Czechoslovak Mathematical Journal

Josef Novák

On convergence groups

Czechoslovak Mathematical Journal, Vol. 20 (1970), No. 3, 357–374

Persistent URL: `http://dml.cz/dmlcz/100973`

Czechoslovak Mathematical Journal, 20 (95) 1970, Praha

ON CONVERGENCE GROUPS

JOSEF NOVÁK, Praha

(Received December 20, 1967)

A convergence group L is a group and a convergence space such that the mapping xy^{-1} on $L \times L$ onto L is sequentially continuous, i.e. if $\lim x_n = x$ and $\lim y_n = y$, then there is a sequence of naturals $n_1 < n_2 < \ldots$ such that $\lim (x_{n_i} y_{n_i}) = xy^{-1}$ [see 5]. O. SCHREIER published a paper [6] where the definition of L-groups is given. According to this definition the group operation xy^{-1} need not be sequentially continuous. Consequently the notions of L-groups and convergence groups differ from each other. Nevertheless, there is a close connexion between both notions (see p. 365).

The present paper deals with convergence groups. In section 1 the necessary notations, definitions and lemmas are given. Section 2 deals with convergence product of two convergence spaces and some point-properties of convergence spaces are studied. In section 3 the theory of convergence groups is developed. In section 4 some relations between convergence topological groups and convergence groups are stated. A problem of E. ČECH is solved concerning the existence of a convergence topological group with uncountable point characters.

1

A closure space (P, v) is a point set P and a map v on the system 2^P of all subsets of P into 2^P such that

$$v\emptyset = \emptyset\,, \quad A \subset vA \quad \text{and} \quad v(A \cup B) = vA \cup vB \quad \text{for each} \quad A \subset P \quad \text{and} \quad B \subset P\,.$$

The map v is called *the closure* and vA *the v-closure*[1]) of the set A in P.

[1]) For the sake of simplicity the symbols of closures, topologies and convergences are sometimes suppressed and in the symbol (x) meaning a one-point set the parenthesis ( ) will often be omitted.

If the axiom

(T$_1$) $$vx = x \quad \text{for each} \quad x \in P$$

is fulfilled, then P is called *a* T_1*-closure space* and v *a* T_1*-closure.*

If the axiom

(F) $$vvA = vA \quad \text{for each} \quad A \subset P$$

is true, then we speak of *a topological space* and the closure v is called *a topology* for P.

Let L be a nonvoid point set. A sequential convergence class (or simply a convergence) $\mathfrak{L}$ on L is the set of elements $(\{x_n\}, x)$ where $\{x_n\}$ denotes a sequence of points of L and x a point of L, fulfilling the three axioms of convergence:

($\mathscr{L}_0$) If $(\{x_n\}, x) \in \mathfrak{L}$ and $(\{x_n\}, y) \in \mathfrak{L}$, then $x = y$.

($\mathscr{L}_1$) If $x_0 \in L$ and $x_n = x_0$ for $n = 1, 2, \ldots$, then $(\{x_n\}, x_0) \in \mathfrak{L}$.

($\mathscr{L}_2$) If $(\{x_n\}, x) \in \mathfrak{L}$ and $\{x_{n_i}\}$ is any subsequence of $\{x_n\}$, then $(\{x_{n_i}\}, x) \in \mathfrak{L}$.

If $(\{x_n\}, x) \in \mathfrak{L}$, then we say that the sequence $\{x_n\}$ *converges to the limit* x, in symbols $\mathfrak{L}$-lim $x_n = x$ or[1]) simply lim $x_n = x$.

The closure λA of a set $A \subset L$ is defined as

(D$_1$) the set of all points lim $x_n \in L$ such[1]) that $\bigcup x_n \subset A$.

From this definition it follows that λ fulfils the axiom (T$_1$). In such a way the convergence $\mathfrak{L}$ induces a T_1-closure for L which will be called *a convegence closure.* The T_1-closure space $(L, \mathfrak{L}, \lambda)$ is called a *convergence space.*

A convergence space $(L, \mathfrak{L}, \lambda)$ need not be a topological space; it is possible to form successive closures of a set A in L:

$$\lambda^0 A \subset \lambda^1 A \subset \lambda^2 A \subset \ldots \subset \lambda^\xi A \subset \ldots$$

where $\lambda^0 A = A$, $\lambda^1 A = \lambda A$, $\lambda^\xi A = \lambda\lambda^{\xi-1} A$ or $= \bigcup_{\eta<\xi} \lambda^\eta A$ according to whether $\xi - 1$ exists or not. It can be easily proved that λ^ξ is a T_1-closure for L and that $\lambda^{\omega_1} A$ is the smallest λ-closed set in L containing A as a subset. Therefore λ^{ω_1} is a topology for L (fulfilling (F)).

The notion of *neighbourhoods* in a T_1-closure space (L, v) is defined in such a way that the following statement is true:

(1) A point $x \in L$ belongs to the v-closure of a set $A \subset L$ if and only if each neighbourhood of x contains at least one point of A.

From this postulate the definition of neighbourhoods in a convergence space $(L, \mathfrak{L}, \lambda)$ follows:

A set $U(x) \subset L$ is a λ-neighbourhood[1]) of a point x if

(D_2) $\mathfrak{L}$-lim $x_n = x$ implies that $x_n \in U(x)$ for nearly all n.

In view of ($\mathscr{L}_1$), each neighbourhood of x contains the point x. From ($\mathscr{L}_2$) it follows that the intersection of two neighbourhoods of a point x is a neighbourhood of the same point x.

Let v_1 and v_2 be T_1-closures for the same point set P. We say that v_1 is *finer* than v_2 or v_2 is *coarser* than v_1 if $v_1A \subset v_2A$ for each $A \subset P$. According to (1), v_1 is finer than v_2 if and only if for each $x \in P$ and every v_2-neighbourhood V_2 of x there is a v_1-neighbourhood V_1 of x such that $V_1 \subset V_2$.

A convergence $\mathfrak{L}$ on L induces on L a convergence closure λ in a unique way. The convergence closure λ, however, can be induced by more than one convergence on L. This is shown by the following example.

Example 1. Let R be the set of all rational numbers and $\mathfrak{R}$ the usual convergence on R. Let $\mathfrak{R}'$ be the set of all elements $(\{x_n\}, x)$, $x \in R$, $x_n \in R$, where $\{x_n\}$ is a monotone sequence of numbers $\mathfrak{R}$-converging to x. Then $(R, \mathfrak{R}, \varrho)$ and $(R, \mathfrak{R}', \varrho')$ are two convergence spaces such that $\mathfrak{R} \neq \mathfrak{R}'$ and $\varrho = \varrho'$.

