrightarrow{\tilde{H}^q_T} \tilde{T}^q(E)\,; \quad q = 1, 2, \dots$$

Both define uniquely a sequence of non-holonomic pseudo-connections on E. The correspondence is in both cases one-to-one. In fact, (4.5) can be written recurrently as

$$(4.7)\qquad NH^1 = \tilde{H}^1_S\,; \quad NH^q = T^1(NH^{q-1})\,\tilde{H}^q_S\,,$$

which can be inverted into

$$(4.8)\qquad \tilde{H}^1_S = NH^1\,; \quad \tilde{H}^q_S = T^1(NH^{q-1})\,NH^q\,.$$

Analogously (4.6) gives rise to the recurrent formulae

$$(4.9)\qquad NH^1 = \tilde{H}^1_T\,; \quad NH^q = \tilde{H}^q_T S^1(NH^{q-1})\,,$$

or

$$(4.10)\qquad \tilde{H}^1_T = NH^1\,; \quad \tilde{H}^q_T = NH^q S^1(NH^{q-1})^{-1}\,.$$

Thus we have got a "one-to-one-to-one" correspondence between sequences of first order pseudo-connections on the non-holonomic jet prolongations, first order pseudo-connections on the non-holonomic tensor prolongations and non-holonomic pseudo-connections (of higher orders) on E. We express this by saying that the corresponding sequences are *associated* and we write briefly $\{\tilde{H}^q_S\} \sim \{NH^q\} \sim \{\tilde{H}^q_T\}$.

Definition 4.2. A sequence $\{NH^q\}$ $(q \geqq 1)$ of bundle morphisms (4.1) is called a *sequence of non-holonomic connections* if NH^1 is a connection on E and each NH^q $(q > 1)$ is a relative connection with respect to NH^{q-1}.

Note that we do not define a non-holonomic connection of a given higher order itself but only sequences of non-holonomic connections. Nevertheless we shall say sometimes that $\{NH^q\}$ "consists of connections". Each sequence of non-holonomic connections is clearly also a sequence of non-holonomic pseudo-connections.

Theorem 4.1. *If $\{\tilde{H}^q_S\} \sim \{NH^q\} \sim \{H^q_T\}$ and one of the sequences consists of connections, then the same is true about the other two.*

Proof. First note that necessarily $\tilde{H}^1_S = NH^1 = \tilde{H}^1_T$ are connections on E. Now if $\{NH^q\}$ is a sequence of non-holonomic connections, each NH^q being therefore an isomorphism, we can apply by virtue of (4.8) and (4.10) Proposition 3.7. If $\{\tilde{H}^q_S\}$ or $\{\tilde{H}^q_T\}$ consists of connections we apply analogously Proposition 3.6.

Now let h be a connection on the cotangent bundle $T(M)^*$ of M and H a connection on E. They induce a connection on each $\tilde{T}^{q-1}(E)$ since this bundle is obtained from $T(M)^*$ and E by "tensor product and direct sum constructions". Since $\tilde{T}^{q-1}(E) = \tilde{T}^{q-2}(E) \oplus \tilde{T}^{q-2}(E) \otimes T(M)^*$, we get a canonical sequence $\{\tilde{H}^q_T\}$ of connections generated by the connections H and h, which can be defined recurrently as $\tilde{H}^1_T = H$ and

$$(4.11) \qquad \tilde{H}^q_T = \tilde{H}^{q-1}_T (\oplus) [\tilde{H}^{q-1}_T (\otimes) h] = T^1(j^1_T)\, \tilde{H}^{q-1}_T\, S^1(\Pi_T) + T^1(j^{1*}_T)\, [\tilde{H}^{q-1}_T (\otimes) h]\, S^1(\Pi^*_T)\,.$$

Combining this with Theorem 4.1 we get

Theorem 4.2. *Each connection on E together with a connection on the tangent bundle $T(M)$ canonically generate a sequence of non-holonomic connections on E and a sequence of connections on the non-holonomic jet (or accordingly tensor) prolongations of E.*

On the other hand if one tries to associate sequences of semi-holonomic pseudo-connections on E with sequences of first order pseudo-connections on semi-holonomic jet and tensor prolongations in a similar way, one finds the situation much more complicated than in the non-holonomic case. In fact, it seems to be necessary to restrict the attention to only *regular* sequences which will be defined below. Let us first introduce the notion of a sequence of semi-holonomic connections on E.

Definition 4.3. A sequence $\{SH^q\}$ $(q \geqq 1)$ of bundle morphisms (4.2) is called a *sequence of semi-holonomic connections* on E if SH^1 is a connection on E and each SH^q $(q > 1)$ satisfies

$$(4.12) \qquad \overline{\Pi}^q_T\, SH^q = SH^{q-1}\overline{\Pi}^q_S$$

and

$$(4.13) \qquad SH^q|_{\mathrm{Ker}\overline{\Pi}^q{}_S} = \bar{I}^q_0\,.$$

Lemma 4.1. *Each sequence of semi-holonomic connections on E is automatically a sequence of semi-holonomic pseudo-connections.*

Proof. We need to show that each SH^q is an isomorphism. Since this is clearly true about SH^1, we can proceed by induction. Thus suppose SH^{q-1} is an isomorphism and let $SH^qX = 0$. From (4.12) we get $SH^{q-1}\overline{\Pi}^q_S X = 0 \Rightarrow X \in \mathrm{Ker}\, \overline{\Pi}^q_S$ and thus from (4.13) we conclude $X = 0$.

Note that we again do not define a semi-holonomic connection of a given higher order itself, although we shall also say sometimes that $\{SH^q\}$ "consists of connections"

Nevertheless, if an SH^q appears in a sequence of semi-holonomic connections, then by virtue of (4.12), (4.13) we can write

$$SH^q = SH^{q-1}\overline{\Pi}^q_S + ZH^q \tag{4.14}$$

where $ZH^q : \bar{S}^q(E) \to E \otimes (\overset{q}{\otimes} T(M)^*)$ and $ZH^q|_{\mathrm{Ker}\overline{\Pi}^q_S} = \bar{I}^q_0$. Thus ZH^q gives rise to a splitting of the exact sequence

$$0 \to E \otimes (\overset{q}{\otimes} T(M)^*) \to \bar{S}^q(E) \to \bar{S}^{q-1}(E) \to 0 \tag{4.15}$$

and hence ZH^q corresponds to a semi-holonomic connection $\bar{C}^q$ of q-th order on E in the sense of [3].

In spite of this we shall not separate the "superfluous" part $SH^{q-1}\overline{\Pi}^q_S$ from SH^q and thus, as a matter of fact, consider only sequences of semi-holonomic connections as introduced in Definition 4.3. The main reason for this is to retain the possibility of comparing semi-holonomic connections with non-holonomic ones, where there is no decomposition analogous to (4.14), since $\mathrm{Ker}\,\tilde{\Pi}^q_S$ is not canonically isomorphic to a "tensor bundle".

Let $\{\overline{H}^q_S\}$ $(q \geqq 1)$ be a sequence of pseudo-connections

$$\overline{H}^q_S : S^1(\bar{S}^{q-1}(E)) \to T^1(\bar{S}^{q-1}(E)) \tag{4.16}$$

on $\bar{S}^{q-1}(E)$, $(q = 1, 2, \ldots)$. We would like to associate with the sequence $\{\overline{H}^q_S\}$ a sequence $\{SH^q\}$ of semi-holonomic pseudo-connections in a similar way as (4.5) does it in the non-holonomic case. It is quite natural to define $SH^1 = \overline{H}^1_S$. Suppose SH^k are given for $k = 1. \ldots, q - 1$ and consider the "semi-holonomic analogue" of (4.7)

$$\tilde{S}H^q = T^1(SH^{q-1})\,\overline{H}^q_S\,.$$

This relation defines $\tilde{S}H^q : S^1(\bar{S}^{q-1}(E)) \to T^1(\bar{T}^{q-1}(E))$ and it may be quite natural to connect with $\tilde{S}H^q$ a semi-holonomic pseudo-connection SH^q subject to $\bar{\imath}^{q'}_T SH^q = \tilde{S}H^q \bar{\imath}^{q'}_S$. These heuristic considerations (c.f. also the "local" formula (2.61)) suggest the following

Definition 4.4. We say that the sequence $\{\overline{H}^q_S\}$, $(q \geqq 1)$ of pseudo-connections (4.16) is *associated* with the sequence $\{SH^q\}$, $(q \geqq 1)$ of semi-holonomic pseudo-connections (4.2) (and conversely) if $SH^1 = \overline{H}^1_S$ and

$$\bar{\imath}^{q'}_T SH^q = T^1(SH^{q-1})\,\overline{H}^q_S \bar{\imath}^{q'}_S \tag{4.17}$$

for each $q > 1$.

As already mentioned, in order to get — in some way — reasonable results, we are compelled to restrict ourselves to regular sequences. They shall be defined next.

Definition 4.5. Call a sequence $\{SH^q\}$ of semi-holonomic pseudo-connections (4.2) *regular* if there exists a sequence $\{A^{q-1}\}, (q \geqq 1)$ of automorphisms $A^{q-1} : \bar{S}^{q-1}(E) \to \bar{S}^{q-1}(E)$ such that

$$\bar{\Pi}^q_T SH^q = SH^{q-1} A^{q-1} \bar{\Pi}^q_S \,. \tag{4.18}$$

Here we have put for convenience $SH^0 = 1_E$.

Note that (4.18) is equivalent to

$$\bar{\Pi}^q_S (SH^q)^{-1} = (SH^{q-1})^{-1} (B^{q-1})^{-1} \bar{\Pi}^q_T \,, \tag{4.19}$$

where $\{B^{q-1}\}, (q \geqq 1)$ is given by

$$SH^{q-1} A^{q-1} = B^{q-1} SH^{q-1} \,. \tag{4.20}$$

If $\{SH^q\}$ is a sequence of semi-holonomic connections, then it is clearly regular with A^{q-1} and B^{q-1} being identities.

Lemma 4.2. *Let $\{SH^q\}$ be a regular sequence of semi-holonomic pseudo-connections. Then for each $q \geqq 1$*

$$(Q^q)^{-1} = \bar{\Pi}^{q*}_T \bar{I}^q_0 (SH^q)^{-1} \bar{j}^{q*}_T \tag{4.21}$$

is an automorphism of $E \otimes (\overset{q}{\otimes} T(M)^*)$ satisfying

$$(\bar{I}^q_0)^{-1} \bar{j}^q_T (Q^q)^{-1} = (SH^q)^{-1} \bar{j}^q_T \,, \tag{4.22}$$

and if $\{SH^q\}$ is a sequence of semi-holonomic connections, then $\{Q^q\}$ is a sequence of identities.

Proof. It suffices to prove that (4.21) is an injection. Thus let $(Q^q)^{-1} X = 0$. Then $\bar{\Pi}^{q*}_T \bar{I}^q_0 (SH^q)^{-1} \bar{j}^{q*}_T X = 0$ and also $\bar{\Pi}^q_T \bar{I}^q_0 (SH^q)^{-1} \bar{j}^{q*}_T X = 0$ since $\operatorname{Im} \bar{I}^q_0 = \operatorname{Ker} \bar{\Pi}^q_T$. Thus $\bar{I}^q_0 (SH^q)^{-1} \bar{j}^{q*}_T X = 0$ and since the mappings here are all injections, we conclude that $X = 0$ and consequently $(Q^q)^{-1}$ is an injection. On the other hand $(\bar{I}^q_0)^{-1} \bar{j}^{q*}_T (Q^q)^{-1} = (\bar{I}^q_0)^{-1} (\bar{j}^{q*}_T \bar{\Pi}^{q*}_T + \bar{j}^q_T \bar{\Pi}^q_T \, \bar{I}^q_0 (SH^q)^{-1} \bar{j}^{q*}_T = (SH^q)^{-1} \bar{j}^{q*}_T$ and this proves (4.22). If $\{SH^q\}$ is a sequence of connections, then $(SH^q)^{-1} \bar{j}^{q*}_T = (\bar{I}^q_0)^{-1} \bar{j}^{q*}_T$ and thus $(Q^q)^{-1}$ is the identity.

