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FREE EXTENSIONS OF (a, b)-SYSTEMS

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The aim of the paper is to transfer some results about free groupoid extensions of halfgroupoids and free planar extensions of partial planes (cf. R. H. Bruck's A survey of binary systems, Springer 1955, pp. 1–8 and G. Pickert's Projektive Ebenen, Springer 1955, pp. 12–26) onto certain "free extensions" of systems consisting of some distinguished a-element subsets of a given set S_1 where each $(b + 1)$-element subset of S_1 is contained in at most one distinguished subset (a, b are integers such that $a \geqq b + 2$).

Let a, b be fixed integers such that $a \geqq b + 2$. An (a, b)-*system* (briefly: a *System*) is defined as a couple $S = (S_1, S_2)$ where S_1 is a set and S_2 is a set of distinguished a-element subsets of S_1 called *blocks* of S such that each $(b + 1)$-element subset of S is contained in at most one block. If moreover each $(b + 1)$-element subset of S_1 is contained in precisely one block then S is said to be *complete*. In the sequel we shall use for any System S the notation $S = (S_1, S_2)$. Further we shall restrict ourselves onto Systems with card $S_1 \geqq b + 1$.

A *sub-System* of a System S is defined as a System S' such that $S'_1 \subset S_1$, $S'_2 \subset S_2$ (notation $S' \Subset S$ or $S \Supset S'$ will mean that S', S are Systems such that S' is a sub-System of S). A sub-System S' of a System S is said to be *closed* in S if $Y \in S_2$, card $(Y \cap S'_1) \geqq b + 1 \Rightarrow Y \in S'_2$ (notation $S' \; \mathbf{cl} \; S$ will mean that $S' \Subset S$ and that S' is closed in S).

We shall start with two simple properties of closed sub-Systems which we shall state without proof:

(a) $S^{(1)}, S^{(2)} \; \mathbf{cl} \; S;\ \exists S' \Subset S^{(1)}, S^{(2)} \Rightarrow (S^{(1)}_1 \cap S^{(2)}_1, S^{(1)}_2 \cap S^{(2)}_2) \; \mathbf{cl} \; S$,

(b) $S^{(1)} \Subset S^{(2)} \Subset S^{(3)},\ S^{(1)} \; \mathbf{cl} \; S^{(2)} \; \mathbf{cl} \; S^{(3)} \Rightarrow S^{(1)} \; \mathbf{cl} \; S^{(3)}$.

If $S^{(1)} \Subset S^{(2)}$ and $(S^{(1)} \Subset S \Subset S^{(2)},\ S \; \mathbf{cl} \; S^{(2)} \Rightarrow S = S^{(2)})$ then we say that $S^{(1)}$ *generates* $S^{(2)}$ (notation $S^{(1)} \; \mathbf{g} \; S^{(2)}$ will mean that $S^{(1)}, S^{(2)}$ are Systems such that $S^{(1)}$ generates $S^{(2)}$).

Remark. Choose $a = 3$, $b = 1$, card $S_1^{(1)} = 3$, card $S_2^{(1)} = 0$, $S_1^{(1)} = S_1^{(2)}$, card $S_1^{(2)} = 1$. Then $S^{(1)}$ **g** $S^{(2)}$ without $S^{(1)} = S^{(2)}$.

Assertion 1. $S^{(1)}$ **g** $S^{(2)}$ **g** $S^{(3)} \Rightarrow S^{(1)}$ **g** $S^{(3)}$.

Proof. Let S be a System such that $S^{(1)} \Subset S \Subset S^{(3)}$ and S ***cl*** $S^{(3)}$. We have to show that $S = S^{(3)}$. In fact, form a System $S' = (S_1^{(2)} \cap S_1, S_2^{(2)} \cap S_2)$. Certainly $S^{(1)} \Subset S' \Subset S^{(2)}$. If $Y \in S_2^{(2)}$, card $(Y \cap S_1') \geqq b + 1$ then $Y \in S_2$ because of S ***cl*** $S^{(3)}$. Therefore $Y \in S_2'$ and we see that S' ***cl*** $S^{(2)}$. As $S^{(1)}$ **g** $S^{(2)}$ it follows $S' = S^{(2)}$ and then $S^{(2)} \Subset S$. Thus from $S^{(2)}$ **g** $S^{(3)}$ it follows $S = S^{(3)}$. Q.E.D.