From this it follows that it is possible to classify convergences on the same point set L by means of the equivalence relation $\sim$:

$$\mathfrak{L} \sim \mathfrak{M} \quad \text{whenever} \quad \lambda = \mu\,.$$

It can be proved [3] that in every class $[\mathfrak{L}]$ of convergences on L there is a largest[2]) convergence $\mathfrak{L}^*$ viz. $\mathfrak{L}^* = \bigcup_{\mathfrak{N} \in [\mathfrak{L}]} \mathfrak{N}$. There is a one-to-one map of the system of all largest convergences on a given point set L onto the system of all convergence closures for L such that

$$\mathfrak{L}^* \subset \mathfrak{M}^* \quad \text{if and only if} \quad \lambda A \subset \mu A \quad \text{for each} \quad A \subset L\,.$$

The largest convergence $\mathfrak{L}^*$ in the class $[\mathfrak{L}]$ is characterized by the axiom [3]:

($\mathscr{L}_3$) If $\{x_n\}$ is a sequence of points in a convergence space $(L, \mathfrak{L}, \lambda)$ and x a point of L such that in each subsequence there is a subsequence $\mathfrak{L}$-converging to x, then $\mathfrak{L}^*$-lim $x_n = x$.

From (D_1) it follows

[2]) The largest convergence is usually denoted by an asterisk; the Greek letters $\lambda, \mu, \ldots$, are used to denote convergence closures induced by convergences $\mathfrak{L}, \mathfrak{M}, \ldots$.

Lemma 1. *Let $(L, \mathfrak{L}_i, \lambda_i)$, $i = 1, 2$, be convergence spaces on the same point set L. Then $\lambda_1 \neq \lambda_2$ and λ_1 is finer than λ_2 if and only if $\mathfrak{L}_1^* \subset \mathfrak{L}_2^*$ and there is a point $x_0 \in L$ and an infinite countable subset $C \subset L$ such that $x_0 \in \lambda_2 C - \lambda_1 C$.*

Let φ be a map on a convergence space $(L, \mathfrak{L}, \lambda)$ into a convergence space $(M, \mathfrak{M}, \mu)$. Then the following statements are equivalent [3]:

(2) $\varphi(\lambda A) \subset \mu\, \varphi(A)$ for each $A \subset L$.

(3) If x is a point of L and $V(\varphi(x))$ a μ-neighbourhood of the point $\varphi(x)$ in M, then there is a λ-neighbourhood $U(x)$ of x such that $\varphi(U(x)) \subset V(\varphi(x))$.

(4) If $\mathfrak{L}$-lim $x_n = x$, then $\mathfrak{M}$-lim $\varphi(x_{n_i}) = \varphi(x)$ where $\{x_{n_i}\}$ is a suitable subsequence of $\{x_n\}$.

The conditions (2) and (3) concern the continuity and the condition (4) the sequential continuity of the map φ.

Lemma 2. *A map φ on a convergence space $(L, \mathfrak{L}, \lambda)$ into a convergence space $(M, \mathfrak{M}, \mu)$ is continuous if and only if*

(5) $$\mathscr{L}\text{-lim } x_n = x \quad \textit{implies} \quad \mathfrak{M}^*\text{-lim } \varphi(x_n) = \varphi(x).$$

Proof. Let φ be continuous on L. If $\mathfrak{L}$-lim $x_n = x$ and if $\{\varphi(x_{n_k})\}_{k=1}^{\infty}$ is any subsequence of $\{\varphi(x_n)\}_{n=1}^{\infty}$ then, in view of (4), $\mathfrak{M}$-lim $\varphi(x_{n_{k_i}}) = \varphi(x)$, $\{\varphi(x_{n_{k_i}})\}$ being a suitable subsequence of $\{\varphi(x_{n_k})\}$. Consequently (5) is true, by $(\mathscr{L}_3)$. Now, suppose that the condition (5) is fulfilled. Since $\mathfrak{M}^* \in [\mathfrak{M}]$, we have $\varphi(x) \in \mu \bigcup_{n=1}^{\infty} \varphi(x_n)$. From this it easily follows that $\varphi(x) = \mathfrak{M}$-lim $\varphi(x_{n_i})$ for a suitable subsequence $\{x_{n_i}\}$ of $\{x_n\}$. Therefore (4) is true.

The map φ is a homeomorphism if φ is a one-to-one continuous map on L onto M such that φ^{-1} is also continuous.

2

Let $(L_1, \mathfrak{L}_1, \lambda_1)$ and $(L_2, \mathfrak{L}_2, \lambda_2)$ be convergence spaces. Denote $L_1 \times L_2$ the Cartesian product of the sets L_1 and L_2. We define a convergence $\mathfrak{L}_{12}$ on $L_1 \times L_2$ as follows:

(D$_3$) $(\{(x_n, y_n)\}, (x, y)) \in \mathfrak{L}_{12}$ if $(\{x_n\}, x) \in \mathfrak{L}_1$ and $(\{y_n\}, y) \in \mathfrak{L}_2$.

In such a way we get a convergence space $(L_1 \times L_2, \mathfrak{L}_{12}, \lambda_{12})$ which is called the convergence product, λ_{12} being a convergence closure for[3] $L_1 \times L_2$ induced by $\mathfrak{L}_{12}$.

[3]) The notation $\mathfrak{L}_{12}$ and λ_{12} will be used also in the case when $L_1 = L_2 = L$.

Notice that there is another T_1-closure $\lambda_1 \times \lambda_2$ for $L_1 \times L_2$; it is defined by means of neighbourhoods as follows:

(D_4) A set $W \subset L_1 \times L_2$ is a $\lambda_1 \times \lambda_2$-neighbourhood of a point (x_0, y_0) if there is a λ_1-neighbourhood U of x_0 in L_1 and a λ_2-neighbourhood V of y_0 in L_2 such that $U \times V \subset W$.

From (1) it follows that the $\lambda_1 \times \lambda_2$-closure of a set $A \subset L_1 \times L_2$ consists of all points $z \in L_1 \times L_2$ such that $A \cap W(z) \neq \emptyset$ for each $\lambda_1 \times \lambda_2$-neighbourhood $W(z)$ of z. It is easy to prove that the map $\lambda_1 \times \lambda_2$ fulfils the axiom (T_1); it will be called a product closure and always denoted by $\lambda_1 \times \lambda_2$. Evidently λ_{12} is finer than $\lambda_1 \times \lambda_2$.

Let us notice that $\lambda_1 \times \lambda_2$ need not be a convergence closure even if both convergence closures λ_1 and λ_2 are topologies (fulfilling the axiom (F)) [3].

Now we are going to find a condition under which the equality $\lambda_1 \times \lambda_2 = \lambda_{12}$ holds. For this purpose we shall define some local properties in a convergence space by means of neighbourhoods, closures and double sequences.