Definition 4.6. Call a sequence $\{\bar{H}^q_S\}$ of pseudo-connections (4.16) *regular* if there exists a sequence $\{A^{q-1}\}$ of automorphisms $A^{q-1} : \bar{S}^{q-1}(E) \to \bar{S}^{q-1}(E)$ such that

$$\Pi_T \bar{H}^q_S \bar{i}^{q'}_S = A^{q-1} \bar{\Pi}^q_S \tag{4.23}$$

and

$$T^1(A^{q-1} \bar{\Pi}^q_S) \, \bar{H}^{q+1}_S \bar{i}^{q+1'}_S = \bar{H}^q_S \bar{i}^{q'}_S A^q \bar{\Pi}^{q+1}_S \tag{4.24}$$

hold for each $q \geqq 1$.

Note that, in distinction to the semi-holonomic connections of higher orders, not each sequence $\{\bar{H}_S^q\}$ of connections is necessarily a regular sequence. However, one easily observes using (2.57) that if $\{\bar{H}_S^q\}$ consists of connections, then the only condition for regularity to be satisfied is

$$T^1(\bar{\Pi}_S^q)\,\bar{H}_S^{q+1}\bar{i}_S^{q+1'} = \bar{H}_S^q S^1(\bar{\Pi}_S^q)\,\bar{i}_S^{q+1'} \tag{4.25}$$

for each $q \geqq 1$. In particular a sequence $\{\bar{H}_S^q\}$ of connections is regular if each pair $\bar{H}_S^{q+1}$, $\bar{H}_S^q$ $(q \geqq 1)$ induces the same R-connection with respect to the projection $\bar{\Pi}_S^q$.

Theorem 4.3. *Let* $\{\bar{H}_S^q\}$ *be a regular sequence of pseudo-connections* (4.16). *Then there exists exactly one sequence* $\{SH^q\}$ *of semi-holonomic pseudo-connections* (4.2) *associated with* $\{\bar{H}_S^q\}$. *This sequence is regular admitting the same automorphisms* A^{q-1} *as the sequence* $\{\bar{H}_S^q\}$.

Proof. The unicity of $\{SH^q\}$ is evident from (4.17) and we need to prove only the existence and the regularity of $\{SH^q\}$. We shall proceed by induction.

Suppose again $SH^0 = 1$ and put $SH^1 = \bar{H}_S^1$. Then (4.17) is satisfied also for $q = 1$. Consider now the relation

$$\bar{i}_T^q SH^q A^q \bar{\Pi}_S^{q+1} = T^1(\bar{i}_T^{q-1})\,T^1(\bar{\Pi}_T^q)\,T^1(SH^q)\,\bar{H}_S^{q+1}\bar{i}_S^{q+1'} . \tag{4.26}$$

If $q = 1$ we have on the left hand side of (4.26) $\bar{H}_S^1 A^1 \bar{i}_S^2$. Observing $\bar{i}_T^1$, $\bar{i}_T^{1'}$, $\bar{i}_T^0$ are identities we see that the right hand side in (4.26) can be given the form $T^1(\Pi_T \bar{H}_S^1)$. $. \bar{H}_S^2 \bar{i}_S^{2'} = T^1(A^0 \Pi_S)\,\bar{H}_S^2 \bar{i}_S^{2'} = \bar{H}_S^1 A^1 \Pi_S^2$. Here we have used (4.23) and (4.24) with $q = 1$. Thus (4.26) holds for $q = 1$.

Suppose now that (4.26) holds for $q = k - 1 \geqq 1$. We shall show that (4.17) defines for $q = k$ an SH^k and that (4.26) holds for $q = k$. According to (2.63) in order to prove that (4.17) defines an SH^k it suffices to show

$$T^1(\bar{i}_T^{k-2}\bar{\Pi}_T^{k-1})\,T^1(SH^{k-1})\,\bar{H}_S^k \bar{i}_S^{k'} = \bar{i}_T^{k-1}\Pi_T T^1(SH^{k-1})\,\bar{H}_S^k \bar{i}_S^{k'} . \tag{4.27}$$

But we have by virtue of (4.26) (with $q = k - 1$)

$$T^1(\bar{i}_T^{k-2})\,T^1(\bar{\Pi}_T^{k-1})\,T^1(SH^{k-1})\,\bar{H}_S^k \bar{i}_S^{k'} = \bar{i}_T^{k-1} SH^{k-1} A^{k-1} \bar{\Pi}_S^k ,$$

and using (4.23), (2.14) and (2.53) with $q = k$ we have also

$$\bar{i}_T^{k-1}\Pi_T T^1(SH^{k-1})\,\bar{H}_S^k \bar{i}_S^{k'} = \bar{i}_T^{k-1} SH^{k-1} A^{k-1} \bar{\Pi}_S^k .$$

Hence (4.27) holds and this means that SH^k is well defined and satisfies

$$\bar{i}_T^{k'} SH^k = T^1(SH^{k-1})\,\bar{H}_S^k \bar{i}_S^{k'} . \tag{4.28}$$

Now we have to prove (4.26) for $q = k$. From (4.26) written with $q = k - 1$ applying T^1 and "multiplying" by $\bar{H}_S^{k+1} \bar{i}_S^{k+1'}$ we get

$$T^1(\bar{i}_T^{k-1})\, T^1(SH^{k-1})\, T^1(A^{k-1}\bar{\Pi}_S^k)\, \bar{H}_S^{k+1} \bar{i}_S^{k+1'} = \tag{4.29}$$
$$= T^1[T^1(\bar{i}_T^{k-2})\, T^1(\bar{\Pi}_T^{k-1})\, T^1(SH^{k-1})\, \bar{H}_S^k \bar{i}_S^{k'}]\, \bar{H}_S^{k+1} \bar{i}_S^{k+1'} .$$

The left hand side of (4.29) can be transformed, using (4.24) with $q = k$, (4.28) and (2.62), into

$$T^1(\bar{i}_T^{k-1})\, T^1(SH^{k-1})\, \bar{H}_S^k \bar{i}_S^{k'} A^k \bar{\Pi}_S^{k+1} = T^1(\bar{i}_T^{k-1})\, \bar{i}_T^{k'} SH^k A^k \bar{\Pi}_S^{k+1} = \bar{i}_T^k SH^k A^k \bar{\Pi}_S^{k+1} ,$$

which is one side of the formula to be proved. The right hand side in (4.29) transforms by virtue of (4.28), (2.64) and (2.62) into

$$T^1[T^1(\bar{i}_T^{k-2})\, T^1(\bar{\Pi}_T^{k-1})\, \bar{i}_T^{k'} SH^k]\, \bar{H}_S^{k+1} \bar{i}_S^{k+1'} = T^1(\bar{i}_T^{k-1} \bar{\Pi}_T^k SH^k)\, \bar{H}_S^{k+1} \bar{i}_S^{k+1'} ,$$

which proves entirely the formula (4.26) for $q = k$.

Now after having proved the existence of $\{SH^q\}$ we need to show that it is regular. Clearly $\Pi_T SH^1 = A^0 \Pi_S$ since $SH^1 = \bar{H}_S^1$. Thus let $q > 1$ and apply $T^1(\bar{\Pi}_T^{q-1})$ to (4.17). Using again (2.64) we get

$$\bar{i}_T^{q-1'} \bar{\Pi}_T^q SH^q = T^1(\bar{\Pi}_T^{q-1} SH^{q-1})\, \bar{H}_S^q \bar{i}_S^{q'}$$

and from (4.26) with $q - 1$ substituted for q, observing that $\bar{i}_T^{q-1} = T^1(\bar{i}_T^{q-2})\, \bar{i}_T^{q-1'}$, we finally get

$$\bar{i}_T^{q-1'} \bar{\Pi}_T^q SH^q = \bar{i}_T^{q-1'} SH^{q-1} A^{q-1} \bar{\Pi}_S^q ,$$

which is (4.18) since $\bar{i}_T^{q-1'}$ is an injection. This completes the proof.

Corollary. *If $\{\bar{H}_S^q\}$ is a sequence of connections then $\{SH^q\}$ given by this theorem is a sequence of semi-holonomic connections.*

Proof. First note that if we write (2.73) in the form $j_T^{1*}(\bar{i}_T^{q-1} \bar{j}_T^{q-1*} \otimes 1) = T^1(\bar{i}_T^{q-1}) \,.\, \bar{i}_T^{q'} \bar{j}_T^{q*}$, $(q > 1)$ and use (2.68), we get

$$j_T^{1*}(\bar{j}_T^{q-1*} \otimes 1) = \bar{i}_T^{q'} \bar{j}_T^{q*} . \tag{4.30}$$

Further (2.66) yields for each $X \in \operatorname{Ker} \bar{\Pi}_T^q$

$$I_0^{-1} T^1(\bar{I}_0^{q-1})^{-1}\, \bar{i}_T^{q'} X = \bar{i}_S^{q'} (\bar{I}_0^q)^{-1}\, X . \tag{4.31}$$

Now since in this case all the A^{q-1} are identities, (4.12) holds and $SH^1 = \bar{H}_S^1$ is a connection. Thus it suffices to show (4.13) for each $q > 1$, or equivalently

$$(SH^q)^{-1}\, \bar{j}_T^{q*} = (\bar{I}_0^q)^{-1}\, \bar{j}_T^{q*} . \tag{4.32}$$

This holds clearly for $q = 1$ and thus supposing that

$$(4.33) \qquad (SH^{q-1})^{-1} \, \bar{j}_T^{q-1*} = (\bar{I}_0^{q-1})^{-1} \, \bar{j}_T^{q-1*}$$

holds, we shall prove (4.32). Writing (4.17) in the equivalent form

$$(4.34) \qquad \bar{i}_S^{q\prime}(SH^q)^{-1} = (\bar{H}_S^q)^{-1} \, T^1(SH^{q-1})^{-1} \, \bar{i}_T^{q\prime}$$

we have, using subsequently the relations (4.34), (4.30), (2.68), (4.33), (3.10), again (2.68), (4.30) and finally (4.31),

$$\bar{i}_S^{q\prime}(SH^q)^{-1} \, \bar{j}_T^{q*} = (\bar{H}_S^q)^{-1} \, T^1(SH^{q-1})^{-1} \, \bar{i}_T^{q\prime} \bar{j}_T^{q*} =$$
$$= (\bar{H}_S^q)^{-1} \, j_T^{1*}((SH^{q-1})^{-1} \, \bar{j}_T^{q-1*} \otimes 1) = (I_0)^{-1} \, j_T^{1*}((\bar{I}_0^{q-1})^{-1} \otimes 1)(\bar{j}_T^{q-1} \otimes 1) =$$
$$= (I_0)^{-1} \, T^1(\bar{I}_0^{q-1})^{-1} \, \bar{i}_T^{q\prime} \, \bar{j}_T^{q*} = \bar{i}_S^{q\prime}(\bar{I}_0^q)^{-1} \bar{j}_T^{q*} \, .$$

But since $\bar{i}_S^{q\prime}$ is an injection, this proves the corollary.

Theorem 4.4. *Let* $\{SH^q\}$ *be a regular sequence of semi-holonomic pseudo-connections* (4.2). *Then all the sequences* $\{\bar{H}_S^q\}$ *of pseudo-connections* (4.16) *associated with* $\{SH^q\}$ *are regular admitting the same automorphisms* A^{q-1} *as* $\{SH^q\}$.