Let $S \Subset S'$. Further let $S^0 = S$. If S^n is a System for some n then define S_2^{n+1} as the set $\{Y \in S_2' \mid \text{card}\,(Y \cap S_1^n) \geqq b + 1\}$ and S_1^{n+1} as the set $S_1^n \cup \bigcup_{Y \in S_2^{n+1}} Y$. Form the System $S_{S'} = \left(\bigcup_{n=0}^{\infty} S_1^n, \bigcup_{n=0}^{\infty} S_2^n\right) \Subset S'$. We shall call $(S^n)_0^\infty$ an *extension chain* over S in S' or an extension chain of $S_{S'}$. $S_{S'}$ is said to be a *closed extension* of S in S'. This notation is justified by the following assertion.

Assertion 2. $S \Subset S' \Rightarrow S_{S'}$ ***cl*** S'.

Proof. Let $Y \in S_2'$, card $(Y \cap (S_{S'})_1) \geqq b + 1$. In the extension chain $(S^n)_{n=0}^\infty$ of $S_{S'}$ there exists a term S^n such that $Y \cap (S_{S'})_1 \subset S_1^n$. Consequently $Y \in S_2^{n+1} \subset (S_{S'})_2$. Q.E.D.

Corollary. *If* $S \Subset S'$ *with* S' *complete then* $S' = S_{S'} \Rightarrow S$ **g** S'.

A *System map* $\sigma : S \to S'$ is defined as a couple (σ_1, σ_2) of maps $\sigma_1 : S_1 \to S_1'$, $\sigma_2 : S_2 \to S_2'$ where S and S' are Systems. If σ is a System map then denote $\sigma = (\sigma_1, \sigma_2)$. A System map $\sigma : S \to S'$ is called a *System surjection* (*bijection*) if both σ_1, σ_2 are surjections (bijections). A System map $\sigma : S \to S'$ is called a *System homomorphism* if $X \in Y \in S_2 \Rightarrow \sigma_1 X \in \sigma_2 Y$. Any surjective System homomorphism is called a *System epimorphism* and any bijective System epimorphism is called a *System isomorphism*. It can be easily verified that each System isomorphism must be a both-sided System epimorphism. If $\sigma : S \to S'$ is a System epimorphism and if there exists an $S'' \Subset S, S'$ such that $\sigma|_{S''}$ is the identity System map then σ is called a System epimorphism *over* S''.

Let S be a System. Put $S^{(0)} = S$. Let us have a System $S^{(n)} \Supset S$ for some n. Then take the set $T^{(n)}$ of all $(b + 1)$-element subsets in $S_1^{(n)}$ such that none of them is contained in any block of $S^{(n)}$. Further let $\varkappa^{(n)}$ be a map assigning to each $Z \in T^{(n)}$ a $(a - b - 1)$-element set such that $\varkappa^{(n)} Z$ for distinct $Z \in T^{(n)}$ are mutually disjoint sets and that each of them is also disjoint to $S_1^{(n)}$. Then define $S_1^{(n+1)}$ to be equal to $S_1^{(n)} \cup \bigcup_{Z \in T^{(n)}} \varkappa^{(n)} Z$ and $S_2^{(n+1)}$ to be equal to $S_2^{(n)} \cup \{Z \cup \varkappa^{(n)} Z \mid Z \in T^{(n)}\}$. Obviously $S^{(n)} \Subset S^{(n+1)}$. Consequently $\left(\bigcup_{n=0}^{\infty} S_1^{(n)}, \bigcup_{n=0}^{\infty} S_2^{(n)}\right)$ must be a complete System. This System will be called a *free extension* over S and $(S^{(n)})_{n=0}^{\infty}$ will be called a *free extension chain* over S. For each free extension over S we shall use the symbol $F(S)$ (up to System isomorphisms).

In the sequel we shall write $(S^n)_{n=0}^{\infty}$, $(S^{(n)})_{n=0}^{\infty}$, $(T^{(n)})_{n=0}^{\infty}$ with the same meaning as above.

Assertion 4. *If S **g** S' where S' is complete then there is a System epimorphism $\varphi : F(S) \to S'$ over S.*