Let N be the set of all naturals. Let L be a convergence space. A map φ on $N \times N$ into L such that $\varphi(m, n) = x_{mn}$ is called *the double sequence* and denoted $\{x_{mn}\}_{m,n=1}^{\infty}$ or simply $\{x_{mn}\}$. A sequence $\{z_m\}$ is a cross-sequence in $\{x_{mn}\}$ provided that there is a function f on N into N such that $z_m = x_{mf(m)}$. Each subsequence $\{z_{m_i}\}$ of $\{z_m\}$ will be called a cross-subsequence in $\{x_{mn}\}$.

Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. Let x_0 be a point in L. We say that x_0 has property α, β, γ, δ respectively if the condition is true:

(α) There is a decreasing sequence $\{V_m\}$ of neighbourhoods of the point x_0 such that each one-to-one sequence of points $\{x_m\}$, $x_m \in V_m$, contains a subsequence converging to x_0.

(β) If $\{x_{mn}\}$ is a double sequence of points in L such that $\lim_n x_{mn} = x_0$ for each m, then there is a function $f(m)$ on N into N such that each cross-sequence $\{x_{mn_m}\}_{m=1}^{\infty}$ where $n_m > f(m)$ contains a subsequence converging to x_0.

(γ) If $\{x_{mn}\}$ is a double sequence of points in L such that $\lim_n x_{mn} = x_0$, for each m, then there is a cross-subsequence in $\{x_{mn}\}$ converging to x_0.

(δ) If $\{A_m\}$ is a decreasing sequence of subsets of L and $x_0 \in \lambda A_m$ for each m, then there is a sequence of points $\{x_m\}$, $x_m \in A_m$, converging to x_0.

We say that the convergence space has property α, β, γ, δ respectively if each of its point has the property in question. The property γ will be called *the cross-subsequence property* and the point x_0 in question a *γ-point*.

In [4] I defined the property ϱ of a point $x_0 \in L$ by the condition:

(ϱ) There is a double sequence of points in L such that $\lim_n x_{mn} = x_0$ for each m and such that no cross-subsequence of it convergences to x_0.

A point with the property ϱ will be called a ϱ-*point*.

Lemma 3. $(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma) \Leftrightarrow (\delta) \Leftrightarrow (\text{non } \varrho)$.

Proof. $(\alpha) \Rightarrow (\beta)$. If $\{x_{mn}\}$ is a double sequence of points such that $\lim_n x_{mn} = x_0$ for each m and if $\{V_m\}$ is a decreasing sequence of neighbourhoods of x_0 satisfying (α), then, by (D_2), there is a function $f_1(m)$, $m \in N$, such that $n > f_1(m)$ implies $x_{mn} \in V_m$.

If the point set $\bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} x_{mn}$ is finite, then the proof is evident; if it is infinite, arrange all x_{mn} distinct from x_0 into a sequence $\{a_k\}_1^\infty$ and choose, for each $m \in N$, a natural $f_2(m)$ such that

$$\bigcup_{k=1}^{m} a_k \cap \bigcup_{n=f_2(m)}^{\infty} x_{mn} = \emptyset .$$

Then put

$$f(m) = \max \{f_1(m), f_2(m)\} .$$

Hence (β) follows from (α).

The implications $(\beta) \Rightarrow (\gamma) \Rightarrow (\delta)$ and the equivalence $(\gamma) \Leftrightarrow \text{non } (\varrho)$ are clear. $(\delta) \Rightarrow (\gamma)$ remains to be proved. Let $\lim_n x_{mn} = x_0$ for each m. Denote $A_k = \bigcup_{m=k}^{\infty} \bigcup_{n=1}^{\infty} x_{mn}$. In view of (δ) there is a sequence of points $\{z_k\}$, $z_k \in A_k$, converging to x_0. Evidently it contains a subsequence which is a cross-subsequence in $\{x_{mn}\}_{m,n=1}^{\infty}$.

It is clear that the point x_0 has the property α if the character[4]) of x_0 is countable; as a matter of fact, if $\{V_m\}$ is a sequence of neighbourhoods of x_0 which form a base at x_0 and if we choose any points $x_m \in V_m$, then $x_0 \in \lambda \bigcup x_m$ so that (α) is true. On the other hand, there are points which have the property α but fail to have a countable character. This is shown by

Example 2. Let L be a point set having an uncountable power $\aleph_\alpha$. Choose a point $z_0 \in L$ and define: $(\{x_n\}, x) \in \mathfrak{L}$ if $x_n = x$ for nearly all n or if $x = z_0$ and there is no constant subsequence $\{y\}$ of $\{x_n\}$, $y \neq z_0$. Evidently $\mathfrak{L}$ is a largest convergence and the convergence space $(L, \mathfrak{L}, \lambda)$ is a topological space each point of which — except z_0 — is isolated and the character of the point z_0 is $\aleph_\alpha$. From this it instantly follows that L has the property α.

---

[4]) The character of a point x is the least cardinal $\mathfrak{m}$ such that there is a base of neighbourhoods at x of power $\mathfrak{m}$.

The conditions (α) and (β) need not be equivalent. This is shown by

Example 3. Let X be a point set of an uncountable power $\aleph_\alpha$. Denote $\mathbf{F}$ the system of all finite subsets of X including $\emptyset$. Then $(\mathbf{F}, \mathfrak{L}, \lambda)$ is a convergence space the convergence $\mathfrak{L}$ being defined in the usual manner[5]).

The space $(\mathbf{F}, \mathfrak{L}, \lambda)$ has the property β: Let $\{X_{mn}\}$ be a double sequence of elements $X_{mn} \in \mathbf{F}$ and X_0 an element of $\mathbf{F}$ such that $\operatorname{Lim} X_{mn} = X_0$ for each m. Since X_0 and X_{mn} are finite, there exists a function $f_1(m)$ on N into N such that $X_0 \subset X_{mn}$ for all $n > f_1(m)$. Denote[6]) $B = \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} X_{mn} - X_0$. If $B = \emptyset$, the proof is evident. If it is not the case, arrange all points of B into a finite or infinite sequence $\{b_k\}$ of points and choose, for each $m \in N$, a natural $f_2(m)$ such that $X_{mn} \cap \bigcup_{k \leq m} b_k = \emptyset$ for all $n > f_2(m)$. Put

$$f(m) = \max \{f_1(m), f_2(m)\} .$$

Let $\{X_{mn_m}\}_{m=1}^{\infty}$ be any cross-sequence in $\{X_{mn}\}$ such that $n_m > f(m)$ for each $m \in N$. Since $X_0 \subset X_{mn_m}$ and because every point of B belongs to X_{mn_m} for at most a finite number of m, it follows that $\operatorname{Lim\,sup} X_{mn_m} - X_0 = \emptyset$ and so $\operatorname{Lim}_m X_{mn_m} = X_0$.