Proof. Let $q \geqq 1$. From

$$(4.35) \qquad \bar{i}_T^{q\prime} SH^q = T^1(SH^{q-1}) \, \bar{H}_S^q \bar{i}_S^{q\prime} \, ,$$

$$(4.36) \qquad \bar{i}_T^{q+1\prime} SH^{q+1} = T^1(SH^q) \, \bar{H}_S^{q+1} \bar{i}_S^{q+1\prime}$$

we get applying $T^1(\bar{\Pi}_T^q)$ to (4.36) and using the regularity of $\{SH^q\}$, (2.64) and (4.35)

$$T^1(\bar{\Pi}_T^q) \, \bar{i}_T^{q+1\prime} SH^{q+1} = T^1(\bar{\Pi}_T^q SH^q) \, \bar{H}_S^{q+1} \bar{i}_S^{q+1\prime} \, ,$$

from there

$$\bar{i}_T^{q\prime} SH^q A^q \bar{\Pi}_S^{q+1} = T^1(SH^{q-1} A^{q-1} \bar{\Pi}_S^q) \, \bar{H}_S^{q+1} \bar{i}_S^{q+1\prime}$$

and finally

$$T^1(SH^{q-1}) \, \bar{H}_S^q \bar{i}_S^{q\prime} A^q \bar{\Pi}_S^{q+1} = T^1(SH^{q-1}) \, T^1(A^{q-1} \bar{\Pi}_S^q) \, \bar{H}_S^{q+1} \bar{i}_S^{q+1\prime}$$

which proves (4.24). On the other hand applying Π_T to (4.35) we get again from the regularity of $\{SH^q\}$ and (2.53)

$$\bar{\Pi}_T^q SH^q = SH^{q-1} \bar{\Pi}_T \bar{H}_S^q \bar{i}_S^{q\prime}$$

or

$$SH^{q-1} A^{q-1} \bar{\Pi}_S^q = SH^{q-1} \Pi_T \bar{H}_S^q \bar{i}_S^{q\prime}$$

which is (4.23) and this completes the proof.

Corollary. *If the sequences* $\{\bar{H}_S^q\}$ *and* $\{SH^q\}$ *are mutually associated and one of them is regular, then the same is true about the other with the same automorphisms* A^{q-1}, $(q \geqq 1)$.

Remark 6. Using direct calculations with coordinate expressions it could be shown that if $\{\bar{H}_S^q\}$ are connections, then — at least in the case of a finite sequence $\bar{H}_S^1$, $\bar{H}_S^2$ — the condition (4.25) is also necessary for the existence of an associated sequence $\{SH^q\}$. Thus our restriction to only regular sequences does not seem to be essential.

We have seen that the regularity of $\{\bar{H}_S^q\}$ guarantees the existence of an associated sequence $\{SH^q\}$ uniquely. Our next task will be to solve the inverse problem, i.e. given a regular sequence $\{SH^q\}$ find a sequence $\{\bar{H}_S^q\}$ associated with $\{SH^q\}$.

First note that if $\{\bar{H}_S^q\}$ is associated with $\{SH^q\}$, then another sequence $\{\bar{H}_S^{q'}\}$ is associated with the same $\{SH^q\}$ if and only if $\bar{H}_S^{q'}\bar{i}_S^{q'} = \bar{H}_S^q\bar{i}_S^{q'}$ for each $q \geqq 1$. Consequently we cannot expect unicity in the inverse problem. In particular, if $\{\bar{H}_S^q\}$ is associated with a sequence $\{SH^q\}$ of semi-holonomic connections, it need not consist of connections itself. Nevertheless in the following theorem we shall construct to each regular sequence $\{SH^q\}$ a *special* regular sequence $\{\bar{H}_S^q\}$ associated with $\{SH^q\}$, which will consist of connections if $\{SH^q\}$ is a sequence of semi-holonomic connections.

Theorem 4.5. *Let $\{SH^q\}$ be a regular sequence of semi-holonomic pseudo-connections* (4.2). *Then there exists a regular sequence $\{\bar{H}_S^q\}$ of pseudo-connections* (4.16) *associated with $\{SH^q\}$. This sequence consists of connections if $\{SH^q\}$ is a sequence of semi-holonomic connections.*

Proof. We shall prove the existence by direct construction. First define for each $q > 1$

$$\bar{R}^q = (\bar{j}_T^{q-1}\bar{\Pi}_T^{q-1} \otimes 1) + (\bar{j}_T^{q-1*}Q^{q-1} \otimes 1)(Q^q)^{-1}(\bar{\Pi}_T^{q-1*} \otimes 1), \tag{4.37}$$

where Q^q, $(q \geqq 1)$ are defined in Lemma 4.2 (c.f. (4.21)). We have

$$\bar{R}^q : \bar{T}^{q-1}(E) \otimes T(M)^* \to \bar{T}^{q-1}(E) \otimes T(M)^*$$

and it is an automorphism. In fact, if $\bar{R}^qX = 0$ we conclude, applying $(\bar{\Pi}_T^{q-1} \otimes 1)$ to (4.37), that $(\bar{\Pi}_T^{q-1} \otimes 1)X = 0$. Using again $\bar{R}^qX = 0$ and observing that $(\bar{j}_T^{q-1*}Q^{q-1} \otimes 1)(Q^q)^{-1}$ is an injection, we get $(\bar{\Pi}_T^{q-1*} \otimes 1)X = 0$ and hence $X = 0$. Consequently $\bar{R}^q$ is an automorphism.

Define now $\bar{H}_S^1 = SH^1$ and if $q > 1$ put

$$(\bar{H}_S^q)^{-1} = \Omega_1\bar{\Pi}_S + \Omega_2\bar{\Pi}_S + \Omega_3\Pi_T^*, \tag{4.38}$$

where

$$\begin{aligned} \Omega_1 &= \bar{i}_S^{q'}(SH^q)^{-1}\,\bar{j}_T^qSH^{q-1} \\ \Omega_2 &= -I_0^{-1}T^1(SH^{q-1})^{-1}\,j_T^{1*}\bar{R}^q\Pi_T^*\bar{i}_T^{q'}\bar{j}_T^qSH^{q-1} \\ \Omega_3 &= I_0^{-1}T^1(SH^{q-1})^{-1}\,j_T^{1*}\bar{R}^q(SH^{q-1} \otimes 1). \end{aligned}$$

We must verify (4.34). We have

$$\Omega_3\Pi_T^*T^1(SH^{q-1})^{-1}\,\bar{i}_T^{q'}\bar{j}_T^q\bar{\Pi}_T^q + \Omega_2\Pi_TT^1(SH^{q-1})^{-1}\,\bar{i}_T^{q'} = 0. \tag{4.39}$$

In fact, we get from (2.14), (2.62) and (2.67)

$$
\begin{aligned}
&-\Omega_2 \Pi_T T^1 (SH^{q-1})^{-1}\, i_T^{q'} = \\
&= I_0^{-1} T^1 (SH^{q-1})^{-1}\, j_T^{1*} \bar{R}^q \Pi_T^* i_T^{q'} j_T^q SH^{q-1} \Pi_T T^1 (SH^{q-1})^{-1}\, i_T^{q'} = \\
&= I_0^{-1} T^1 (SH^{q-1})^{-1}\, j_T^{1*} \bar{R}^q \Pi_T^* i_T^{q'} j_T^q \bar{\Pi}_T^q = \\
&= \Omega_3 \Pi_T^* T^1 (SH^{q-1})^{-1}\, i_T^{q'}\, j_T^q \bar{\Pi}_T^q .
\end{aligned}
$$

This proves (4.39). On the other hand we have from (2.67), (4.34), (2.68)

$$
\begin{aligned}
&\Omega_3 \Pi_T^* T^1 (SH^{q-1})^{-1}\, i_T^{q'}\, j_T^{q*} \bar{\Pi}_T^{q*} = \Omega_3 ((SH^{q-1})^{-1} \otimes 1)\, (j_T^{q-1*} \otimes 1)\, \bar{\Pi}_T^{q*} = \\
&= I_0^{-1} T^1 (SH^{q-1})^{-1}\, j_T^{1*} \{ (j_T^{q-1} \bar{\Pi}_T^{q-1} \otimes 1) + \\
&+ (j_T^{q-1*} Q^{q-1} \otimes 1)\, (Q^q)^{-1}\, (\bar{\Pi}_T^{q-1*} \otimes 1) \}\, (j_T^{q-1*} \otimes 1)\, \bar{\Pi}_T^{q*} = \\
&= I_0^{-1} T^1 (SH^{q-1})^{-1}\, j_T^{1*} (j_T^{q-1*} Q^{q-1} \otimes 1)\, (Q^q)^{-1}\, \bar{\Pi}_T^{q*} = \\
&= I_0^{-1} T^1 [(SH^{q-1})^{-1}\, j_T^{q-1*} Q^{q-1}]\, j_T^{1*} (Q^q)^{-1}\, \bar{\Pi}_T^{q*} .
\end{aligned}
$$

But this can be further transformed by virtue of (4.22), (2.68), (4.30) and (4.31), into

$$
\begin{aligned}
&I_0^{-1} T^1 (\bar{I}_0^{q-1})^{-1}\, j_T^{1*\prime} (\bar{j}_T^{q-1*} \otimes 1)\, (Q^q)^{-1}\, \bar{\Pi}_T^{q*} = \\
&= I_0^{-1} T^1 (\bar{I}_0^{q-1})^{-1}\, \bar{i}_T^{q'}\, \bar{j}_T^{q*} (Q^q)^{-1}\, \bar{\Pi}_T^{q*} = \bar{i}_S^{q'} (\bar{I}_0^q)^{-1}\, \bar{j}_T^{q*} (Q^q)^{-1}\, \bar{\Pi}_T^{q*} .
\end{aligned}
$$

Using now again (4.22) we obtain

$$
\Omega_3 \Pi_T^* T^1 (SH^{q-1})^{-1}\, \bar{i}_T^{q'}\, \bar{j}_T^{q*} \bar{\Pi}_T^{q*} = \bar{i}_S^{q'} (SH^q)^{-1}\, \bar{j}_T^{q*} \bar{\Pi}_T^{q*} . \tag{4.40}
$$

Finally we have from (2.14) and (2.53)

$$
\Omega_1 \Pi_T T^1 (SH^{q-1})^{-1}\, \bar{i}_T^{q'} = \bar{i}_S^{q'} (SH^q)^{-1}\, \bar{j}_T^q \bar{\Pi}_T^q . \tag{4.41}
$$

Adding the relations (4.39), (4.40), (4.41) we obtain the required result

$$
[\Omega_1 \Pi_T + \Omega_2 \Pi_T + \Omega_3 \Pi_T^*]\; T^1 (SH^{q-1})^{-1}\, \bar{i}_T^{q'} = \bar{i}_S^{q'} (SH^q)^{-1} .
$$

It remains to show that $\bar{H}_S^q$ is really defined by (4.38), i.e. that (4.38) is an isomorphism. But this is almost evident. If $(\bar{H}_S^q)^{-1} X = 0$ then $\Pi_S (\bar{H}_S^q)^{-1} X = 0$ and this implies $\Pi_T X = 0$, since $\Pi_S \Omega_3 = \Pi_S \Omega_2 = 0$ and $\Pi_S \Omega_1 = 1$. Thus we get again from $(\bar{H}_S^q)^{-1} X = 0$ that $\Omega_3 \Pi_T^* X = 0$ and since Ω_3 consists of injections only, we conclude that $\Pi_T^* X = 0$ and hence $X = 0$. Thus (4.38) is an injection and consequently it defines a pseudo-connection.

Suppose now that $\{SH^q\}$ is a sequence of connections. If $q = 1$ the proof is evident and for $q > 1$ applying Π_S to (4.38), as we have already seen, we get $\Pi_S (\bar{H}_S^q)^{-1} = \Pi_T$. On the other hand we see immediately that $(\bar{H}_S^q)^{-1}\, j_T^{1*} = \Omega_3$. But if $\{SH^q\}$ is a sequence of connections then each Q^q, and also each $\bar{R}^q$, are identities and we see that in this case $\Omega_3 = I_0^{-1} j_T^{1*}$. This completes the proof of Theorem 4.5.

