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INNER GEOMETRY OF SUBMANIFOLDS OF HOMOGENEOUS SPACES

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It is well known from the classical differential geometry that the central role in the theory of hypersurfaces of Euclidean, affine and projective spaces is played by certain quadratic and cubic differential forms. The definition of these forms is always given by the structure of the space, and presumably they have nothing in common for different spaces. In this paper I show that there are certain invariant G-structures on submanifolds in homogeneous spaces, these G-structures being just the generalisation of the forms which occur in the known situations.

1. Let G be a Lie group and H its closed subgroup. Assume that G is a linear group, and let the normalizer of H coincide with H, i.e., let

$$\mathrm{ad}\,(g)\,\mathfrak{h} \subset \mathfrak{h} \Rightarrow g \in H\,, \tag{1.1}$$

$$\mathrm{ad}\,(A)\,\mathfrak{h} \subset \mathfrak{h} \Rightarrow A \in \mathfrak{h}\,. \tag{1.2}$$

In the Lie algebra $\mathfrak{g}$, we have

$$[A, B] = AB - BA\,, \quad \mathrm{ad}\,(g)\,A = gAg^{-1}\,, \quad \mathrm{ad}\,(B)\,A = [B, A] \quad \text{for} \quad A, B \in \mathfrak{g}\,, \quad g \in G\,. \tag{1.3}$$

Recall the known relation

$$\mathrm{ad}\,(g)\,[A, B] = [\mathrm{ad}\,(g)\,A, \mathrm{ad}\,(g)\,B] \quad \text{for} \quad A, B \in \mathfrak{g}\,, \quad g \in G\,. \tag{1.4}$$

Let N be a natural number, $N \leqq \dim G$. Denote by $\mathrm{St}\,(N)$ the Stiefel manifold of all N-frames in $\mathfrak{g}$; analoguously, let $\mathrm{Gr}\,(N)$ be the Grassman manifold of subspaces of dimension N in $\mathfrak{g}$, $\pi : \mathrm{St}\,(N) \to \mathrm{Gr}\,(N)$ be the natural projection. The full linear group $GL(N, \mathbf{R})$ operates on $\mathrm{St}\,(N)$ to the right according to the rule

$$(A_1, \ldots, A_N)\,(a_i^j) = \Big(\sum_{i=1}^{N} a_1^i A_i, \ldots, \sum_{i=1}^{N} a_N^i A_i\Big)\,; \quad (A_1, \ldots, A_N) \in \mathrm{St}\,(N)\,, \quad (a_i^j) \in GL(N, \mathbf{R})\,. \tag{1.5}$$

The Lie algebra $\mathfrak{gl}(N, \mathbf{R})$ operates on $\mathfrak{g}^N = \mathfrak{g} \times \ldots \times \mathfrak{g}$ (N-times) to the right according to the same rule.

Consider the homogeneous space G/H; let $\pi : G \to G/H$ be the natural projection. The group G acts in the well known manner on G/H according to the rule $L_g(g_1 H) = (gg_1) H$.

2. Let M be a manifold, $\dim M \leq \dim G/H$, and be given an embedding $V : M \to G/H$. The lift of this embedding is any mapping $v : M \to G$ such that the diagram

$$\begin{array}{ccc} & \overset{v}{\nearrow} & G \\ M & & \downarrow \pi \\ & \underset{V}{\searrow} & G/H \end{array} \tag{2.1}$$

commutes. To each lift $v : M \to G$, we associate the $\mathfrak{g}$-valued 1-form ω on M defined by

$$\omega = v^{-1} \, \mathrm{d}v . \tag{2.2}$$

The form ω satisfies the integrability condition

$$\mathrm{d}\omega = -\omega \wedge \omega . \quad \text{i.e.,} \quad (\mathrm{d}\omega)(X, Y) = -\tfrac{1}{2}[\omega(X), \omega(Y)] . \tag{2.3}$$

Further, we have

$$\omega(T_m(M)) \cap \mathfrak{h} = 0 \quad \text{for each} \quad m \in M . \tag{2.4}$$

$v : M \to G$ being a lift of V, any other lift of V is obtained as follows: choose any mapping $h : M \to H$ and set

$$v'(m) = v(m) \, h(m) \quad \text{for} \quad m \in M . \tag{2.5}$$

If ω' is the 1-form associated to the lift v' (2.5) then

$$\omega' = \mathrm{ad}\,(h(m)^{-1}) \, \omega + h(m)^{-1} \, \mathrm{d}h\,(m) , \tag{2.6}$$

this being well known.