The space $(\mathbf{F}, \mathfrak{L}, \lambda)$ has not the property α: Let $\{\mathbf{V}_m\}$ be a decreasing sequence of neighbourhoods of the element $\emptyset \in \mathbf{F}$. There is a disjoint uncountable collection $\mathbf{C}$ of countable infinite subsets C_ι of X. Choose a point c_0^ι in C_ι and denote c_n^ι, $n = 1, 2, \ldots$, all remaining points of C_ι. In view of (D_2), in the complement of every neighbourhood $\mathbf{V}_m$ there is at most a finite number of one-point elements $(x) \in \mathbf{F}$ and at most a countable number of two-point elements $(c_0^\iota, c_n^\iota) \in \mathbf{F}$. Because $\mathbf{C}$ is uncountable, $\bigcap_{m=1}^{\infty} \mathbf{V}_m$ contains an element $(c_0^k) \in \mathbf{F}$ and all two-point elements $(c_0^k, c_m^k) \in \mathbf{F}$, $m = 1, 2, \ldots$ Since $\operatorname{Lim}_m (c_0^k, c_m^k) = (c_0^k)$, (α) is not fulfilled.

Notice that the subspace of $(\mathbf{F}, \mathfrak{L}, \lambda)$ consisting of $\emptyset$ and all one-point elements of $\mathbf{F}$ is homeomorph to the topological space above (Example 2). Consequently the character of the element $\emptyset$ in $\mathbf{F}$ is $\geqq \aleph_\alpha$.

Theorem 1. *Let $(L_i, \mathscr{L}_i, \lambda_i)$, $i = 1, 2$, be convergence spaces. If L_1 has the property δ and L_2 the property α, then $\lambda_1 \times \lambda_2 = \lambda_{12}$.*

Proof. Let (x_0, y_0) be a point in $L_1 \times L_2$. Assume that, on the contrary, there is a λ_{12}-neighbourhood $O(x_0, y_0)$ of (x_0, y_0) such that $U \times V \not\subset O(x_0, y_0)$ for each

---

[5]) $\operatorname{Lim} A_n = A$ whenever $A = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$.

[6]) If the set operations $\bigcap$ and $\bigcup$ are used, then the elements X_{mn} are considered as subsets of X.

λ_1-neighbourhood U of x_0 in L_1 and each λ_2-neighbourhood V of y_0 in L_2. We shall construct in the complement of $O(x_0, y_0)$ a sequence of points $\mathfrak{L}_{12}$-converging to (x_0, y_0). Let $\{V_m\}$ be a decreasing sequence of λ_2-neighbourhoods of y_0 in L_2 having the property α. Let A_m be the set of all points $x \in L_1$ such that $(x, y) \notin O(x_0, y_0)$ for some $y \in V_m$. Then $A_1 \supset A_2 \supset \ldots$ and each λ_1-neighbourhood of x_0 in L_1 contains at least one point of each A_m; otherwise there would be a λ_1-neighbourhood U' of x_0 in L_1 and a natural p such that $U' \cap A_p = \emptyset$ so that $U' \times V_p \subset O(x_0, y_0)$ and this would be a contradiction. Consequently, $x_0 \in \lambda_1 A_m$ for each m and, by (δ), there is a sequence $\{x_m\}$, $x_m \in A_m$, converging to x_0 in L_1. Choose points $y_m^{(1)} \in V_m$ such that $(x_m, y_m^{(1)}) \notin O(x_0, y_0)$, $m = 1, 2, \ldots$ If there is a one-to-one subsequence or a constant subsequence $\{y_0\}$ of $\{y_m^{(1)}\}$ then, in view of (α), the proof is finished. If it is not the case, then – without loss of generality – we may suppose that the sequence $\{y_m^{(1)}\}$ is constant, say $y_m^{(1)} = y^{(1)}$. Evidently $y^{(1)} \neq y_0$. It is easy to see that the sequence of neighbourhoods $V_m - (y^{(1)})$ of the point y_0 in L_2 fulfils the condition (α) and $x_0 \in \bigcap \lambda A_m^{(2)}$, $A_m^{(2)}$ being the set of all $x \in L_1$ such that $(x, y) \notin O(x_0, y_0)$, $y \in V_m - (y^{(1)})$. Now, suppose we have just found sequences $\{(x_m^{(i)}, y^{(i)})\}_{m=1}^{\infty}$, $i = 1, 2, \ldots, k$, such that $\mathfrak{L}_1\text{-}\lim_n x_n^{(i)} = x_0$ and $y^{(i)}$ are distinct points of $\bigcap V_m$. Consider the sequence $\{V_m - \bigcup_{i=1}^{k} (y^{(i)})\}_{m=1}^{\infty}$ of neighbourhoods of y_0 and choose a sequence $\{(x_m^{(k+1)}, y_m^{(k+1)})\}_{m=1}^{\infty}$ such that $\lim_m x_m^{(k+1)} = x_0$ and $\{y_m^{(k+1)}\}_{m=1}^{\infty}$ is either a one-to-one or a constant sequence of points $y_m^{(k+1)} \in V_m - \bigcup_{i=1}^{k} (y^{(i)})$. If it is one-to-one, then the proof is finished. If it is not one-to-one, then continue in constructing sequences $\{(x_m^{(n)}, y^{(n)})\}_{m=1}^{\infty}$, $n = 1, 2, \ldots$ If no $\{y_m^{(n)}\}_{m=1}^{\infty}$ is one-to-one, we have a double sequence $\{x_m^{(n)}\}$ and a one-to-one sequence of points $y^{(n)} \in \bigcap V_m$. Since the point x_0 has the cross-subsequence property, there is, by (α), a cross-subsequence in $\{x_m^{(n)}, y^{(n)}\}_{m,n=1}^{\infty}$ of points in the complement of $O(x_0, y_0)$ which $\mathfrak{L}_{12}$-converges to the point (x_0, y_0).

Corollary 1. *Let $(L_i, \mathfrak{L}_i, \lambda_i)$, $i = 1, 2$, be convergence spaces fulfilling the first axiom of countability. Then $\lambda_1 \times \lambda_2 = \lambda_{12}$.*

Corollary 2. *Let $(L, \mathfrak{L}, \lambda)$ be a convergence space having the property α. Let $(L \times L, \mathfrak{L}_{12}, \lambda_{12})$ be the convergence product of the space L. Then $\lambda \times \lambda = \lambda_{12}$.*

The proofs of both corollaries follow instantly from Theorem 1 and Lemma 3.