Now we shall associate with sequences $\{SH^q\}$ of semi-holonomic pseudo-connections sequences $\{\overline{H}^q_T\}$, $(q \geqq 1)$ of pseudo-connections

$$(4.42) \qquad \overline{H}^q_T : S^1(\overline{T}^{q-1}(E)) \to T^1(\overline{T}^{q-1}(E)) .$$

Analogous considerations as in the case of pseudo-connections on jet prolongations lead to

Definition 4.7. We say that the sequence $\{\overline{H}^q_T\}$, $(q \geqq 1)$ of pseudo-connections (4.42) is *associated* with the sequence $\{SH^q\}$, $(q \geqq 1)$ of semi-holonomic pseudo-connections (4.2) (and conversely), if $SH^1 = \overline{H}^1_T$ and

$$(4.43) \qquad i^{q\prime}_T SH^q = \overline{H}^q_T S^1(SH^{q-1})\, i^{q\prime}_S$$

for $q > 1$.

On the other hand each sequence $\{SH^q\}$ of semi-holonomic pseudo-connections defines a on-to-one correspondence between all the sequences $\{\overline{H}^q_S\}$ of pseudo-connections (4.16) and all the sequences $\{\overline{H}^q_T\}$ of pseudo-connections (4.42). A sequence $\{\overline{H}^q_S\}$ corresponds to $\{\overline{H}^q_T\}$, and conversely, if each $\overline{H}^q_S$ induces the same R-pseudo-connection as $\overline{H}^q_T$ with respect to SH^{q-1}, i.e. for each $q \geqq 1$,

$$(4.44) \qquad T^1(SH^{q-1})\, \overline{H}^q_S = \overline{H}^q_T S^1(SH^{q-1}) .$$

We shall say briefly that $\{\overline{H}^q_S\}$ *corresponds to* $\{\overline{H}^q_T\}$ (and conversely) *by means of* $\{SH^q\}$. Note that we put $SH^0 = 1$ and therefore (4.44) implies $\overline{H}^1_S = \overline{H}^1_T$. We have then the evident

Theorem 4.6. *Let* $\{\overline{H}^q_S\}$ *correspond to* $\{\overline{H}^q_T\}$ *by means of* $\{SH^q\}$. *Then* $\{\overline{H}^q_S\}$ *is associated with* $\{SH^q\}$ *if and only if* $\{\overline{H}^q_T\}$ *is associated with* $\{SH^q\}$.

Definition 4.8. Call a sequence $\{\overline{H}^q_T\}$ of pseudo-connections (4.42) *regular* if there exists a sequence $\{B^{q-1}\}$, $(q \geqq 1)$ of automorphisms $B^{q-1} : \overline{T}^{q-1}(E) \to \overline{T}^{q-1}(E)$ such that

$$(4.45) \qquad \Pi_S(\overline{H}^q_T)^{-1}\, i^{q\prime}_T = (B^{q-1})^{-1}\, \overline{\Pi}^q_T$$

and

$$(4.46) \qquad S^1((B^{q-1})^{-1}\, \overline{\Pi}^q_T)\, (\overline{H}^{q+1}_T)^{-1}\, i^{q+1\prime}_T = (\overline{H}^q_T)^{-1}\, i^{q\prime}_T (B^q)^{-1}\, \overline{\Pi}^{q+1}_T$$

hold for each $q \geqq 1$.

Again, not each sequence $\{\overline{H}^q_T\}$ consisting of connections is regular, however if $\{\overline{H}^q_T\}$ consists of connections, then its regularity is guaranteed by the condition

$$(4.47) \qquad S^1(\overline{\Pi}^q_T)\, (\overline{H}^{q+1}_T)^{-1}\, i^{q+1\prime}_T = (\overline{H}^q_T)^{-1}\, T^1(\overline{\Pi}^q_T)\, i^{q+1\prime}_T ,$$

which is fulfilled especially if each pair $\overline{H}^{q+1}_T$, $\overline{H}^q_T$, $(q \geqq 1)$ induce the same R-connection with respect to $\overline{\Pi}^q_T$.

Theorem 4.7. *Let the sequence $\{\bar{H}_S^q\}$ correspond to the sequence $\{\bar{H}_T^q\}$ by means of $\{SH^q\}$. Then $\{\bar{H}_S^q\}$ consists of connections if and only if $\{\bar{H}_T^q\}$ consists of connections.*

This theorem is an immediate consequence of Propositions 3.6 and 3.7.

Theorem 4.8. (c.f. Theorem 4.3). *Let $\{\bar{H}_T^q\}$ be a regular sequence of pseudo-connections* (4.42). *Then there exists exactly one sequence $\{SH^q\}$ of semi-holonomic pseudo-connections* (4.2) *associated with $\{\bar{H}_T^q\}$. This sequence $\{SH^q\}$ is regular admitting the same automorphisms B^{q-1} in* (4.19) *as the sequence $\{\bar{H}_T^q\}$.*

Proof. First transform the relation (4.43) into the equivalent form

$$\bar{\imath}_S^{q'}(SH^q)^{-1} = S^1(SH^{q-1})^{-1}(\bar{H}_T^q)^{-1}\,\bar{\imath}_T^{q'}\,. \tag{4.48}$$

A comparison with (4.17) shows that the proof can be made in a similar manner as that employed to prove Theorem 4.3. Therefore we give only an outline of it.

First note that (4.43) or (4.48) holds for $q = 1$ if we put $SH^1 = \bar{H}_T^1$ and also

$$\bar{\imath}_S^q(SH^q)^{-1}(B^q)^{-1}\,\bar{\Pi}_T^{q+1} = S^1(\bar{\imath}_S^{q-1})\,S^1(\bar{\Pi}_S^q)\,S^1(SH^q)^{-1}(\bar{H}_T^{q+1})^{-1}\,\bar{\imath}_T^{q+1'} \tag{4.49}$$

holds for $q = 1$. Suppose now that (4.48) and (4.49) hold for $q = k - 1$ and we shall prove them for $q = k$. We must again verify (c.f. (2.54))

$$\begin{aligned} &S^1(\bar{\imath}_S^{k-2}\bar{\Pi}_S^{k-1})\,S^1(SH^{k-1})^{-1}(\bar{H}_T^k)^{-1}\,\bar{\imath}_T^{k'} = \\ &= \bar{\imath}_S^{k-1}\bar{\Pi}_S S^1(SH^{k-1})^{-1}(\bar{H}_T^k)^{-1}\,\bar{\imath}_T^{k'}\,. \end{aligned}$$

But both the sides here can be brought to the form $\bar{\imath}_S^{k-1}(SH^{k-1})^{-1}(B^{k-1})^{-1}\,\bar{\Pi}_T^k$ and thus SH^k is well defined.

Now applying S^1 to (4.49) with $q = k - 1$ and "multiplying" this by $(\bar{H}_T^{k+1})^{-1}\,\bar{\imath}_T^{k+1'}$ we get

$$\begin{aligned} &S^1(\bar{\imath}_S^{k-1})\,S^1(SH^{k-1})^{-1}\,S^1(B^{k-1})^{-1}\,S^1(\bar{\Pi}_T^k)(\bar{H}_T^{k+1})^{-1}\,\bar{\imath}_T^{k+1'} = \\ &= S^1[S^1(\bar{\imath}_S^{k-2}\bar{\Pi}_S^{k-1}(SH^{k-1})^{-1})(\bar{H}_T^k)^{-1}\,\bar{\imath}_T^{k'}](\bar{H}_T^{k+1})^{-1}\,\bar{\imath}_T^{k+1'}\,, \end{aligned}$$

and from there

$$\begin{aligned} &S^1(\bar{\imath}_S^{k-1})\,S^1(SH^{k-1})^{-1}(\bar{H}_T^k)^{-1}\,\bar{\imath}_T^{k'}(B^k)^{-1}\,\bar{\Pi}_T^{k+1} = \\ &= S^1[S^1(\bar{\imath}_S^{k-2})\,S^1(\bar{\Pi}_S^{k-1})\,\bar{\imath}_S^{k'}(SH^k)^{-1}](\bar{H}_T^{k+1})^{-1}\,\bar{\imath}_T^{k+1'}\,, \end{aligned}$$

and further

$$\bar{\imath}_S^k(SH^k)^{-1}(B^k)^{-1}\,\bar{\Pi}_T^{k+1} = S^1\,(\bar{\imath}_S^{k-1})S^1(\bar{\Pi}_S^k)\,S^1(SH^k)^{-1}(\bar{H}_T^{k+1})^{-1}\,\bar{\imath}_T^{k+1'})\,.$$

This proves the existence of $\{SH^q\}$. Applying $S^1(\bar{\Pi}_S^{q-1})$ to (4.48) we get

$$\bar{\imath}_S^{q-1'}\bar{\Pi}_S^q(SH^q)^{-1} = \bar{\imath}_S^{q-1'}(SH^{q-1})^{-1}(B^{q-1})^{-1}\,\bar{\Pi}_T^q$$

and this completes the proof.

Corollary. *If $\{\overline{H}^q_T\}$ is a sequence of connections, then $\{SH^q\}$ given in this theorem is a sequence of semi-holonomic connections.*

Proof. In fact, denoting by $\{\overline{H}^q_S\}$ the sequence corresponding to $\{\overline{H}^q_T\}$ by means of $\{SH^q\}$, then according to Theorem 4.7, $\{\overline{H}^q_S\}$ consists of connections and according to Theorem 4.6, $\{\overline{H}^q_S\}$ is associated with $\{SH^q\}$. From there and the Corollary of Theorem 4.3 we conclude that $\{SH^q\}$ is a sequence of semi-holonomic connections.

Theorem 4.9. *Let $\{SH^q\}$ be a regular sequence of semi-holonomic pseudo-connections* (4.2). *Then all the sequences $\{\overline{H}^q_T\}$ of pseudo-connections* (4.42) *associated with $\{SH^q\}$ are regular admitting the same automorphisms B^{q-1} as $\{SH^q\}$ in* (4.19).

We omit the proof since it is quite analogous to that of Theorem 4.4. We have also the

Corollary. *If the sequences $\{\overline{H}^q_T\}$ and $\{SH^q\}$ are mutually associated and one of them is regular, then the same is true about the other with the same automorphisms B^{q-1}, $(q \geqq 1)$.*

The corresponding inverse problem in the "tensor case" is de facto already solved. We have namely

Theorem 4.10. *Let $\{SH^q\}$ be a regular sequence of semi-holonomic pseudo-connections* (4.2). *Then there exists a regular sequence $\{\overline{H}^q_T\}$ of pseudo-connections* (4.42) *associated with $\{SH^q\}$.This sequence consists of connections if $\{SH^q\}$ is a sequence of semi-holonomic connections.*

Proof. It suffices to take for $\{\overline{H}^q_T\}$ the sequence corresponding to the special sequence $\{\overline{H}^q_S\}$ by means of $\{SH^q\}$, where $\{\overline{H}^q_S\}$ has been defined in Theorem 4.5.