Proof Let $\varphi^0 : S \to S$ be the identity System map. Further let there be given a System epimorphism $\varphi^n : S^{(n)} \to S^n$ over S for some n. Then construct a System map $\varphi^{n+1} : S^{(n+1)} \to S^{n+1}$ extending φ^n as follows. If $Z \in T^{(n)}$ then let φ_2^{n+1} assign to each block $\hat{Z} \in S_2^{(n+1)}$, $\hat{Z} \supset Z$ a block $\tilde{Z} \in S_2^{n+1}$, $\tilde{Z} \supset \varphi_1^n Z$: In case card $\varphi_1^n Z = b + 1$, $\tilde{Z} \supset \varphi_1^n Z$ implies that $\tilde{Z}$ is uniquely determined. In this case choose $\varphi_1^{n+1}|_{\hat{Z}\setminus Z} : \hat{Z} \setminus Z \to \tilde{Z} \setminus \varphi_1^n Z$ to be a surjection. When card $\varphi_1^n Z < b + 1$ then choose $\tilde{Z}$ as an arbitrary block of S^{n+1} containing $\varphi_1^n Z$ and define $\varphi_1^{n+1}|_{\hat{Z}\setminus Z}$ as an arbitrary map of $\hat{Z} \setminus Z$ into $\tilde{Z}$. As φ^{n+1} extends φ^n we have $X \in Y \in S_2^{(n)} \Rightarrow \varphi_1^{n+1} X \in \varphi_2^{n+1} Y$. For remaining $X \in Y \in S_2^{(n)}$ the validity of $\varphi_1^{n+1} X \in \varphi_2^{n+1} Y$ is guaranteed by the preceding construction. Now prove that $\varphi^{n+1} : S^{(n+1)} \to S^{n+1}$ is a System surjection. Take an arbitrary block $\bar{Y} \in S_2^{n+1} \setminus S_2^n$ so that necessarily card $(\bar{Y} \cap S_1^n) \geqq b + 1$. In $(\varphi_1^n)^{-1} \,.\, (\bar{Y} \cap S_1^n)$ choose a $(b + 1)$-element subset V such that also card $\varphi_1^n V = b + 1$ (this is always possible). If $V \notin T^{(n)}$ then there exists a block $W \in S_2^n$, $W \supset V$ and we have $\varphi_2^n W = \bar{Y} \in S_2^n$, a contradiction. Thus $V \in T^{(n)}$ and the starting block $\bar{Y}$ is the image of $\hat{V} \in S_2^{(n+1)}$, $\hat{V} \supset V$ in φ_2^{n+1}. From this it follows also that $\varphi_1^{n+1} : S_1^{(n+1)} \to S_1^{n+1}$ is a surjection. Thus $\varphi^{n+1} : S^{(n+1)} \to S^{n+1}$ must be a System epimorphism over S extending φ^n. The common prolongation of $\varphi^0, \varphi^1, \varphi^2, \dots$ is then the required System epimorphism $\varphi : F(S) \to S'$ over S. Q.E.D.

Assertion 5. *Let S **g** S' where S' is complete. Further let $\psi : S' \to F(S)$ be a System epimorphism over S such that $\psi|_{S^n} : S^n \to S^{(n)}$ is a System epimorphism over S for all $n = 0, 1, 2, \dots$. Then ψ is a System isomorphism.*

Proof. We shall prove by induction that $\psi^n = \psi|_{S^n}$ are System isomorphisms for all $n = 0, 1, 2, \dots$ This is true for $n = 0$ since ψ^0 is the identity System map. Let ψ^n be already a System isomorphism for some n. Take an arbitrary block $Y \in S_2^{n+1} \setminus S_2^n$. Then card $(Y \cap S_1^n) \geqq b + 1$ so that consequently card $\psi_1^n(Y \cap S_1^n) \geqq b + 1$. From card $\psi_1^n(Y \cap S_1^n) > b + 1$ follows that there is no block from $S_2^{(n+1)} \setminus S_2^{(n)}$ containing $\psi_1^n(Y \cap S_1^n)$ which contradicts the fact that ψ is a System epimorphism. Thus card $\psi_1^n(Y \cap S_1^n) = b + 1$ and consequently also card $(Y \cap S_1^n) = b + 1$. Denote by $\tilde{Y}$ the uniquely determined block from $S_2^{(n+1)} \setminus S_2^{(n)}$ which contains $\psi_1^n(Y \cap S_1^n)$. By the preceding and from the fact that $\psi_2^{n+1} : S_2^{n+1} \to S_2^{(n+1)}$ is a surjection it follows that the map $\psi_2^{n+1}|_{S_2^{n+1} \setminus S_2^n} : S_2^{n+1} \setminus S_2^n \to S_2^{(n+1)} \setminus S_2^n$ with $Y \to \tilde{Y}$ for all $Y \in S_2^{n+1} \setminus S_2^n$ is a bijection. Now suppose that $\psi_1^{n+1} : S_1^{n+1} \to S_1^{(n+1)}$ is not bijective. Then there are elements $a, b \in S_1^{n+1} \setminus S_1^n$, $c \in S_1^{(n+1)} \setminus S_1^{(n)}$ such that $a \neq b$, $\psi_1^{n+1} a = \psi_1^{n+1} b = c$. When a, b are not in the same block from S_2^{n+1} then $\psi_1^{n+1} a = \psi_1^{n+1} b$ contradicts the fact that ψ_2^{n+1} is a bijection and that distinct blocks from $S_2^{(n+1)}$ must be disjoint outside $S_1^{(n)}$. If a, b are in the same block from S_2^{n+1} then $\psi_1^{n+1} a = \psi_1^{n+1} b$ contradicts by the preceding the fact that ψ_1^{n+1} is a surjection. Consequently ψ_1^{n+1} must be a bijection. We conclude that all $\psi^0, \psi^1, \psi^2, \ldots$ are System isomorphisms so that ψ is a System isomorphism, too. Q.E.D.