3

Definition. Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. Let $(L, .)$ be a group with a group operation . on L. We say that $(L, \mathfrak{L}, \lambda, .)$ is *a convergence closure group* or simply *a convergence group* if the map $\delta(x, y) = xy^{-1}$ on the convergence product $(L, \mathfrak{L}, \lambda) \times$

$\times (L, \mathfrak{L}, \lambda)$ onto $(L, \mathfrak{L}, \lambda)$ is sequentially continuous, i.e. if the condition (SG) is fulfilled [5]:

(SG) If $\mathfrak{L}$-lim $x_n = x$ and $\mathfrak{L}$-lim $y_n = y$, then there is a subsequence $\{n_i\}$ of $\{n\}$ such that $\mathfrak{L}\text{-}\lim\limits_i (x_{n_i} y_{n_i}^{-1}) = xy^{-1}$.

According to Lemma 2 the condition (SG) is equivalent to the condition

(S′G) If $\mathfrak{L}$-lim $x_n = x$ and $\mathfrak{L}$-lim $y_n = y$, then $\mathfrak{L}^*$-lim $(x_n y_n^{-1}) = xy^{-1}$.

If $\mathfrak{L}$ is a largest convergence on L, then from (S′G) it follows that $L$is a convergence group if and only if

(S*G) $\mathfrak{L}$-lim $x_n = x$ and $\mathfrak{L}$-lim $y_n = y$ implies $\mathfrak{L}$-lim $(x_n y_n^{-1}) = xy^{-1}$.

Example 4. $(R, \mathfrak{R}', \varrho, +)$ is a convergence group by (SG) (see Example 1). Since the sequences $\{2^{-n} + (-4)^{-n}\}_{n=3}^{\infty}$ and $\{2^{-n}\}_{n=3}^{\infty}$ $\mathfrak{R}'$-converge to 0 and because $\{(-4)^{-n}\}_{n=3}^{\infty}$ does not $\mathfrak{R}'$-converge at all, it follows that the condition (SG) is and (S*G) is not fulfilled; therefore they are not equivalent.

Example 5. Let M be a set the elements of which are classes $[f]$ of B-measurable functions f of real argument. Let $\mathfrak{M}$ denote the convergence almost everywhere. Define $[f] + [g] = [f + g]$. Then M is a commutative group fulfilling the condition (S*G) and consequently also (SG). Therefore $(M, \mathfrak{M}, \mu, +)$ is a convergence group. It is well known that the largest convergence $\mathfrak{M}^*$ is the convergence in measure and that $\mathfrak{M} \neq \mathfrak{M}^*$. From this it follows that (S*G) need not imply $\mathfrak{L} = \mathfrak{L}^*$.

Remark. In the literature ([6] and [7]) we find the following definition of a convergence group (so called L-group):

Eine Gruppe heisst L-Gruppe, wenn in ihr ein Grenzbegriff definiert ist, der folgenden Forderungen genügt:

1) Ist lim $a_n = a$ und lim $a_n = b$, so ist $a = b$.
2) Ist lim $a_n = a$ und lim $n_\nu = \infty$, so ist lim $a_{n_\nu} = a$.
3) Ist lim $a_{n+1} = a$, so ist lim $a_n = a$.
4) Ist $a_n = a$ für alle n, so ist lim $a_n = a$.
5) Ist lim $a_n = a$ und lim $b_n = b$, so ist lim $a_n \, . \, b_n = a \, . \, b$.
6) Ist lim $a_n = a$, so ist lim $a_n^{-1} = a^{-1}$.

The operation xy^{-1} on an L-group defined by 1)–6) need not be sequentially continuous in the sense of the continuity defined by (4). For example, let $(R, \mathfrak{R}, \varrho, +)$ be the topological group of rational numbers, $\mathfrak{R}$ being the usual convergence of rational numbers. Define a convergence $\mathfrak{S}$ on R : $\mathfrak{S}$-lim $x_n = x$ if $\{x_n\}_{n=1}^{\infty}$ can be arranged into a nearly monotone[7]) sequence $\{x_{f(n)}\}_{n=1}^{\infty}$ and $\lim (x_n - x) = 0$, $f(n)$ being a map on N into N such that $\lim f(n) = \infty$. Denote σ the convergence closure induced by $\mathfrak{S}$. It is easy to see that $\mathfrak{S} \in [\mathfrak{R}]$. Hence $\sigma = \varrho$. The operation $x - y$ is continuous on $(R, \mathfrak{R}, \varrho, +)$, however, it fails to be continuous on $(R, \mathfrak{S}, \varrho, +)$. As a matter of fact, if we put $x_n = 2^{-n} + (-4)^{-n}$ and $y_n = 2^{-n}$, then $\mathfrak{S}$-$\lim x_n = 0 = \mathfrak{S}$-$\lim y_n$, but the sequence $x_n - y_n$ does not $\mathfrak{S}$-converge at all. From this it follows that the notion of the L-group depends on the choice of the representative in the class of equivalent convergences. This undesirable result makes the above definition of an L-group unsuitable from the topological point of view. It can be avoided if the postulate 3) is replaced by the axiom $(\mathscr{L}_3)$ or if both 5) and 6) are replaced by (SG).

Lemma 4. *Let $(L, \mathfrak{L}, \lambda, .)$ be a convergence group. Let $U(x)$ be a λ-neighbourhood of a point $x \in L$. Let y be a point of L. Then the set $y(U(x))^{-1}$ is a λ-neighbourhood of the point yx^{-1} and $(U(x))^{-1}y$ a λ-neighbourhood of the point $x^{-1}y$.*

Proof. If $\{z_n\}$ is a sequence converging to yx^{-1} and $\{z_{n_i}\}$ any subsequence of $\{z_n\}$ then, by (SG), there is a subsequence $\{z_{n_{i_k}}\}$ of $\{z_{n_i}\}$ such that $\lim_k (z_{n_{i_k}}^{-1} y) = x$ and so $z_{n_{i_k}}^{-1} y \in U(x)$ and $z_{n_{i_k}} \in y(U(x))^{-1}$ for nearly all k. From this it follows that $z_n \in y(U(x))^{-1}$ for nearly all n. Hence $y(U(x))^{-1}$ is a neighbourhood of yx^{-1} by (D_2). The proof of the second assertion is analogous. (Cf. [2]).

Lemma 5. *Let $(L, \mathfrak{L}, \lambda, .)$ be a convergence group. Let x_0 be a point of L. Then the maps*

$$h_1(x) = x_0 x\,, \quad h_2(x) = x x_0\,, \quad h_3(x) = x^{-1}\,, \quad x \in L$$

are homeomorphisms on L onto L.

Proof. From (SG) it instantly follows that the operations h_j and h_j^{-1}, $j = 1, 2, 3$, are continuous.