Given a connection H on E together with a connection h on the cotangent bundle $T(M)^*$, we have again a canonical sequence $\{\overline{H}^q_T\}$ generated by the pair H, h. In fact, since $\overline{T}^{q-1}(E) = \overline{T}^{q-2}(E) \oplus E \otimes (\overset{q-1}{\otimes} T(M)^*)$ we have for this sequence the recurrent formula $\overline{H}^1_T = H$ and

$$\begin{aligned} \text{(4.50)} \qquad \overline{H}^q_T &= \overline{H}^{q-1}_T (\oplus) H (\oplus) [(\overset{q-1}{\oplus}) h] = \\ &= T^1(j^{q-1}_T) \overline{H}^{q-1}_T S^1(\overline{\Pi}^{q-1}_T) + T^1(j^{q-1*}_T) \hat{H}^{q-1}_T S^1(\overline{\Pi}^{q-1*}_T), \end{aligned}$$

where we write briefly

$$\text{(4.51)} \qquad \hat{H}^{q-1}_T = H (\otimes) [(\overset{q-1}{\otimes}) h].$$

Theorem 4.11. *The canonical sequence $\{\overline{H}^q_T\}$ generated by an arbitrary connection H on E and an arbitrary connection h on $T(M)^*$ is regular.*

Proof. Since $\{\overline{H}_T^q\}$ consists of connections it suffices to show that $\overline{H}_T^{q-1}$, $\overline{H}_T^q$, $(q > 1)$ induce the same R-connection with respect to $\overline{\Pi}_T^{q-1}$. But applying $T^1(\overline{\Pi}_T^{q-1})$ to (4.50) we get

$$T^1(\overline{\Pi}_T^{q-1})\, \overline{H}_T^q = \overline{H}_T^{q-1}\, S^1(\overline{\Pi}_T^{q-1}) + 0\,,$$

and this completes the proof.

Combining again this result with the preceding we have

Theorem 4.12. *Each connection on E together with a connection on the tangent bundle T(M) canonically generate a sequence of semi-holonomic connections of higher orders on E and sequences of connections on the semi-holonomic jet and semi-holonomic tensor prolongations of E.*

Proof. The canonical sequence $\{\overline{H}_T^q\}$ has been just defined, the canonical sequence $\{SH^q\}$ being then given by Theorem 4.8 and its corollary. But there are *a priori* two ways in defining the canonical sequence $\{\overline{H}_S^q\}$. It is either the sequence corresponding to $\{\overline{H}_T^q\}$ by means of $\{SH^q\}$, or the special sequence defined in Theorem 4.5 associated with $\{SH^q\}$. However we shall prove later the following lemma which states that there is *de facto* no difference between these two possibilities of defining th: canonical sequence $\{\overline{H}_S^q\}$.

Lemma 4.3. *Let $\{SH^q\}$ and $\{\overline{H}_T^q\}$ be two canonical sequences generated by a pair H and h of connections on E and $T(M)^*$ respectively. Then the sequence $\{\overline{H}_S^q\}$ constructed from $\{SH^q\}$ according to Theorem* 4.5 *coincides with that corresponding to $\{\overline{H}_S^q\}$ by means of $\{SH^q\}$.*

Now we are passing to the problem of reducibility of non-holonomic pseudo-connections to semi-holonomic ones.

Definition 4.9. A non-holonomic pseudo-connection NH^q, $(q \geqq 1)$ is said to be reducible to the semi-holonomic pseudo-connection SH^q if the diagram

$$\begin{array}{ccc} \tilde{S}^q(E) & \xrightarrow{NH^q} & \tilde{T}^q(E) \\ {\scriptstyle i_S^q}\uparrow & & \uparrow{\scriptstyle i_T^q} \\ \overline{S}^q(E) & \xrightarrow{SH^q} & \overline{T}^q(E) \end{array}$$

is commutative, i.e. if

(4.52) $$i_T^q SH^q = NH^q i_S^q\,.$$

Note that NH^q is reducible to one SH^q at the most.

Definition 4.10. A pseudo-connection $\tilde{H}^q_S$ given in (4.3) (accordingly a pseudo-connection $\tilde{H}^q_T$ given in (4.4)) is said to be *reducible* to a pseudo-connection $\bar{H}^q_S$ in (4.16), (accordingly to a pseudo-connection $\bar{H}^q_T$ in (4.42)) if $\tilde{H}^q_S$ and $\bar{H}^q_S$ induce the same R-pseudo-connection with respect to $\bar{i}^{q-1}_S$ (accordingly if $\tilde{H}^q_T$ and $\bar{H}^q_T$ induce the same R-pseudo-connection with respect to $\bar{i}^{q-1}_T$), i.e. if

$$\tilde{H}^q_S\, S^1(\bar{i}^{q-1}_S) = T^1(\bar{i}^{q-1}_S)\, \bar{H}^q_S\,, \tag{4.53}$$

(accordingly if

$$\tilde{H}^q_T\, S^1(\bar{i}^{q-1}_T) = T^1(\bar{i}^{q-1}_T)\, \bar{H}^q_T)\,. \tag{4.54}$$

Note again that $\tilde{H}^q_S$ and $\tilde{H}^q_T$ are reducible at the most to one $\bar{H}^q_S$ and $\bar{H}^q_T$ respectively.

Similarly one could define the reduction of SH^q or $\bar{H}^q_S$ or $\bar{H}^q_T$ to a holonomic pseudo-connection or a pseudo-connection on holonomic jet or tensor prolongations of E. However we do not bring these definitions explicitly since we are not concerned with the "holonomic case" in this paper (but c.f. [3]).

Theorem 4.13. *Let $\tilde{H}^q_S$ (or $\tilde{H}^q_T$) be a connection which is reducible to $\bar{H}^q_S$ (or to $\bar{H}^q_T$). Then $\bar{H}^q_S$ (or $\bar{H}^q_T$) is also a connection.*

Proof. Applying Π_T to (4.53) we get by virtue of (2.13) and (2.57)

$$\Pi_S\, S^1(\bar{i}^{q-1}_S) = \bar{i}^{q-1}_S \Pi_T \bar{H}^q_S\,,$$

i.e.

$$\bar{i}^{q-1}_S \Pi_S = \bar{i}^{q-1}_S \Pi_T \bar{H}^q_S$$

and thus $\Pi_T \bar{H}^q_S = \Pi_S$ since $\bar{i}^{q-1}_S$ is an injection. On the other hand (4.53) implies

$$S^1(\bar{i}^{q-1}_S)\,(\bar{H}^q_S)^{-1}\, j^{1*}_T = (\tilde{H}^q_S)^{-1}\, T^1(\bar{i}^{q-1}_S)\, j^{1*}_T$$

and hence from (2.68) and (2.65)

$$\begin{aligned} S^1(\bar{i}^{q-1}_S)\,(\bar{H}^q_S)^{-1}\, j^{1*}_T &= (\tilde{H}^q_S)^{-1}\, j^{1*}_T(\bar{i}^{q-1}_S \otimes 1) = \\ &= I_0^{-1}\, T^1(\bar{i}^{q-1}_S)\, j^{1*}_T = S^1(\bar{i}^{q-1}_S)\, I_0^{-1}\, j^{1*}_T \end{aligned}$$

and this completes the proof since $S^1(\bar{i}^{q-1}_S)$ is an injection. A similar argument applies to the case with $\tilde{H}^q_T$, $\bar{H}^q_T$.

We shall say that the sequence $\{NH^q\}$ or $\{\tilde{H}^q_S\}$ or $\{\tilde{H}^q_T\}$ is reducible to a sequence $\{SH^q\}$ or $\{\bar{H}^q_S\}$ or $\{\bar{H}^q_T\}$ if (4.52) or (4.53) or (4.54) respectively hold for each $q \geqq 1$. We have just proved that if $\{\tilde{H}^q_S\}$ or $\{\tilde{H}^q_T\}$ consist of connections and they are reducible to $\{\bar{H}^q_S\}$ or $\{\bar{H}^q_T\}$ respectively, then these sequences also consist of connections.

Theorem 4.14. *Let $\{NH^q\}$ be a sequence of non-holonomic connections which is reducible to a sequence $\{SH^q\}$. Then $\{SH^q\}$ is a sequence of semi-holonomic connections.*

Proof. We have for each $q > 1$, $\Pi_T NH^q = NH^{q-1}\Pi_S$ and

$$(NH^q)^{-1} j_T^{1*} = I_0^{-1} T^1(NH^{q-1})^{-1} j_T^{1*}. \tag{4.55}$$

Hence applying Π_T to (4.52) and using Lemma 2.3 we get first

$$NH^{q-1}\Pi_S \bar{i}_S^q = \Pi_T \bar{i}_T^q SH^q, \quad NH^{q-1}\bar{i}_S^{q-1}\Pi_S^q = \bar{i}_T^{q-1}\bar{\Pi}_T^q SH^q,$$
$$\bar{i}_T^{q-1} SH^{q-1}\bar{\Pi}_S^q = \bar{i}_T^{q-1}\bar{\Pi}_T^q SH^q$$

and this is (4.12). In order to prove (4.13), i.e. (4.32), note that (4.32) clearly holds for $q = 1$. Thus suppose that it holds for some $q - 1 \geqq 1$, i.e.

$$(SH^{q-1})^{-1} \bar{j}_T^{q-1*} = (\bar{I}_0^{q-1})^{-1} \bar{j}_T^{q-1*}. \tag{4.56}$$

Further from (4.52) we get

$$\bar{i}_S^q (SH^q)^{-1} \bar{j}_T^{q*} = (NH^q)^{-1} \bar{i}_T^q \bar{j}_T^{q*}.$$

But using subsequently (2.73), (4.55), (4.52), (4.56), (4.30), (2.65), (4.31) we have

$$\begin{aligned}
(NH^q)^{-1} \bar{i}_T^q \bar{j}_T^{q*} &= (NH^q)^{-1} j_T^{1*}(\bar{i}_{T_1}^{q-1} \bar{j}_T^{q-1*} \otimes 1) = \\
&= I_0^{-1} T^1(NH^{q-1})^{-1} T^1(\bar{i}_T^{q-1}) T^1(\bar{j}_T^{q-1*}) j_T^{1*} = \\
&= I_0^{-1} T^1(\bar{i}_S^{q-1}) T^1(SH^{q-1})^{-1} T^1(\bar{j}_T^{q-1*}) j_T^{1*} = \\
&= I_0^{-1} T^1(\bar{i}_S^{q-1}) T^1(\bar{I}_0^{q-1})^{-1} j_T^{1*}(\bar{j}_T^{q-1*} \otimes 1) = \\
&= S^1(\bar{i}_S^{q-1}) I_0^{-1} T^1(\bar{I}_0^{q-1})^{-1} \bar{i}_T^{q'} \bar{j}_T^{q*} = \bar{i}_S^q (\bar{I}_0^q)^{-1} \bar{j}_T^{q*}
\end{aligned}$$

and from there we conclude (4.32), which completes the proof.

Theorem 4.15a. *Let $\{\tilde{H}_S^q\} \sim \{NH^q\} \sim \{\tilde{H}_T^q\}$ and suppose $\{\tilde{H}_S^q\}$ is reducible to a sequence $\{\bar{H}_S^q\}$ associated with $\{SH^q\}$. Then $\{NH^q\}$ is reducible to $\{SH^q\}$ and $\{\tilde{H}_T^q\}$ is reducible to the sequence $\{\bar{H}_T^q\}$ corresponding to $\{\bar{H}_S^q\}$ by means of $\{SH^q\}$.*

Theorem 4.15b. *Let $\{\tilde{H}_S^q\} \sim \{NH^q\} \sim \{\tilde{H}_T^q\}$ and suppose that $\{H_T^q\}$ is reducible to a sequence $\{\bar{H}_T^q\}$ associated with $\{SH^q\}$. Then $\{NH^q\}$ is reducible to $\{SH^q\}$ and $\{\tilde{H}_S^q\}$ is reducible to the sequence $\{\bar{H}_S^q\}$ corresponding to $\{\bar{H}_T^q\}$ by means of $\{SH^q\}$.*

Proof. Both the theorems are similar and so we give the proof of only the first.