For the sake of simplicity, let us write

$$K(m) = \omega(T_m(M)) , \quad L(m) = K(m) \oplus \mathfrak{h} ; \tag{2.7}$$

of course, $K(m) \in \mathrm{Gr}\,(\dim M)$ according to (2.4). Denoting by $\mathrm{Gr}\,(\mathfrak{h}, N)$ the set of all subspaces L of the algebra $\mathfrak{g}$ such that $\mathfrak{h} \subset L$ and $\dim L = \dim \mathfrak{h} + N$, we have $L(m) \in \mathrm{Gr}\,(\mathfrak{h}, \dim M)$. Replacing the lift v by the lift v' (2.5), we get

$$L'(m) = \mathrm{ad}\,(h(m)^{-1}) \, L(m) \tag{2.8}$$

as follows from (2.6).

Let $L \in \mathrm{Gr}(\mathfrak{h}, N)$. Denote by $H(L)$ the set of all $h \in H$ such that

$$L = \mathrm{ad}(h^{-1}) L. \tag{2.9}$$

The set $H(L)$ is a Lie group and its Lie algebra $\mathfrak{h}(L)$ consists of all vectors $A \in \mathfrak{h}$ satisfying

$$[A, L] \subset L; \tag{2.10}$$

see [1].

Suppose that, for a given embedding $V: M \to G/H$, there is a lift $v: M \to G$ such that

$$L(m) = L \quad \text{for each} \quad m \in M, \tag{2.11}$$

$L \in \mathrm{Gr}(\mathfrak{h}, \dim M)$ being a fixed space. The lifts with this property are called the *tangent lifts* of the embedding V; the construction of a tangent lift from an arbitrary lift v'' consists of the construction of a mapping $h: M \to H$ such that $\mathrm{ad}(h(m)^{-1})\,.\,L''(m) = L$ for each $m \in M$. In each concrete case, we have to decide whether such a map h exists; the only known general result is contained in [1] according to which such an h exists, at least in a neighborhood of a point $m_0 \in M$, if $\dim \mathfrak{h} - \dim \mathfrak{h}(L(m_0))$ is sufficiently large. Now, let there be given a tangent lift $v: M \to G$ satisfying (2.11). We obtain every other tangent lift $v': M \to G$ satisfying $L'(m) = L$ if we choose a mapping $h: M \to H(L)$ and set (2.5). If $P(V(M), H)$ is the principal fiber bundle generated from the homogeneous space $G(G/H, H)$ by the restriction of the base manifold to $V(M)$, the existence of a tangent lift ensures the existence of the reduction of the structural group H of P to $H(L)$.

Let us consider the vector space $L/\mathfrak{h}$; of course, $\dim L/\mathfrak{h} = \dim M$. Let $\pi: L \to L/\mathfrak{h}$ be the natural projection. If $A, B \in L$, $A - B \in \mathfrak{h}$, $h \in H(L)$ then $\mathrm{ad}(h)(A - B) \in \mathfrak{h}$, i.e., $\pi\,\mathrm{ad}(h) A = \pi\,\mathrm{ad}(h) B$. We have thus proved that there is a representation of the group $H(L)$ in $L/\mathfrak{h}$ denoted by

$$\mathrm{ad}: H(L) \to \mathrm{Aut}(L/\mathfrak{h}), \tag{2.12}$$

its precise definition being as follows: Let $A_0 \in L/\mathfrak{h}$, and let $A \in L$ be any element such that $\pi A = A_0$; if $h \in H(L)$ then $\mathrm{ad}(h) A_0 = \pi(\mathrm{ad}(h) A)$. Let $V: M \to G/H$ be an embedding and $v: M \to G$ its tangent lift satisfying (2.11). Because the map $\omega: T_m(M) \to L/\mathfrak{h}$ is an isomorphism for each $m \in M$ we get naturally an $\mathrm{ad}(H(L))$-structure on M. This structure may be called *the inner geometry of the first order* on M. The just introduced structure does not depend on the choice of the space L in (2.11). Indeed, let $v': M \to G$ be another tangent lift of the imbedding $V: M \to G$ such that

