# Czechoslovak Mathematical Journal

Vlastimil Pták Openness of linear mappings in LF-spaces

Czechoslovak Mathematical Journal, Vol. 19 (1969), No. 3, 547-552

Persistent URL: http://dml.cz/dmlcz/100921

## Terms of use:

© Institute of Mathematics AS CR, 1969

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

### OPENNESS OF LINEAR MAPPINGS IN LF-SPACES

VLASTIMIL РТА́к, Praha Received January 15, 1969

In a recent paper we have introduced and discussed the notion of orthogonality for pairs of subspaces of a locally convex space. There are many ways of characterizing this notion; these have been discussed at some length in [3]. For our purposes and for the particular case of Fréchet spaces, the following two will be sufficient.

Let Y and R be two closed subspaces of a Fréchet space E. We shall say that Y and R are *orthogonal* if one of the following two equivalent conditions is satisfied:

1° the natural mapping of  $R \oplus Y$  onto R + Y is open,

 $2^{\circ}$  given two continuous linear functionals on Y and R which coincide on  $R \cap Y$ , there exists a continuous linear functional on E which is their common extension.

Clearly the relation of orthogonality is symmetric; we shall write simply  $Y \perp R$  or  $R \perp Y$ .

In the present note we intend to introduce, for LF-spaces, a notion which describes the position of a subspace with respect to the spaces of a defining sequence. If R is a sequentially closed subspace of an LF-space F such that - roughly speaking - R is orthogonal to each element of a defining sequence for F, it turns out that R has many nice properties. In particular, this notion enables us to formulate a simple condition for a sequentially open mapping to be open. This condition - given in section two - is essentially equivalent to condition (4,4) of [2]. However, using the notion of orthogonality, both the statement and the proof of the theorem become exceedingly simple and transparent.

Some remarks concerning terminology and notation. If E is a locally convex space, we denote by U(E) the system of all closed absolutely convex neighbourhoods of zero in E. If Y is a subspace of E, we denote by P(Y) the operator which assigns to each  $x' \in E'$  its restriction to Y. If E is a subspace of an E-space E, we say that E is sequentially closed in E if E and E implies E implies E and E continuous linear mapping of an E-space E into an E-space E is said to be sequentially open if its range is sequentially closed in E. For other equivalent descriptions of this notion see Proposition (3,2) of [2].

#### 1. ORTHOGONAL SUBSPACES

(1,1) Proposition. If R is a sequentially closed subspace of an LF-space F, the following conditions are equivalent:

1° there exists a defining sequence  $F_j$  such that  $F_j \perp (R \cap F_k)$  for each k > j; 2° for each defining sequence  $F_j$  there exists a sequence p(j) of natural numbers,  $p(j) \geq j$ , with the following property: given k > p(j),  $y' \in F'_{p(j)}$  and  $r' \in (R \cap F_k)'$  which coincide on  $R \cap F_{p(j)}$ , there exists an  $x' \in F_k'$  such that  $P(F_j)$   $x' = P(F_j)$  y' and  $P(R \cap F_k)$  x' = r';

3° for each defining sequence  $F_j$  there exists another defining sequence  $P_j \supset F_j$  such that  $P_i \perp (R \cap P_k)$  for each k > j.

If one of these conditions is fulfilled we shall say that R is orthogonal in F.

Proof. Assume 1° and consider a defining sequence  $H_j$ . Given j, there exist indices m, r such that

$$H_i \subset F_m \subset H_r$$
.

Suppose that k > r and consider a  $y' \in H'_r$  and an  $r' \in (R \cap H_k)'$  such that y' and r' coincide on  $R \cap H_r$ .

Take an  $F_s \supset H_k$  so that s > m and consider an extension p' of r' to  $R \cap F_s$ . Since  $F_m \subset H_r$ , the functionals  $P(F_m)$  y' and p' coincide on  $R \cap F_m$ . Condition 1° being satisfied, there exists an  $x' \in F_s'$  such that  $P(F_m)$   $x' = P(F_m)$  y' and

$$P(R \cap F_s) x' = p'.$$

Put  $z' = P(H_k) x'$  so that  $z' \in H'_k$ . Since  $H_k \subset F_s$ , we have

$$P(R \cap H_k) z' = P(R \cap H_k) P(H_k) x' = P(R \cap H_k) x' =$$

$$= P(R \cap H_k) P(R \cap F_s) x' = P(R \cap H_k) p' = r'$$

on the other hand, since  $H_i \subset F_m \subset H_k$ 

$$P(H_j) z' = P(H_j) P(H_k) x' = P(H_j) x' = P(H_j) P(F_m) x' = P(H_j) P(F_m) y' = P(H_j) y'.$$

Hence it suffices to take p(j) = r and condition  $2^{\circ}$  is satisfied.

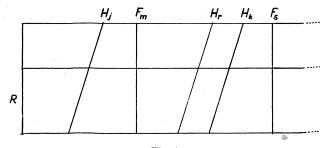


Fig. 1.