We have $SH^1 = NH^1$. In general suppose

$$\bar{i}_T^{q-1} SH^{q-1} = NH^{q-1}\bar{i}_S^{q-1} \tag{4.57}$$

and prove (4.52). From (4.53) using (2.53) we get

$$T^1(\bar{i}_S^{q-1}) \bar{H}_S^q \bar{i}_S^{q'} = \tilde{H}_S^q S^1(\bar{i}_S^{q-1}) \bar{i}_S^{q'} = \tilde{H}_S^q \bar{i}_S^q. \tag{4.58}$$

Now applying subsequently (2.53), (4.57), (4.58) and (4.7) to (4.17), we get

$$\bar{\imath}^q_T SH^q = T^1(\bar{\imath}^{q-1}_T SH^{q-1})\,\bar{H}^q_S \bar{\imath}^{q'}_S =$$
$$= T^1(NH^{q-1})\,T^1(\bar{\imath}^{q-1}_S)\,\bar{H}^q_S \bar{\imath}^{q'}_S = T^1(NH^{q-1})\,\tilde{H}^q_S \bar{\imath}^q_S = NH^q \bar{\imath}^q_S$$

and this proves the reducibility of $\{NH^q\}$ to $\{SH^q\}$.

Further applying the functors S^1 and T^1 to (4.57) we get

$$S^1(\bar{\imath}^{q-1}_T)\,S^1(SH^{q-1}) = S^1(NH^{q-1})\,S^1(\bar{\imath}^{q-1}_S)\,, \tag{4.59}$$

$$T^1(\bar{\imath}^{q-1}_T)\,T^1(SH^{q-1}) = T^1(NH^{q-1})\,T^1(\bar{\imath}^{q-1}_S) \tag{4.60}$$

and from (4.7)

$$NH^q\,S^1(\bar{\imath}^{q-1}_S)\,S^1(SH^{q-1})^{-1} = T^1(NH^{q-1})\,\tilde{H}^q_S\,S^1(\bar{\imath}^{q-1}_S)\,S^1(SH^{q-1})^{-1}\,. \tag{4.61}$$

But the left hand side of (4.61) can be transformed by (4.59) and (4.10) into

$$NH^q\,S^1(NH^{q-1})^{-1}\,S^1(\bar{\imath}^{q-1}_T) = \tilde{H}^q_T\,S^1(\bar{\imath}^{q-1}_T)\,.$$

The right hand side of (4.61) goes by (4.53), (4.60) and (4.44) into

$$T^1(NH^{q-1})\,T^1(\bar{\imath}^{q-1}_S)\,\bar{H}^q_S\,S^1(SH^{q-1})^{-1} =$$
$$= T^1(\bar{\imath}^{q-1}_T)\,T^1(SH^{q-1})\,\bar{H}^q_S\,S^1(SH^{q-1})^{-1} = T^1(\bar{\imath}^{q-1}_T)\,\bar{H}^q_T\,.$$

Hence the sequence $\{\tilde{H}^q_T\}$ is reducible to $\{\bar{H}^q_T\}$ and this completes the proof.

Note that by virtue of Theorem 4.6 the sequence $\{\bar{H}^q_T\}$ given in Theorem 4.15a (accordingly the sequence $\{\bar{H}^q_S\}$ given in Theorem 4.15b) is associated with $\{SH^q\}$. Further from Theorems 4.3 and 4.8 we deduce the

Corollary. *Let* $\{\tilde{H}^q_S\} \sim \{NH^q\} \sim \{\tilde{H}^q_T\}$. *If* $\{\tilde{H}^q_S\}$ *or* $\{\tilde{H}^q_T\}$ *is reducible to a regular sequence* $\{\bar{H}^q_S\}$ *or* $\{\bar{H}^q_T\}$ *respectively, then* $\{NH^q\}$ *is reducible to a regular sequence* $\{SH^q\}$.

Theorem 4.16. *Let* $\{\tilde{H}^q_S\} \sim \{NH^q\} \sim \{\tilde{H}^q_T\}$ *and let* $\{NH^q\}$ *be a sequence of non-holonomic connections reducible to a sequence* $\{SH^q\}$ *of semi-holonomic pseudo-connections. Then* $\{\tilde{H}^q_S\}$ *and* $\{\tilde{H}^q_T\}$ *are reducible to regular sequence* $\{\bar{H}^q_S\}$ *and* $\{\bar{H}^q_T\}$ *consisting of connections.*

Proof. First note that $\{SH^q\}$ is necessarily a sequence of semi-holonomic connections (see Theorem 4.14) and hence it is regular. Our task is to prove the relation (4.53), i.e.

$$S^1(\bar{\imath}^{q-1}_S)\,(\bar{H}^q_S)^{-1} = (\tilde{H}^q_S)^{-1}\,T^1(\bar{\imath}^{q-1}_S) \tag{4.62}$$

for some sequence $\{\bar{H}^q_S\}$. In fact, we shall show that namely the special sequence associated with $\{SH^q\}$ defined in Theorem 4.5, i.e. (4.38), satisfies this relation. Note that in our case this $\{\bar{H}_S^q\}$ consists of connections. First it is evident, that (4.62) holds for $q = 1$, since $\bar{H}_S^1 = SH^1 = NH^1 = \tilde{H}_S^1$. Thus suppose $q > 1$. By virtue of (4.8) and (4.52) or (4.57) we have

$$(\tilde{H}_S^q)^{-1}\, T^1(\bar{i}_S^{q-1}) = (NH^q)^{-1}\, T^1(NH^{q-1})\, T^1(\bar{i}_S^{q-1}) = \\ = (NH^q)^{-1}\, T^1(\bar{i}_T^{q-1})\, T^1(SH^{q-1}).$$

This means that (4.62) with (4.38) goes into the equivalent relation

$$(4.63)\qquad S^1(\bar{i}_S^{q-1})\, \Omega_1 \Pi_T\, T^1(SH^{q-1})^{-1} + S^1(\bar{i}_S^{q-1})\, \Omega_2 \Pi_T\, T^1(SH^{q-1})^{-1} + \\ + S^1(\bar{i}_S^{q-1})\, \Omega_3 \Pi_T^*\, T^1(SH^{q-1})^{-1} = (NH^q)^{-1}\, T^1(\bar{i}_T^{q-1})$$

and our next task is to prove it.

First we shall show that

$$(4.64)\qquad S^1(\bar{i}_S^{q-1})\, \Omega_1 \Pi_T\, T^1(SH^{q-1})^{-1} + \\ + S^1(\bar{i}_S^{q-1})\, \Omega_2 \Pi_T\, T^1(SH^{q-1})^{-1} = (NH^q)^{-1}\, j_T^1 \Pi_T\, T^1(\bar{i}_T^{q-1}).$$

In fact, we have from (2.53) and (4.52)

$$(4.65)\qquad S^1(\bar{i}_S^{q-1})\, \Omega_1 \Pi_T\, T^1(SH^{q-1})^{-1} = \\ = \bar{i}_S^q (SH^q)^{-1}\, \bar{j}_T^q \Pi_T = (NH^q)^{-1}\, \bar{i}_T^q \bar{j}_T^q \Pi_T.$$

But since, according to Lemma 2.3,

$$\bar{i}_T^q \bar{j}_T^q = (j_T^1 \Pi_T + j_T^{1*} \Pi_T^*)\, \bar{i}_T^q \bar{j}_T^q = j_T^1 \bar{i}_T^{q-1} + j_T^{1*} \Pi_T^* \bar{i}_T^q \bar{j}_T^q$$

and $\{NH^q\}$ is a sequence of connections, i.e.

$$(4.66)\qquad (NH^q)^{-1}\, j_T^{1*} = I_0^{-1}\, T^1(NH^{q-1})^{-1}\, j_T^{1*},$$

we transform the right hand side of (4.65) by the relations (2.14), (4.66), (2.62), (2.67), (2.68), (4.57) and (2.65) into

$$(NH^q)^{-1}\, j_T^1 \Pi_T\, T^1(\bar{i}_T^{q-1}) + I_0^{-1}\, T^1(NH^{q-1})^{-1}\, j_T^{1*} \Pi_T^* \bar{i}_T^q \bar{j}_T^q \Pi_T = \\ = (NH^q)^{-1}\, j_T^1 \Pi_T\, T^1(\bar{i}_T^{q-1}) + I_0^{-1}\, T^1((NH^{q-1})^{-1}\, \bar{i}_T^{q-1})\, j_T^{1*} \Pi_T^* \bar{i}_T^{q'} \bar{j}_T^q \Pi_T = \\ = (NH^q)^{-1}\, j_T^1 \Pi_T\, T^1(\bar{i}_T^{q-1}) + S^1(\bar{i}_S^{q-1})\, I_0^{-1}\, T^1(SH^{q-1})^{-1}\, j_T^{1*} \Pi_T^* \bar{i}_T^{q'}\, \bar{j}_T^q \Pi_T.$$

Noticing that – since $\{SH^q\}$ is a sequence of connections – $\bar{R}^q = 1$, we establish (4.64). On the other hand we get applying (2.65), (4.57), (2.67), (2.68) and (4.66)

$$S^1(\bar{i}_S^{q-1})\, \Omega_3 \Pi_T^*\, T^1(SH^{q-1})^{-1} = \\ = I_0^{-1}\, T^1(NH^{q-1})^{-1}\, j_T^{1*} \Pi_T^* T^1(\bar{i}_T^{q-1}) = (NH^q)^{-1}\, j_T^{1*} \Pi_T^*\, T^1(\bar{i}_T^{q-1}),$$

and adding this result to (4.64) we get finally (4.63).

Thus we have seen that the sequence of connections $\{\tilde{H}^q_S\}$ is reducible to the sequence $\{\bar{H}^q_S\}$ of Theorem 4.5. But from Theorem 4.15a we conclude that the sequence $\{\tilde{H}^q_T\}$ is reducible to the sequence $\{\bar{H}^q_T\}$ corresponding to $\{\bar{H}^q_S\}$ by means of $\{SH^q\}$, i.e. to the special sequence given in Theorem 4.10. Observing that both $\{\bar{H}^q_S\}$ and $\{\bar{H}^q_T\}$ are sequences of connections, we complete the proof.

Lemma 4.4. *Let $\{\tilde{H}^q_S\} \sim \{NH^q\} \sim \{\tilde{H}^q_T\}$ and let $\{\tilde{H}^q_T\}$ be a sequence of connections reducible to a sequence $\{H^q_T\}$ (and thus $\{NH^q\}$ reducible to the corresponding sequence $\{SH^q\}$). Then the sequence $\{\bar{H}^q_S\}$ connected with $\{SH^q\}$ by means of Theorem 4.5 coincides with the sequence corresponding to $\{\bar{H}^q_T\}$ by means of $\{SH^q\}$.*

Proof. We easily conclude that all the sequences in view are sequences of connections. By Theorem 4.16 the sequence $\{\tilde{H}^q_S\}$ is reducible to the sequence $\{\bar{H}^q_S\}$ obtained from $\{SH^q\}$ by means of Theorem 4.5. On the other hand Theorem 14b states that $\{\tilde{H}^q_S\}$ is reducible to the sequence corresponding to $\{\bar{H}^q_T\}$ by means of $\{SH^q\}$. Noticing that $\{\tilde{H}^q_S\}$ is reducible at the most to one sequence $\{\bar{H}^q_S\}$, we establish the lemma.

Up from now to the end of the paper suppose that we are given fixed connections H and h on E and the cotangent bundle $T(M)^*$ respectively. Our purpose is to show that the canonical sequence $\{\tilde{H}^q_T\}$ defined in (4.11) is reducible to the canonical sequence $\{\bar{H}^q_T\}$ defined in (4.50).