Lemma 6. *Let $(L, \mathfrak{L}, \lambda, .)$ be a convergence group, A a subset and x_0 a point of L. Then*

$$\lambda(x_0 A^{-1}) = x_0(\lambda A)^{-1} \quad \text{and} \quad \lambda(A^{-1} x_0) = (\lambda A)^{-1}\, x_0\,.$$

Proof. The maps $g_1(x) = x_0 x^{-1}$ and $g_2(x) = x^{-1} x_0$ are homeomorphisms on L onto L. Therefore $\lambda\, g_j(A) = g_j(\lambda A)$, $j = 1, 2$. (Cf. [2]).

According to Lemma 5 a convergence group is topologically homogeneous, i.e. for any two points a and b of L there is a homeomorphism h on L onto L such that $h(a) = b$. There is another notion of homogeneity in convergence groups viz. the homogeneity of a convergence, which is defined as follows:

[7]) $\{x_n\}_{n=1}^{\infty}$, is nearly monotone if there is a natural p such that $\{x_n\}_{n=p}^{\infty}$ is monotone (either increasing or decreasing).

Let $(L, \mathfrak{L}, \lambda, .)$ be a convergence group. We say that the convergence $\mathfrak{L}$ is left or right homogeneous if for each point $a \in L$ the condition (h_l) or (h_r) is fulfilled:

$$(h_l) \qquad \mathfrak{L}\text{-}\lim x_n = e \quad \text{if and only if} \quad \mathfrak{L}\text{-}\lim (x_n a) = a$$

$$(h_r) \qquad \mathfrak{L}\text{-}\lim x_n = e \quad \text{if and only if} \quad \mathfrak{L}\text{-}\lim (a x_n) = a\,.$$

The convergence $\mathfrak{L}$ is called homogeneous if it is both left and right homogeneous. If it is so then we say that the convergence group L is convergence homogeneous.

Lemma 7. *Let $(L, \mathfrak{L}, \lambda, .)$ be a convergence group. If the convergence $\mathfrak{L}$ fulfils the condition* (S*G), *then $\mathfrak{L}$ is homogeneous.*

Proof. If $(\{x_n\}, e) \in \mathfrak{L}$ and $a \in L$, then $(\{x_n a\}, a) \in \mathfrak{L}$, and $(\{a x_n\}, a) \in \mathfrak{L}$, by (S*G). If $(\{x_n a\}, a) \in \mathfrak{L}$ or $(\{a x_n\}, a) \in \mathfrak{L}$, then from (S*G) it easily follows that $(\{x_n\}, e) \in \mathfrak{L}$.

From Lemma 7 it instantly follows that every largest convergence defined on a convergence group is homogeneous.

Lemma 8. *Let $(L, \mathfrak{L}, \lambda, .)$ be a convergence group. Then there is the smallest homogeneous convergence $\mathfrak{L}^h$ containing $\mathfrak{L}$ as a subset.*

Proof. Denote $\mathfrak{L}' = \bigcap_{\mathfrak{H} \supset \mathfrak{L}} \mathfrak{H}$ where $\mathfrak{H}$ are homogeneous convergences on L containing $\mathfrak{L}$ as a subset (for example $\mathfrak{H} = \mathfrak{L}^*$). Since each $\mathfrak{H}$ fulfils axioms $(\mathscr{L}_0)$, $(\mathscr{L}_1)$ and $(\mathscr{L}_2)$ and satisfies conditions (h_l) and (h_r), it follows that $\mathfrak{L}'$ is a homogeneous convergence. Consequently we may put $\mathfrak{L}^h = \mathfrak{L}'$.

Since $\mathfrak{L}^*$ is a homogeneous convergence, we have $\mathfrak{L} \subset \mathfrak{L}^h \subset \mathfrak{L}^*$ so that $\mathfrak{L}^h \in [\mathfrak{L}]$. Hence $\lambda^h = \lambda$.

Let us notice that a homogeneous convergence need not fulfil the condition (S*G). This is shown by the convergence group $(R, \mathfrak{R}', \varrho', +)$ in Example 4.

If a convergence group $(L, \mathfrak{L}, \lambda, .)$ is convergence homogeneous, then it is possible to investigate local convergence properties by studying convergence properties only at one point, for example at the neutral element e of L. In this case it suffices to study "neutral" sequences, i.e. the elements $(\{x_n\}, e) \in \mathfrak{L}$.

If G is a commutative group, then a homogeneous convergence on G can be defined by means of zero sequences in the following way:

Theorem 2. *Let $(G, +)$ be a commutative group. Let $\mathfrak{G}(0)$ be the set of elements $(\{x_n\}, 0)$ where $\{x_n\}$ is a sequence of points of G, fulfilling the conditions*:

1° $\mathfrak{G}(0) \neq \emptyset$.

2° *If $(\{x_n\}, 0) \in \mathfrak{G}(0)$, then $(\{x_{n_i}\}, 0) \in \mathfrak{G}(0)$ for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$.*

3° *If $(\{x_n\}, 0) \in \mathfrak{G}(0)$ and $a \neq 0$, then $(\{x_n + a\}, 0)$ does not belong to $\mathfrak{G}(0)$.*

4° *If* $(\{x_n\}, 0) \in \mathfrak{G}(0)$ *and* $(\{y_n\}, 0) \in \mathfrak{G}(0)$, *then there is a subsequence* $\{n_i\}$ *of* $\{n\}$ *such that* $(\{x_{n_i} - y_{n_i}\}, 0) \in \mathfrak{G}(0)$.

Let $\mathfrak{G}$ *be the set of elements* $(\{x_n\}, x)$ *defined as follows*:

$$(\mathrm{D}_5) \qquad (\{x_n\}, x) \in \mathfrak{G} \quad \textit{if and only if} \quad (\{x_n - x\}, 0) \in \mathfrak{G}(0)\,.$$

Then $\mathfrak{G}$ *is a homogeneous convergence on* G *and* $(G, \mathfrak{G}, \gamma, +)$ *is a convergence commutative group.*

Proof. From 1° and 4° it follows that $(\{0\}, 0) \in \mathfrak{G}(0)$. Therefore $(\{x\}, x) \in \mathfrak{G}$ for each $x \in G$, by (D_5). If $(\{x_n\}, x) \in \mathfrak{G}$ and $(\{x_n\}, y) \in \mathfrak{G}$, then $(\{x_n - x\}, 0) \in \mathfrak{G}$ and $(\{x_n - x + a\}, 0) \in \mathfrak{G}$ where $a = x - y$. Therefore $x = y$, by 3°. If $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ and $(\{x_n\}, x) \in \mathfrak{G}$ then $(\{x_{n_i}\}, x) \in \mathfrak{G}$, by 2°. Therefore axioms $(\mathscr{L}_0)$, $(\mathscr{L}_1)$ and $(\mathscr{L}_2)$ are fulfilled. The homogeneity of $\mathfrak{G}$ immediately follows from (D_5), (h_l) and (h_r).