$$L'(m) = L_1. \tag{2.13}$$

There is a mapping $h : M \to H$ such that we have (2.5) and (2.8); in our case, $L_1 = \operatorname{ad}(h(m)^{-1}) L$. Let $h_1 \in H(L_1)$. Then $\operatorname{ad}(h_1^{-1}) L_1 = L_1$ yields $\operatorname{ad}(h(m) . h_1^{-1} h(m)^{-1}) L = L$, i.e., $h(m) h_1 h(m)^{-1} \in H(L)$, i.e., $h_1 \in h(m)^{-1} H(L) h(m)$, and we have $H(L_1) = h(m)^{-1} H(L) h(m)$. If we write $h(m_0) = h_0$ for an arbitrary but fixed point $m_0 \in M$, the lift $v''(m) = v(m) h_0$ is a tangent lift, and we have $L''(m) = L_1$ for $m \in M$. Because of that we have $H(L_1) = h_0^{-1} H(L) h_0$ what is to be proved.

Let $V: M \to G/H$ be an embedding and $v : M \to G$ its tangent lift satisfying (2.11). Because $\omega(X) \in L$ for each vector $X \in T_m(M)$ and each $m \in M$, we have $(\mathrm{d}\omega)(X, Y) \in L$ identically. From (2.3), we have

$$[\omega(X), \omega(Y)] \in L \quad \text{for each} \quad X, Y \in T_m(M), \quad m \in M. \tag{2.14}$$

Denote by $\operatorname{Gr}(L, N)$ the set of all subspaces K such that $K \subset L \subset \mathfrak{g}$, $\dim K = N$ and $[K, K] \subset L$. It is therefore obvious that $\omega(T_m(M)) \in \operatorname{Gr}(L, \dim M)$ for each $m \in M$. It is well possible that there is a lift $v : M \to G$ such that $\omega(T_m(M)) \subset H(L) \oplus K_0$ for each $m_0 \in M$, $K_0 \in \operatorname{Gr}(L, \dim M)$ being a fixed element. We may call such a lift the *osculating lift*. The main thing is that we are now able to repeat all previous considerations replacing the group H by $H(L)$ and the space L by the space $H(L) \oplus K_0$. On the manifold M, we get the inner geometry of the second order. In the optimal case, we continue in the reduction of the group H up to the moment when H is reduced to a finite group; the reduction to the identity would mean an orientation of M.

3. In this section, I will present another more geometric definition of the inner geometry of the first order.

Be given an embedding $V: M \to G/H$ and an arbitrary lift $v : M \to G$. We associate to V a mapping $\mathfrak{V} : M \to \operatorname{Gr}(\dim H)$ defined by

$$\mathfrak{V}(m) = \operatorname{ad}(v(m))\, \mathfrak{h} \quad \text{for} \quad m \in M. \tag{3.1}$$

The mapping $\mathfrak{V}$ does not depend on the choice of the lift v. Let $V' : M' \to G/H$ be another embedding, suppose $\dim M = \dim M'$. Let $m_0 \in M$ be a fixed point, and let there be given a local diffeomorphism $F : U \to M'$, $U \subset M$ being a neighborhood of the point m_0. The map F is the *deformation of order k at the point m_0 realized by $g \in G$* if the following is true: Consider the mappings

$$(g\mathfrak{V}) \circ F : U \to \operatorname{Gr}(\dim H) \quad \text{where} \quad (g\mathfrak{V})(m) = \operatorname{ad}(g)\, \mathfrak{V}(m); \qquad \mathfrak{V}' : U \to \operatorname{Gr}(\dim H); \tag{3.2}$$

then there are lifts

$$\varrho : U \to \operatorname{St}(\dim H), \quad \sigma : U \to \operatorname{St}(\dim H) \tag{3.3}$$

of these mappings such that

$$j^k_{m_0}(\varrho) = j^k_{m_0}(\sigma). \tag{3.4}$$

Because $\mathrm{St}(\dim H) \subset \mathfrak{g}^{\dim H}$, ϱ and σ are mappings of the neighborhood U into the vector space $\mathfrak{g}^{\dim H}$; $j^k_{m_0}(\varrho)$ is the jet of order k of ϱ at the point m_0.