Lemma 4.5. *Let $q > 1$. Then the canonical connections satisfy*

$$[\bar{H}^{q-1}_T (\otimes) h]\, S^1(\Pi^*_T i^{q-1'}_T) = T^1(\Pi^*_T i^{q-1'}_T)\, \bar{H}^q_T\,, \tag{4.67}$$

i.e. $\bar{H}^{q-1}_T (\otimes) h$ and $\bar{H}^q_T$ induce the same R*-connection with respect to $\Pi^*_T i^{q-1'}_T$.*

Proof. We shall proceed by induction. If $q = 2$ then (4.67) becomes

$$[H (\otimes) h]\, S^1(\Pi^*_T) = T^1(\Pi^*_T)\, \bar{H}^2_T\,,$$

which follows immediately from (4.50). Thus let $q \geqq 3$ and suppose

$$[\bar{H}^{q-2}_T (\otimes) h]\, S^1(\Pi^*_T i^{q-2'}_T) = T^1(\Pi^*_T i^{q-2'}_T)\, \bar{H}^{q-1}_T\,. \tag{4.68}$$

Applying Lemma 3.4 to the "direct sum" $\bar{H}^{q-1}_T = \bar{H}^{q-2}_T (\oplus) \hat{H}^{q-2}_T$ and noticing from (4.51) that $\hat{H}^{q-2}_T (\otimes) h = \hat{H}^{q-1}_T$ we get

$$\bar{H}^{q-1}_T (\otimes) h = T^1(j^{q-2}_T \otimes 1)\,(\bar{H}^{q-2}_T (\otimes) h)\, S^1(\bar{\Pi}^{q-2}_T \otimes 1) + \\ + T^1(j^{q-2*}_T \otimes 1)\, \hat{H}^{q-1}_T\, S^1(\bar{\Pi}^{q-2*}_T \otimes 1)\,.$$

From this relation we get using succeedingly (2.67), (2.64), (2.74), (2.75), (4.68) and (2.69)

$$[\bar{H}^{q-1}_T (\otimes) h]\, S^1(\Pi^*_T)\, S^1(i^{q-1'}_T) = \\ = T^1(j^{q-2}_T \otimes 1)\,(\bar{H}^{q-2}_T (\otimes) h)\, S^1(\Pi^*_T)\, S^1(i^{q-2'}_T)\, S^1(\bar{\Pi}^{q-1}_T) +$$

$$+ T^1(\Pi_T^*)\, T^1(\bar{\imath}_T^{q-1'})\, T^1(j_T^{q-1*})\, \hat{H}_T^{q-1}\, S^1(\bar{\Pi}_T^{q-1*}) =$$
$$= T^1(j_T^{q-2} \otimes 1)\, T^1(\Pi_T^*)\, T^1(\bar{\imath}_T^{q-2'})\, \bar{H}_T^{q-1}\, S^1(\bar{\Pi}_T^{q-1}) +$$
$$+ T^1(\Pi_T^*)\, T^1(\bar{\imath}_T^{q-1'})\, T^1(j_T^{q-1*})\, \hat{H}_T^{q-1}\, S^1(\bar{\Pi}_T^{q-1*}) =$$
$$= T^1(\Pi_T^*)\, T^1(\bar{\imath}_T^{q-1'})\, \{T^1(j_T^{q-1})\, \bar{H}_T^{q-1}\, S^1(\bar{\Pi}_T^{q-1}) +$$
$$+ T^1(j_T^{q-1*})\, \hat{H}_T^{q-1}\, S^1(\Pi_T^{q-1*})\} = T^1(\Pi_T^*)\, T^1(\bar{\imath}_T^{q-1'})\, \bar{H}_T^q$$

and this completes the proof.

Theorem 4.17. *The canonical sequence of connections $\{\tilde{H}_T^q\}$ is reducible to the canonical sequence of connections $\{\bar{H}_T^q\}$ for any generating connections H and h.*

Proof. Since $\tilde{H}_T^1 = \bar{H}_T^1 = H$, (4.53) holds for $q = 1$ and $q = 2$. Thus let $q > 2$ and suppose again

$$\tilde{H}_T^{q-1}\, S^1(\bar{\imath}_T^{q-2}) = T^1(\bar{\imath}_T^{q-2})\, \bar{H}_T^{q-1}\,. \tag{4.69}$$

First from (2.70) and (2.76) we have

$$\Pi_T^* \bar{\imath}_T^{q-1} = \Pi_T^* \bar{\imath}_T^{q-1}(j_T^{q-1*}\bar{\Pi}_T^{q-1*} + j_T^{q-1}\bar{\Pi}_T^{q-1}) =$$
$$= (\bar{\imath}_T^{q-2}\, j_T^{q-2*} \otimes 1)\, \bar{\Pi}_T^{q-1*} + (\bar{\imath}_T^{q-2}\, j_T^{q-2} \otimes 1)\, \Pi_T^* \bar{\imath}_T^{q-2'} \bar{\Pi}_T^{q-1}$$

and hence we have the decomposition

$$T^1(j_T^{1*})\, [\tilde{H}_T^{q-1}\, (\otimes)\, h]\, S^1(\Pi_T^*)\, S^1(\bar{\imath}_T^{q-1}) = \omega_1 + \omega_2\,,$$

where

$$\omega_1 = T^1(j_T^{1*})\, [\tilde{H}_T^{q-1}\, (\oplus)\, h]\, S^1(\bar{\imath}_T^{q-2} j_T^{q-2*} \otimes 1)\, S^1(\bar{\Pi}_T^{q-1*})$$
$$\omega_2 = T^1(j_T^{1*})\, [\tilde{H}_T^{q-1}\, (\otimes)\, h]\, S^1(\bar{\imath}_T^{q-2} j_T^{q-2} \otimes 1)\, S^1(\Pi_T^*)\, S^1(\bar{\imath}_T^{q-2'})\, S^1(\bar{\Pi}_T^{q-1})\,.$$

From (4.69) and the recurrent formula for $\bar{H}_T^{q-1}$ we get

$$\tilde{H}_T^{q-1}\, S^1(\bar{\imath}_T^{q-2})\, S^1(j_T^{q-2*}) = T^1(\bar{\imath}_T^{q-2})\, \bar{H}_T^{q-1}\, S^1(j_T^{q-2*}) =$$
$$= T^1(\bar{\imath}_T^{q-2})\, T^1(j_T^{q-2*})\, \hat{H}_T^{q-2}$$

and

$$\tilde{H}_T^{q-1}\, S^1(\bar{\imath}_T^{q-2})\, S^1(j_T^{q-2}) = T^1(\bar{\imath}_T^{q-2})\, T^1(j_T^{q-2})\, \bar{H}_T^{q-2}\,.$$

Both these relations justify the application of Lemma 3.3 to $\Phi = \bar{\imath}_T^{q-2} j_T^{q-2*}$ and $\Phi = \bar{\imath}_T^{q-2} j_T^{q-2}$ respectively. So we get

$$\omega_1 = T^1(j_T^{1*})\, T^1(\bar{\imath}_T^{q-2} j_T^{q-2*} \otimes 1)\, \hat{H}_T^{q-1}\, S^1(\bar{\Pi}_T^{q-1*})$$

and

$$\omega_2 = T^1(j_T^{1*})\, T^1(\bar{\imath}_T^{q-2} j_T^{q-2} \otimes 1)\, [\bar{H}_T^{q-2}\, (\otimes)\, h] \times \\ \times S^1(\Pi_T^*)\, S^1(\bar{\imath}_T^{q-2'})\, S^1(\bar{\Pi}_T^{q-1})\,. \tag{4.70}$$

Combining these results we easily see that the relation (4.53) to be proved now becomes

$$T^1(j_T^1)\,\tilde{H}_T^{q-1}\,S^1(\Pi_T)\,S^1(\bar{i}_T^{q-1}) + (\omega_1 + \omega_2) =$$
$$= T^1(\bar{i}_T^{q-1})\,T^1(\bar{j}_T^{q-1})\,\bar{H}_T^{q-1}\,S^1(\bar{\Pi}_T^{q-1}) + \omega_1$$

and it remains to show that ω_2 equals

$$T^1(\bar{i}_T^{q-1})\,T^1(\bar{j}_T^{q-1})\,\bar{H}_T^{q-1}\,S^1(\Pi_T^{q-1}) - T^1(j_T^1)\,\tilde{H}_T^{q-1}\,S^1(\Pi_T)\,S^1(\bar{i}_T^{q-1}),$$

which can be transformed by Lemma 2.3, (4.69) and (2.78) into

$$\{T^1(\bar{i}_T^{q-1})\,T^1(\bar{j}_T^{q-1}) - T^1(j_T^1)\,T^1(\bar{i}_T^{q-2})\}\,\bar{H}_T^{q-1}\,S^1(\bar{\Pi}_T^{q-1}) =$$
$$= T^1(j_T^1)\,T^1(\bar{i}_T^{q-2}\bar{j}_T^{q-2} \otimes 1)\,T^1(\Pi_T^*\bar{i}_T^{q-2'})\,\bar{H}_T^{q-1}\,S^1(\bar{\Pi}_T^{q-1}).$$

But applying Lemma 4.5 to ω_2 in (4.70) we get immediately the required equation and this completes the proof.

An evident combination of this theorem and Lemma 4.4 establishes now the proof of Lemma 4.3.

From Theorems 4.17 and 4.15b we conclude

Theorem 4.18. *The canonical sequence of non-holonomic connections on E is reducible to the corresponding canonical sequence of semi-holonomic connections on E, whatever be both the generating connections on E and on the tangent bundle of M.*

We finish the paper with this result. Note only that it states de facto that "*iterating the computation of covariant derivatives* (to obtain higher order derivatives) *one need not care about the position of the derivatives of order zero*", a rule which one could quite expect by intuition.

However on the other hand an analogous result, concerning the reducibility of canonical semi-holonomic or non-holonomic connections to holonomic ones, cannot be expected in general. In fact, in this case the problem of reduction depends, roughly said, on the *curvature* of the generating connections. See for this purpose e.g. [3], where the question is discussed from a point of view very near to that employed in this paper.

References

[1] *Ch. Ehresmann:* Les prolongements d'une variété différentiable, C. R. Acad. Sci., Paris *233* (1951), 598—600, 777—779, 1081—1083; *239* (1954), 1762—1764; *240* (1955), 397—399, 1755—1757.

[2] *W. Greub:* Zur Theorie der linearen Übertragungen, Ann. Ac. Sci. Fennicae, Helsinki 1964.

[3] *P. Libermann:* Sur la géommétrie des prolongements des espaces fibrés vectroiels, Ann. Inst. Fourier, Grenoble *14* (1964), 145—172.

[4] *A. Lichnerowicz:* Théorie globale des connexions et des groupes d'holonomie, Roma 1955.
[5] *S. Mac Lane:* Homology, Berlin 1963.
[6] *J. M. Singer, S. Sternberg:* The infinitive Groups of Lie and Cartan, Part 1 (The Transitive Groups), Inst. of Technology, Harvard Univ. 1965 (polycopied notes.)
[7] *L. Schwartz:* Variedades Analiticas Complejas. Ecuaciones Diferenciales Parciales Elipticas, Bogota 1956.
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Резюме

НЕГОЛОНОМНЫЕ СВЯЗНОСТИ НА ВЕКТОРНЫХ РАССЛОЕННЫХ ПРОСТРАНСТВАХ

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Изучаются связности и псевдосвязности высших порядков на векторном расслоенном пространстве при помощи продолжений расслоенных пространств (в смысле Эресмана, см. [1]). Главное внимание обращается на неголономные и полуголономные продолжения и связанные с ними неголономные и полуголономные псевдосвязности и связности высших порядков. Главные понятия и результаты теории продолжений Эресмана векторных расслоенных пространств кратко приводятся в работе. Некоторые свойства этих продолжений получаются их сравнением с соответствуюими, так называемыми тензорными щпродолжениями исходного векторного пространства. Поставлена и решена проблема приведения последовательности неголономных псевдосвязностей всех порядков к последовательности полуголономных псевдосвязностей. Показывается, что данная связность на исходном векторном расслоенном пространстве вместе со связностью на базе порождают каноническую последовательность неголономных связностей высших порядков и каноническую последовательность полуголономных связностей, и что первая последовательность всегда приводима ко второй.