Suppose now that we are given a defining sequence  $F_j$  which satisfies condition  $2^\circ$ . Define first an increasing sequence of natural numbers z(n),  $n \in \mathbb{N}$ , as follows: z(1) = 1 and z(n+1) = p(z(n)) + 1. It follows that z(n+1) > z(n) and  $p(z(n+1)) \ge 2$  z(n+1) > p(z(n)). Define  $P_j$  as the closure in  $F_{p(z(j))}$  of the space

$$F_{z(j)} + (R \cap F_{p(z(j))});$$

it follows that

$$(1) F_{z(j)} \subset P_j \subset F_{p(z(j))}$$

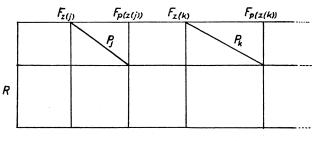


Fig. 2.

whence  $R \cap P_j \subset R \cap F_{p(z(j))}$ . Since, on the other hand,  $P_j$  contains, by its definition, the intersection  $R \cap F_{p(z(j))}$ , we have

$$(2) R \cap P_j = R \cap F_{p(\mathbf{z}(j))}.$$

Let k > j and consider a pair of functionals  $y' \in P'_j$  and  $r' \in (R \cap P_k)'$  such that

$$(3) P(R \cap P_i) y' = P(R \cap P_i) r'.$$

Since  $P_j \subset F_{p(z(j))}$ , there exists a  $u' \subset F'_{p(z(j))}$  such that  $P(P_j) u' = y'$ . We have p(z(k)) > p(z(j)) and two functionals  $u' \in F'_{p(z(j))}$  and  $r' \in (R \cap P_k)' = (R \cap F_{p(z(k))})'$  which coincide on  $R \cap P_j = R \cap F_{p(z(j))}$ . It follows from condition  $2^\circ$  that there exists an  $x' \in F'_{p(z(k))}$  with the following properties

(4) 
$$P(F_{z(j)}) x' = P(F_{z(j)}) u'$$

(5) 
$$P(R \cap F_{p(z(k))}) x' = r'.$$

Since  $P_k \subset F_{p(z(k))}$ , we may form  $z' = P(P_k) x' \in P'_k$ . According to (4), we have

$$P(F_{z(j)}) z' = P(F_{z(j)}) P(P_k) x' = P(F_{z(j)}) x' = P(F_{z(j)}) u' =$$

$$= P(F_{z(j)}) P(P_j) u' = P(F_{z(j)}) y'.$$

It follows from (5) and (3) that

$$P(R \cap F_{p(z(j))}) z' = P(R \cap F_{p(z(j))}) x' = P(R \cap F_{p(z(j))}) r' =$$

$$= P(R \cap P_j) r' = P(R \cap P_j) y'.$$

It follows that  $z' \in P'_k$  coincides with y' on  $F_{z(j)} + (R \cap F_{p(z(j))})$  and hence on  $P_{j^*}$  Further, again by (5),

$$P(R \cap P_k) z' = P(R \cap F_{p(z(k))}) x' = r'.$$

This completes the proof of the orthogonality of  $P_j$  and  $(R \cap P_k)$ . Since 3° implies 1° immediately, the proof is complete.

We shall need the following simple lemma.

(1,2) Let H be an absolutely convex neighbourhood of zero in a locally convex space E. If  $0 < \xi < 1$ , then  $\xi \overline{H} \subset H$ .

Proof. If  $x \in \xi \overline{H}$ , the set  $x + (1 - \xi)H$  is a neighbourhood of x and hence intersects  $\xi H$ . It follows that  $x + (1 - \xi)h_1 = \xi h_2$  for suitable  $h_1, h_2 \in H$ . The set H being absolutely convex,  $h_3 = -h_1 \in H$  so that  $x = \xi h_2 + (1 - \xi)h_3 \in H$ .

#### 2. OPEN MAPPINGS

- **(2,1) Theorem.** Let E and F be two LF-spaces and let T be a continuous linear mapping of E into F. Suppose that the following two conditions are satisfied:
  - 1° the mapping T is sequentially open,
  - 2° the range of T is orthogonal in F.

Then T is open.