Let us specialize the previous situation. Be given an embedding $V: M \to G/H$, its fixed lift $v: M \to G$, the mapping $\mathfrak{V}$ (3.1) and a fixed point $m_0 \in M$. Let G_k the set of $g \in G$ such that there is a neighborhood $U \subset M$ of the point m_0 and a mapping $F: U \to M$, $F(m_0) = m_0$, which is the deformation of order k of the embeddings V' and gV at m_0 realized by g. Now, let $\mathscr{B}$ be a fixed basis of the space $\mathfrak{h}$.

$$\varrho(m) = \mathrm{ad}\,(gv(F(m)))\,\mathscr{B} \tag{3.5}$$

is a lift of the mapping $(g\mathfrak{V}) \circ F$ and $\sigma'(m) = \mathrm{ad}\,(v/m))\,\mathscr{B}$ is a lift of the mapping $\mathfrak{V}$. An arbitrary lift of the mapping $\mathfrak{V}$ is

$$\sigma(m) = \mathrm{ad}\,(v(m))\,\mathscr{B}S(m) \tag{3.6}$$

where $S: U \to GL(\dim H, \mathbf{R})$ is arbitrary. Thus we have $g \in G_k$ if and only if there is a neighborhood $U \subset M$ of m_0 and mappings $F: U \to M$, $F(m_0) = m_0$, $S: U \to \to GL(\dim H, \mathbf{R})$ such that

$$j^k_{m_0}(\varrho) = j^k_{m_0}(\sigma)\,. \tag{3.7}$$

We have $\varrho_0 = \mathrm{ad}\,(gv_0)\,\mathscr{B}$, $\sigma_0 = \mathrm{ad}\,(v_0)\,\mathscr{B}S_0$ where $\varrho(m_0) = \varrho_0$ etc. The condition $(3.7)_{k=0}$ yields $\varrho_0 = \sigma_0$, i.e.,

$$\mathrm{ad}\,(v_0^{-1}gv_0)\,\mathscr{B} = \mathscr{B}S_0\,. \tag{3.8}$$

G_0 consists of the elements $g \in G$ such that there is $S_0 \in GL(\dim H, \mathbf{R})$ satisfying (3.8). $\mathscr{B}$ being a basis of $\mathfrak{h}$, we have $v_0^{-1}gv_0 \in H$ and $g \in v_0Hv_0^{-1}$. This yields

$$G_0 = v_0Hv_0^{-1}\,, \quad \mathfrak{g}_0 = \mathrm{ad}\,(v_0)\,\mathfrak{h}\,, \tag{3.9}$$

and we may write $g = v_0hv_0^{-1}$, $h \in H$,

$$\varrho(m) = \mathrm{ad}\,(v_0hv_0^{-1}\,v(F(m)))\,\mathscr{B}\,, \tag{3.10}$$

and (3.8) reduces to

$$\mathrm{ad}\,(h)\,\mathscr{B} = \mathscr{B}S_0\,. \tag{3.11}$$

Let us now determine G_1. From (3.10), we get

$$\varrho(m)\,v_0hv_0^{-1}\,v(F(m)) = v_0hv_0^{-1}\,v(F(m))\,\mathscr{B}\,. \tag{3.12}$$