Пусть E – векторное расслоенное пространство с базой M. Пусть $S^1(E)$ – векторное расслоенное пространство, являющееся продолжением Эресмана первого порядка пространства E. Пусть, далее, $T^1(E) = E \oplus E.\otimes T(M)^*$, где $T(M)$ – векторное расслоенное пространство, касательное к базе M, и $T(M)^*$ сорпяжено с $T(M)$. Обозначим соответственно через Π_S и Π_T естественные проекции $S^1(E)$ и $T^1(E)$ на E, и через I_0 еэтественный изоморфизм ядра $\operatorname{Ker} \Pi_S$ проекции Π_S на ядро $\operatorname{Ker} \Pi_T$ проекции Π_T. Изоморфизм векторных расслоенных пространств $H : S^1(E) \to T^1(E)$ называется *псевдосвязностью* (первого порядка) на E. В частности, этот изоморфизм называется *связностью* на E, если $\Pi_T H = \Pi_S$ и если сужение H на $\operatorname{Ker} \Pi_S$ совпадает с I_0. Показывается, что в этом случае задание H эквивалентно заданию системы локальных диффферен-

циальных форм, и законы преобразования этих форм соответствуют законам преобразования компонент связности в классическом смысле слова. Получаются некоторые свойства псевдосвязностей и связностей, соответствующие известным фактам из классической теории связностей. Показывается в частности, как данная связность порождает связность на сопряженном пространстве и как связности на E и F порождают связность на $E \otimes F$.

Гоморфизм векторных расслоенных пространств $\Phi : E \to F$ порождает естественным образом гомоморфизмы $S^1(\Phi) : S^1(E) \to S^1(F)$ и $T^1(\Phi) : T^1(E) \to T^1(F)$ так, что S^1 и T^1 являются ковариантными функторами из категории векторных расслоенных пространств в себя. Изоморфизм $H_\Phi : S^1(E) \to T^1(F)$ называется связностью относительно гомоморфизма $\Phi : E \to F$, если $\Pi_T H_\Phi = \Phi \Pi_S$ и сужение H_Φ на $\operatorname{Ker} \Pi_S$ совпадает с $T^1(\Phi)\, I_0$.

Соответственно голономное, полуголономное и неголономное продолжения в смысле Эресмана порядка $q \geqq 1$ пространства E обозначаются через $S^q(E)$, $\bar{S}^q(E)$ и $\tilde{S}^q(E)$. С другой стороны $T^q(E), \bar{T}^q(E)$ и $\tilde{T}^q(E)$ обозначают соответственно голономное, полуголономное и неголономное тензорные продолжения пространства E, определенные рекуррентно (см. (2.16), (2.17) и (2.18) в работе. Имеются естественные вложения $\bar{i}^q_T : \bar{T}^q(E) \to \tilde{T}^q(E)$, $\bar{i}^q_S : \bar{S}^q(E) \to \tilde{S}^q(E)$ и проекции $\bar{\Pi}^q_T : \bar{T}^q(E) \to \bar{T}^{q-1}(E)$, $\bar{\Pi}^q_S : \bar{S}^q(E) \to \bar{S}^{q-1}(E)$. Через $\bar{I}^q_0$ обозначается изоморфизм ядра $\operatorname{Ker} \bar{\Pi}^q_S$ на $\operatorname{Ker} \bar{\Pi}^q_T$.

Голономной, полуголономной или неголономной псевдосвязностью порядка q на E называется соответственно изоморфизм $HH^q : S^q(E) \to T^q(E)$, $SH^q : \bar{S}^q(E) \to \bar{T}^q(E)$ и $NH^q : \tilde{S}^q(E) \to \tilde{T}^q(E)$. Но оказывается, что более удобным является рассматривать не отдельные псевдосвязности высших порядков на E, но их *последовательности* постепенных порядков, начиная с порядка первого, или конечных или бесконечных. Все результаты в работе формулированы для бесконечных последовательностей (т.е. содержащих одновременно псевдосвязности всех порядков), но нетрудно „переводить“ эти результаты на случай последовательностей, оканчивающихся псевдосвязностью определенного порядка.

Последовательность $\{NH^q\}$ называется *последовательностью неголономных связностей* на E, если NH^1 является связностью (первого порядка) на E и если свякое NH^q, $(q > 1)$ представляет собой связность относительно NH^{q-1}. Заметим, что не определяется понятие отдельной неголономной связности данного порядка, хотя иногда и говорим, что $\{NH^q\}$ „состоит из связностей“. Показывается, что существует „одно-одно-однозначное“ соответствие между всеми последовательностями $\{NH^q\}$ неголономных псевдосвязностей и последовательностями $\{\tilde{H}^q_S\}$ и $\{\tilde{H}^q_T\}$, обозначаемое как $\{\tilde{H}^q_S\} \sim \{NH^q\} \sim \{\tilde{H}^q_T\}$. Здесь всякое $\tilde{H}^q_S$ или $\tilde{H}^q_T$ есть некоторая псевдосвязность (первого порядка) соответственно на $\tilde{S}^{q-1}(E)$ или $\tilde{T}^{q-1}(E)$. Доказывается, что если $\{\tilde{H}^q_S\} \sim \{NH^q\} \sim \{\tilde{H}^q_T\}$ и одна из этих последовательностей состоит из связностей, то и остальные две состоят из связностей. Но так как определенная связность H первого порядка на E вместе со связностью h на $T(M)$ очевидным образом порождают некоторую связность $\tilde{H}^q_T$ на тензорном продолжении $\tilde{T}^{q-1}(E)$ для всякого $q \geqq 1$, мы получаем *канонические последовательности* $\{\tilde{H}^q_S\} \sim \{NH^q\} \sim \{\tilde{H}^T_q\}$, *порожденные парой* (H, h).

Однако в полуголономном случае положение не так просто. Последовательность $\{SH^q\}$ называется *последовательностью полуголономных связностей* на E, если SH^1 является связностью на E (первого порядка) и всякое SH^q $(q > 1)$ удовлетворяет отношению $\Pi_T\, SH^q = SH^{q-1}\, \Pi_S$ и сужение SH^q на ядро $\operatorname{Ker} \overline{\Pi}{}^q_S$ совпадает с $\bar{I}^q_0$. Опять не определяется понятие отдельной полуголономной связности данного порядка, хотя иногда и говорим что $\{SH^q\}$ „состоит из связностей". Однако если SH^q принадлежит некоторой последовательности полуголономных связностей, то имеем $SH^q = SH^{q-1}\, \overline{\Pi}{}^q_S + ZH^q$, и здесь слагаемое ZH^q точно соответствует понятию полуголономной связности порядка q, рассматриваемой в [3].

Вводится понятие последовательности $\{\overline{H}{}^q_S\}$ или последобательности $\{\overline{H}{}^q_T\}$, сопряженной с последовательностью полуголономных псевдосвязностей $\{SH^q\}$. При этом $\overline{H}{}^q_S$ и $\overline{H}{}^q_T$ $(q \geqq 1)$ обозначают некоторую псевдосвязность (первого порядка) соответственно на $\bar{S}^{q-1}(E)$ и $\bar{T}^{q-1}(E)$. Определяются *регулярные* последовательности $\{\overline{H}{}^q_S\}$, $\{\overline{H}{}^q_T\}$ и $\{SH^q\}$. При этом всякая последовательность $\{SH^q\}$, состоящая из связностей, автоматически является регулярной. При данной последовательности $\{SH^q\}$ вводится понятие соответствия последовательностей $\{\overline{H}{}^q_S\}$ и $\{\overline{H}{}^q_T\}$. Это соответствие (зависящее от $\{SH^q\}$) является одно-однозначным. Если при данной последовательности $\{SH^q\}$ последовательность $\{\overline{H}{}^q_S\}$ соответствует $\{\overline{H}{}^q_T\}$, то $\{\overline{H}{}^q_S\}$ сопряжена с $\{SH^q\}$, регулярна или состоит из связностей тогда и только тогда, если последовательность $\{\overline{H}{}^q_T\}$ соответственно сопряжена с $\{SH^q\}$, регулярна или состоит из связностей.

Доказывается, что к данной регулярной последовательности$\{\overline{H}{}^q_S\}$ существует точно одна последовательность $\{SH^q\}$, сопряженная с ней. Последняя тоже регулярна и состоит из связностей, если это верно для исходной $\{\overline{H}{}^q_S\}$. Более того, если $\{\overline{H}{}^q_S\}$ сопряжена с $\{SH^q\}$ и одна из этих последовательностей регулярна, то вторая тоже регулярна. Однако, регулряная последовательность $\{SH^q\}$ не определяет однозначно последовательность $\{\overline{H}{}^q_S\}$ с ней сопряженную, но в работе доказывается существование всегда хотя бы одной такой последовательности $\{\overline{H}{}^q_S\}$. Именно, для данной регулярной последовательности $\{SH^q\}$ строится явно последовательность $\{\overline{H}{}^q_S\}$, обладающая тем свойством, что она состоит из связностей, если $\{SH^q\}$ состоит из связностей.

Аналогичные результаты имеют место и в случае сопряженности последовательностей $\{SH^q\}$ и $\{\overline{H}{}^q_T\}$.

Скажем, что последовательность неголономных псевдосвязностей $\{NH^q\}$ *приводима к последовательности полуголономных псевдосвязностей* $\{SH^q\}$, если для всякого $q \geqq 1$ имеем $\bar{\imath}^q_T\, SH^q = NH^q\, \bar{\imath}^q_S$. Последовательность $\{NH^q\}$ приводима не более чем к одной последовательности $\{SH^q\}$, и если $\{NH^q\}$ состоит из связностей, то то же самое верно для $\{SH^q\}$. Последовательность $\{\tilde{H}^q_S\}$ или $\{\tilde{H}^q_T\}$ называется *приводимой к последовательности* $\{\bar{H}^q_S\}$ или $\{\bar{H}^q_T\}$, если соответственно $\tilde{H}^q_S\, S^1(\bar{\iota}^{q-1}_S) = T^1(\bar{\iota}^{q-1}_S)\, \bar{H}^q_S$ или $\tilde{H}^q_T\, S^1(\bar{\iota}^{q-1}_T) = T^1(\bar{\iota}^{q-1}_T)\, \bar{H}^q_T$ для всякого $q \geqq 1$, и в этом случае последовательность, состоящая из связностей приводима только к последовательности связностей. Доказывается, что если $\{\tilde{H}^q_S\} \sim \{NH^q\} \sim \{\tilde{H}^q_T\}$ и $\{\tilde{H}^q_S\}$ приводима к регулярной последовательности $\{\bar{H}^q_S\}$ (или $\{\tilde{H}^q_T\}$ приводима к регулярной последовательности $\{\bar{H}^q_T\}$), то $\{NH^q\}$ приводима к регулярной последовательности $\{SH^q\}$. Наоборот, если $\{\tilde{H}^q_S\} \sim \sim \{NH^q\} \sim \{\tilde{H}^q_T\}$, $\{NH^q\}$ состоит из *связностей* и приводима к $\{SH^q\}$, то и последовательности $\{\tilde{H}^q_S\}$ и $\{\tilde{H}^q_T\}$ приводимы к некоторым последовательностям $\{\bar{H}^q_S\}$ и $\{\bar{H}^q_T\}$, состоящим из связностей.

Опять пара (H, h) связностей соответственно на E и $T(M)$ порождает *каноническую последовательность* $\{\bar{H}^q_T\}$. Эта последовательность регулярна, и каноническая последовательность $\{\tilde{H}^q_T\}$, порожденная той же самой парой (H, h), всегда к ней приводима. Тем самым пара (H, h) однозначно прождает и канонические последовательности $\{SH^q\}$ и $\{\bar{H}^q_S\}$.

Наконец заметим, что аналогичное утверждение о приводимости всякой канонической последовательности полуголономных связностей к соответствующей канонической последовательности *голономных* связностей (определяемой очевидным образом) вообще не верно. Здесь проблема связана с „кривизной" порождающих связностей и в работе не решается, но см. [3].