Proof. Denote by R the range of T so that R is sequentially closed by proposition (3,2) of [2]. Let  $U \in U(E)$  and let us show that there exists a  $V \in U(F)$  such that  $V \cap R \subset TU$ . We shall apply lemma (4,2) of [2]. Since R is orthogonal in F there exists, by (1,1), a defining sequence  $F_j$  of F such that  $F_j \perp R_k$  for each k > j where  $R_k = R \cap F_k$ . Since T is sequentially open, there exists, by (3,2) of [2], for each n a  $H_n \subset U(F)$  such that  $H_n \cap R_n \subset TU$ . Suppose now we are given f, f, f, f, with the following properties: f of f and f of f and f of f and f of f of f such that

$$(1) V_{j+1} \cap R_{j+1} \subset TU$$

$$(1-\varepsilon)(V_j\cap F_j)\subset V_{j+1}.$$

Take a  $H_{j+1} \in \mathbf{U}(F)$  such that  $H_{j+1} \cap R_{j+1} \subset TU$ . Since  $F_j \perp R_{j+1}$ , the mapping

 $F_j \oplus R_{j+1} \to F_j + R_{j+1}$  is open so that  $(V_j \cap F_j) + (H_{j+1} \cap R_{j+1})$  is a neighbourhood of zero in  $F_j + R_{j+1}$ . Accordingly, there exists a  $W \in U(F)$  such that

(3) 
$$W \cap (F_i + R_{i+1}) \subset (V_i \cap F_i) + (H_{i+1} \cap R_{i+1}).$$

With  $\sigma = \frac{1}{2}\varepsilon$  define

$$H = \operatorname{conv} ((1 - \sigma) (V_j \cap F_j), \sigma W).$$

Let us show now that  $H \cap R_{j+1} \subset TU$ . Suppose that  $p \in H \cap R_{j+1}$  so that p may be written in the form

$$p = \lambda \sigma w + (1 - \lambda)(1 - \sigma)z$$

with  $0 \le \lambda \le 1$ ,  $w \in W$ ,  $z \in V_j \cap F_j$ . If  $\lambda = 0$ , we have  $p = (1 - \sigma)z \in V_j \cap F_j$  and  $p \in R$  so that  $p \in V_j \cap R_j \subset TU$ . Hence we may suppose  $\lambda > 0$ . It follows that

$$\lambda \sigma w = -(1-\lambda)(1-\sigma)z + p \in F_i + R_{i+1}$$

whence  $w \in F_j + R_{j+1}$  so that, according to (3), the vector w may be written in the form  $w = z_0 + Tu_0$  for a suitable  $z_0 \in V_j \cap F_j$  and  $u_0 \in U$ . Since  $0 < \varepsilon < 1$ , we have  $0 < \sigma/(1 - \sigma) < 1$  so that  $z_{00} = [\sigma/(1 - \sigma)] z_0 \in V_j \cap F_j$  as well. We have thus

$$p = (1 - \lambda)(1 - \sigma)z + \lambda\sigma z_0 + \lambda\sigma Tu_0 =$$

$$= (1 - \lambda)(1 - \sigma)z + \lambda(1 - \sigma)z_{00} + \lambda\sigma Tu_0 = (1 - \sigma)z_{000} + \lambda\sigma Tu_0$$

with  $z_{000} \in V_j \cap F_j$ . It follows that

 $z_{000} \in V_j \cap F_j \cap R \subset TU$  so that  $z_{000} = Tu$  for some  $u \in U$ .

Hence

$$p = T[(1 - \sigma) u + \sigma \lambda u_0] \in TU.$$

This proves the inclusion  $H \cap R_{j+1} \subset TU$ .

To sum up: we have constructed an absolutely convex set H with the following properties:

(4) 
$$H$$
 is a neighbourhood of zero if  $F$  (since  $H \supset \sigma W$ )

$$(1-\sigma)(V_j\cap F_j)\subset H.$$

$$(6) H \cap R_{j+1} \subset TU$$

We intend to show now that it suffices to take  $V_{j+1} = \xi \overline{H}$  where  $\xi = (1 - \varepsilon)/(1 - \sigma)$ . First of all, the inclusion  $V_{j+1} \in U(F)$  is obvious. Since  $V_{j+1} = \xi \overline{H} \subset H$ , we have  $V_{j+1} \cap R_{j+1} \subset H \cap R_{j+1} \subset TU$  and

$$\left(1-\varepsilon\right)\left(V_{j}\cap F_{j}\right)=\xi\left(1-\sigma\right)\left(V_{j}\cap F_{j}\right)\subset \xi H\subset \xi\overline{H}=V_{j+1}\,.$$

The proof is complete.

(2,2) Corollary. Let  $F_j$  be an increasing sequence of Fréchet spaces such that the topology of  $F_{j+1}$  induces the topology of  $F_j$  on  $F_j$ . Let (F, u) be the inductive

limit of the sequence  $F_j$ . Let R be a sequentially closed subspace of F. Let (R, v) be the inductive limit of the sequence  $R \cap F_j$  so that v is finer than the restriction  $u_R$  of u to R. If R is orthogonal in F then  $v = u_R$ .

#### References

- [1] L. Hörmander: On the range of differential and convolution operators, Ann. of Math. 76 (1962), 148-170.
- [2] V. Pták: Some open mapping theorems in LF-spaces and their application to existence theorems for convolution equations, Math. Scand. 16 (1965), 75—93.
- [3] V. Pták: Simultaneous extensions of two linear functionals (in print).
- [4] W. Slowikowski: On the theory of inductive families, Studia Mathematica 26 (1965), 1-10.

Author's address: Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).