Now, let $(\{x_n\}, x) \in \mathfrak{G}$ and $(\{y_n\}, y) \in \mathfrak{G}$. Then $(\{x_n - x\}, 0) \in \mathfrak{G}(0)$ and $(\{y_n - y\}, 0) \in \mathfrak{G}(0)$. According to 4° there is a subsequence $\{n_i\}$ of $\{n\}$ such that $(\{x_{n_i} - x - (y_{n_i} - y)\}, 0) \in \mathfrak{G}(0)$ and since $+$ is a commutative operation, we have $(\{x_{n_i} - y_{n_i}\}, x - y) \in \mathfrak{G}$. Hence $(G, \mathfrak{G}, \gamma, +)$ is a convergence commutative group.

It is well known that each topological group is a completely regular space. The notion of complete regularity of topological spaces corresponds to the notion of sequential regularity[8]) of convergence spaces. Now we shall construct convergence groups which fail to be sequentially regular.

Lemma 9. *Let* $(L, \mathfrak{L}, \lambda, .)$ *be a convergence group and* $\{L_n\}_1^\infty$ *an increasing sequence of subgroups of* L *such that* $\bigcup L_n = L$. *Let* $\mathfrak{L}_a$ *be the set of all* $(\{x_n\}, x)$ *with the property*

(D_6) $(\{x_n\}, x) \in \mathfrak{L}$ *and* $x_n \in L_p$ *for nearly all* n *where* p *is a suitable natural (depending on* $\{x_n\}$*).*

Then $(L, \mathfrak{L}_a, \lambda_a, .)$ *is a convergence group and* λ_a *is finer than* λ.

Proof. It can be easily proved that the axioms $\mathscr{L}_0$, $\mathscr{L}_1$ and $\mathscr{L}_2$ are satisfied and the condition (SG) is fulfilled. Since $\mathfrak{L}_a \subset \mathfrak{L}$, λ_a is finer than λ.

Lemma 10. *Let* $(L, \mathfrak{L}, \lambda, .)$ *be a convergence group. Let the convergence closure* λ *be a topology. Let* $\{L_n\}_{n=1}^\infty$ *be a strictly increasing sequence of subgroups such that* $\bigcup L_n = L = \lambda L_1$. *Then the convergence group* $(L, \mathfrak{L}_a, \lambda_a, .)$ *fails to be either regular or sequentially regular.*

---

[8]) A convergence space $(L, \mathfrak{L}, \lambda)$ is sequentially regular if for each point x_0 and each sequence of points $x_n \in L$ such that no subsequence of $\{x_n\}$ converges to x_0 there is a (sequentially) continuous function f on L such that $\{f(x_n)\}$ does not converge to $f(x_0)$.

Proof. First prove that $\lambda \neq \lambda_a$: Choose a point y_m in $L_{m+1} - L_m$ for each m. Since $L = \lambda L_1$, there is a one-to-one sequence $\{y_{mn}\}_{n=1}^{\infty}$ of points $y_{mn} \in L_1$ which $\mathfrak{L}$-converges to y_m. By (SG) there is a subsequence $\{y_m y_{mn_i}^{-1}\}_{i=1}^{\infty}$ of $\{y_m y_{mn}^{-1}\}_{n=1}^{\infty}$ which $\mathfrak{L}$-converges to e. From (D_6) it follows that $\mathfrak{L}_a\text{-}\lim_i y_m y_{mn_i}^{-1} = e$ for each m and that no cross-subsequence of the double sequence $\{y_m y_{mn_i}^{-1}\}$ $\mathfrak{L}_a$-converges to e. Consequently, e is a ϱ-point in $(L, \mathfrak{L}_a, \lambda_a)$. Hence, by the remark on the page 371, λ_a is not a topology. Therefore $\lambda \neq \lambda_a$.

$(L, \mathfrak{L}_a, \lambda_a, .)$ is not sequentially regular: Since λ_a is finer than λ and $\lambda \neq \lambda_a$, it follows from Lemma 1 that there is a point x_0 and an $\mathfrak{L}$-convergent sequence of points z_m in L such that $x_0 \in \lambda \bigcup z_m - \lambda_a \bigcup z_m$. Then $z_m \neq x_0$ for each m and no subsequence of $\{z_m\}$ $\mathfrak{L}_a$-converges to x_0. Let g be a real-valued function which is sequentially continuous on $(L, \mathfrak{L}_a, \lambda_a, .)$. We prove that $\lim g(z_m) = g(x_0)$. By $(\mathscr{L}_3)$ it is sufficient to prove that for any subsequence $\{x_m\}$ of $\{z_m\}$ there is a subsequence $\{x_{m_i}\}$ of $\{x_m\}$ such that $\lim g(x_{m_i}) = g(x_0)$. Because $L = \lambda L_1$, there is a sequence $\{x_{mn}\}_{n=1}^{\infty}$ of points $x_{mn} \in L_1$, $x_{mn} \neq x_0$, which $\mathfrak{L}$-converges and, by (D_6), also $\mathfrak{L}_a$-converges to x_m for each m By (5) there is a natural k_m such that

$$(+) \qquad |g(x_m) - g(x_{mn})| < m^{-1} \quad \text{for all} \quad n \geqq k_m .$$

Denote $A = \bigcup_{m=1}^{\infty} \bigcup_{n=k_m}^{\infty} x_{mn}$. Since $\mathfrak{L}\text{-}\lim x_m = x_0$, $x_{mn} \neq x_0$ and because λ fulfils (F), we have $x_0 \in \lambda A$. Hence there is a cross-subsequence $\{x_{m_i n_{m_i}}\}_{i=1}^{\infty}$ in $\{x_{mn}\}_{m=1,n=k_m}^{\infty,\ \infty}$ which $\mathfrak{L}$-converges and consequently, by (D_6), also $\mathfrak{L}_a$-converges to x_0. Therefore $\lim_i g(x_{m_i n_{m_i}}) = g(x_0)$, by Lemma 2. Because $n_{m_i} \geqq k_{m_i}$ and in view of $(+)$ we have $\lim g(x_{m_i}) = g(x_0)$.

$(L, \mathfrak{L}_a, \lambda_a, .)$ is not regular: Let $\{z_m\}$, $\{x_m\}$ and $\{x_{mn}\}$ have the same meaning as above. Since no subsequence of $\{x_m\}$ $\mathfrak{L}_a$-converges to x_0, from (D_2) it follows that $L - \bigcup x_m$ is a λ_a-neighbourhood of x_0 in L. If there were a λ_a-neighbourhood V of x_0 such that $\lambda_a V \subset L - \bigcup x_m$, then $\lambda_a V \cap \bigcup x_m = \emptyset$ and there would be, by (D_1), a function f on N into N such that no point x_{mn}, $n \geqq f(m)$ would belong to V. However $x_0 \in \lambda \bigcup_{m=1}^{\infty} \bigcup_{n=f(m)}^{\infty} x_{mn}$. This is, by (1), a contradiction.