Let X be a vector field on M; it is sufficient to consider X only on a neighborhood $U \subset M$ of the point m_0. From (2.2), we get

$$Xv = v\,\omega(X)\,. \tag{3.13}$$

Denote by

$$J(m) = (\mathrm{d}F)_m : T_m(M) \to T_m(M) \tag{3.14}$$

the differential of F at the point $m \in U$; evidently,

$$J_0 \equiv J(m_0) \in \operatorname{Aut}(T_{m_0}(M)) . \tag{3.15}$$

The known formula for the differential of a composed mapping and (3.13) yield

$$X(v_0 F) = ((\mathrm{d}F)\, X)\, v = v\, \omega((\mathrm{d}F)\, X) ,$$

i.e.,

$$X v(F(m)) = v(F(m))\, \omega(J(m)\, X) . \tag{3.16}$$

Applying the vector field X to (3.12), we get

$$\begin{aligned} X\, \varrho(m)\, v_0 h v_0^{-1}\, v(F(m)) + \varrho(m)\, v_0 h v_0^{-1}\, v(F(m))\, \omega(J(m)\, X) = \\ = v_0 h v_0^{-1}\, v(F(m))\, \omega(J(m)\, X) \mathscr{B} , \end{aligned}$$

i.e.,

$$X\, \varrho(m) = \operatorname{ad}(v_0 h v_0^{-1}\, v(F(m)))\, [\omega(J(m)\, X), \mathscr{B}] , \tag{3.17}$$

$$X\, \varrho(m_0) = \operatorname{ad}(v_0 h)\, [\omega(J_0 X), \mathscr{B}] . \tag{3.18}$$

From (3.6), we get

$$\sigma(m)\, v(m) = v(m) \mathscr{B}\, S(m) ;$$

applying X, we obtain

$$\begin{aligned} X\, \sigma(m)\, v(m) + \sigma(m)\, v(m)\, \omega(X) = \\ = v(m)\, \omega(X) \mathscr{B}\, S(m) + v(m) \mathscr{B} X\, S(m) , \end{aligned}$$

i.e.,

$$X\, \sigma(m) = \operatorname{ad}(v(m))\, [\omega(X), \mathscr{B}\, S(m)] + \operatorname{ad}(v(m)) \mathscr{B} X\, S(m) , \tag{3.19}$$

$$X\, \sigma(m_0) = \operatorname{ad}(v_0)\, [\omega(X), \operatorname{ad}(h) \mathscr{B}] + \operatorname{ad}(v)_0) \mathscr{B} X\, S(m_0) . \tag{3.20}$$

The condition $(3.7)_{k=1}$ is now equivalent to

$$X_{\varrho(m_0)} = X_{\sigma(m_0)} \quad \text{for each } X . \tag{3.21}$$

According to (3.18) and (3.20), we may rewrite (3.21) as

$$[\omega(J_0 X) - \operatorname{ad}(h^{-1})\, \omega(X), B] = \operatorname{ad}(h^{-1}) \mathscr{B} X\, S(m_0) , \tag{3.22}$$

and we get: The element $g = v_0 h v_0^{-1}$, $h \in H$, belongs to G_1 if and only if there is an automorphism $J : T_{m_0}(M) \to T_{m_0}(M)$ such that

$$\omega(JX) - \operatorname{ad}(h^{-1})\,\omega(X) \in h \quad \text{for each} \quad X \in T_{m_0}(M)\,. \tag{3.23}$$

From (3.23), it is clear that $g = v_0 h v_0^{-1}$, $h \in H$, belongs to G_1 if and only if we have $\omega(T_{m_0}(M)) \oplus \mathfrak{h} = \omega'(T_{m_0}(M)) \oplus \mathfrak{h}$ for the lifts $v : M \to G$ and $v'(m) = v(m)\ h$. If $v : M \to G$ is a tangent lift satisfying (2.11) we get

$$G_1 = v_0\, H(L)\, v_0^{-1}\,, \quad \mathfrak{g}_1 = \operatorname{ad}(v_0)\,\mathfrak{h}(L)\,. \tag{3.24}$$

Thus we have explained the role of the group $H(L)$.

Bibliography

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Резюме

ВНУТРЕННАЯ ГЕОМЕТРИЯ ПОДМНОГООБРАЗИЙ ОДНОРОДНОГО ПРОСТРАНСТВА

АЛОИС ШВЕЦ (Alois Švec), Прага

Показывается, что на подмногообразиях однородного пространства существуют некоторые G-структуры, которые в частных случаях хорошо известны в классической дифференциальной геометрии.