A System S is called *finite* if S_1 is finite.

Assertion 6. *Let S', S'' be finite Systems such that there exists a System isomorphism $\sigma : F(S') \to F(S'')$. Then there exist Systems $'S$, $''S$ such that* (i) $'S = S'_{('S)}$, $''S = S''_{(''S)}$, (ii) *there is a System isomorphism $\varkappa : {}'S \to {}''S$ and* (iii) $F(S') = F('S)$, $F(S'') = F(''S)$.

Proof. Let $(S'^{(n)})_{n=0}^{\infty}$, $(S''^{(n)})_{n=0}^{\infty}$ be free extension chains over S' and S'' respectively. Let there exist a System isomorphism $\sigma : F(S') \to F(S'')$. As S', S'' are finite an index m exists such that $\sigma_i S'_i \cup S''_i \subset S''^{(m)}_i$ $(i = 1, 2)$ and that $Y \cap S''^{(n)}_1 \subset S''^{(m)}_1$ or $Y \cap \cap\, \sigma_1 S'^{(n)}_1 \subset S'^{(m)}_1$ for every Y and every n for which $Y \in S''^{(m)}_2 \cap (S''^{(n+1)}_2 \setminus S''^{(n)}_2)$ or $Y \in S''^{(m)}_2 \cap (\sigma_2 S'^{(n+1)}_2 \setminus \sigma_2 S'^{(n)}_2)$ respectively. The rest of the proof follows readily. Q.E.D.

A System $F(S)$ is said to be *free* if S_2 is void.

Assertion 7. *Every complete sub-System of a free System is free.*

Proof. Let S' be a complete sub-System of $F(S)$ where S a System with $S_2 = \emptyset$. Put $V^0 = S_1 \cap S'_1$. If V^n is already determined for some n then define $V^{n+1} = S_1^{(n+1)} \cap (S'_1 \setminus S_1^{[n]})$ where $S^{[n]} = F((\bigcup_{m=0}^{n} V^m, \emptyset))$. Thus a sequence $(V^n)_{n=0}^{\infty}$ is well defined by induction. Since $S' = F((\bigcup_{n=0}^{\infty} V^n, \emptyset))$, the proof is complete. Q.E.D.

Assertion 8. *To every complete System S there is a System epimorphism $\sigma : F(S') \to S$ where $S' = (S_1, \emptyset)$.*

Proof. Let $(S^n)_{n=0}^{\infty}$ and $(S^{(n)})_{n=0}^{\infty}$ be the extension chain over S' in S and the free extension chain over S' respectively. Certainly $S = S^1 = S^2 = \ldots$ Let $\sigma^0 : S^{(0)} \to S^0$ be the identity System map. Further suppose that a System epimorphism $\sigma^n : S^{(n)} \to S^n$ is determined for some n. Then define a System map $\sigma^{n+1} : S^{(n+1)} \to S^{n+1}$ prolonging σ^n as follows: For $Z \in T^{(n)}$ let $\sigma_2^{n+1} \hat{Z} = \tilde{Z}$ where $\hat{Z} \in S_2^{(n+1)}$, $\hat{Z} \supset Z$ and $\tilde{Z} \in S_2^{n+1}$, $\tilde{Z} \supset \sigma_1^n Z$. The remaining map $\sigma_1^{n+1} : S_1^{(n+1)} \to S_1^{n+1}$ can be chosen so that $\hat{Z} \setminus Z$ is mapped anywhere into $\tilde{Z}$. Thus by induction a sequence $(\sigma^n)_{n=0}^{\infty}$ is well defined. The common prolongation of all $\sigma^0, \sigma^1, \sigma^2, \ldots$ is the required System isomorphism $\sigma : F(S') \to S$. Q.E.D.

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