Example 6. Let $(R, \mathfrak{R}, \varrho, +)$ be the topological group of rational numbers (see Example 1). Let R_k be the set of all rational numbers of the form rs^{-1} where r denotes an integer and $s = \prod_{i=1}^{k} p_i^{m_i}$, p_i being the i-th prime number and m_i a non-negative integer, $1 \leqq i \leqq k$. Then $\{R_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of subgroups of R such that $R = \bigcup_{k=1}^{\infty} R_k$ and $R = \varrho R_1$. Therefore, by Lemma 10, $(R, \mathfrak{R}_a, \varrho_a, +)$ is a convergence group which is neither regular nor sequentially regular.

Let $(L, \mathfrak{L}, \lambda)$ be a convergence space. Then (L, λ^{ω_1}) is a topological space [3] where λ^{ω_1} is the finest topology coarser than λ. Let . be a group operation on L such that $(L, \mathfrak{L}, \lambda, .)$ is a convergence group. There arises a question whether $(L, \lambda^{\omega_1}, .)$ is always a topological group. The answer is negative. As a matter of fact, $(R, \mathfrak{R}_a, \varrho_a, +)$ is a convergence group (see Example 6) in which the set $P = \bigcup_{i=1}^{\infty} p_i^{-1}$ is ϱ_a-closed; consequently it is $\varrho_a^{\omega_1}$-closed in $(R, \varrho_a^{\omega_1})$. Therefore $R - P$ is a $\varrho_a^{\omega_1}$-neighbourhood of the zero-element 0 in $(R, \varrho_a^{\omega_1})$. Suppose that $U(0)$ is a $\varrho_a^{\omega_1}$-neighbourhood contained in $R - P$. Then $U(0)$ is a ϱ_a-neighbourhood as well, ϱ_a being finer than $\varrho_a^{\omega_1}$. Since $P \cap \varrho_a U(0) \neq 0$ and $\varrho_a U(0) \subset \varrho_a^{\omega_1} U(0)$, we have $\varrho_a^{\omega_1} U(0) \not\subset R - P$. Therefore $(R, \varrho_a^{\omega_1})$ fails to be a regular topological space. For this reason $(R, \varrho_a^{\omega_1}, +)$ cannot be a topological group.

Let $(G, u, .)$ be a topological group. Define a convergence $\mathfrak{L}_u$ as follows: $(\{x_n\}, x) \in \in \mathfrak{L}_u$ whenever each u-neighbourhood of the point x contains nearly all x_n. Then $\mathfrak{L}_u$ induces a convergence closure λ_u which is the coarsest convergence closure for G finer than u [3].

Lemma 11. *Let $(G, u, .)$ be a topological group. Then $(G, \mathfrak{L}_u, \lambda_u, .)$ is a convergence group.*

Proof. Let $\mathfrak{L}_u$-lim $x_n = x$ and $\mathfrak{L}_u$-lim $y_n = y$. Let W be a u-neighbourhood of the point xy^{-1}. Then there are u-neighbourhoods U and V of points x and y such that $UV^{-1} \subset W$. Hence $x_n y_n^{-1} \in W$ for nearly all n. Consequently $\mathfrak{L}_u$-lim $x_n . y_n^{-1} = x.y^{-1}$ and (SG) holds true.

Now we shall define the notion of a convergence subgroup. Let $(L, \mathfrak{L}, \lambda, .)$ be a convergence group. Each λ-closed subgroup of L will be called a convergence subgroup of L. It can be easily proved that a non-empty subset H of a convergence group is a convergence subgroup if and only if the following condition is satisfied:

(D_7) If $\{x_n\}$ and $\{y_n\}$ are sequences of points of H such that lim $x_n = x$ and lim $y_n = y$, then $xy^{-1} \in H$.

Theorem 3. *Let $(L, \mathfrak{L}, \lambda, .)$ be a convergence group. Let H be a subgroup of L and ξ an ordinal. Then $\lambda^\xi H$ is a subgroup of L and $\lambda^{\omega_1} H$ is the smallest closed subgroup containing H as a subgroup.*

Proof. Suppose, the assertion is true for all ordinals ξ such that $\xi < \alpha$. Let x, y be points of $\lambda^\alpha H$. If $\alpha - 1$ exists, then $\mathfrak{L}$-lim $x_n = x$ and $\mathfrak{L}$-lim $y_n = y$ where $\{x_n\}$ and $\{y_n\}$ are suitable sequences of points in $\lambda^{\alpha-1}H$. Then $x_n y_n^{-1} \in \lambda^{\alpha-1}H$; consequently $xy^{-1} \in \lambda^\alpha H$. If $\alpha \neq 0$ and $\alpha - 1$ does not exist then $\lambda^\alpha H = \bigcup_{\xi<\alpha} \lambda^\xi H$; therefore $\lambda^\alpha H$ is a group. From this it follows that $\lambda^{\omega_1} H$ is the smallest λ-closed group containing H as a subset.

Let $(L, \mathfrak{L}, \lambda, .)$ be a convergence commutative group and H an invariant convergence subgroup of L. Define the convergence on the quotient group L/H as follows:

(D_8) If T_n and T are cosets of H then $\lim T_n = T$ whenever there is a sequence $\{b_n\}$ of points $b_n \in T_n$ and a point $b \in T$ such that $\mathfrak{L}$-$\lim b_n = b$.

The axioms $(\mathscr{L}_1)$ and $(\mathscr{L}_2)$ are evidently fulfilled. Now, prove that also the axiom $(\mathscr{L}_0)$ is true. Let (D_8) hold. Let $c_n \in T_n$ and $c \in T_0$ and $\mathfrak{L}$-$\lim c_n = c$. Since H is closed, we have $bc^{-1} \in H$, by (SG). It follows $c \in T$ and $T = T_0$.

It is easy to see that the convergence on the quotient group L/H fulfils the condition (SG). Consequently L/H is a convergence group.

4

Now, consider the relation between convergence and topological groups. The topological group $(G, u, .)$ is a group and a topological space such that the map xy^{-1} on the Cartesian product $G \times G$ onto G is continuous. This condition is equivalent to the following well known condition:

(TG) If x and y are points of G and W a neighbourhood of the point xy^{-1}, then there are neighbourhoods U of x and V of y such that $UV^{-1} \subset W$.

In the definition of the topological group the topology u cannot be replaced by a T_1-closure (especially by a convergence closure) v which does not fulfil the axiom (F). This is shown by the following

Lemma 12. *Let $(G, \mathfrak{L}, \lambda)$ be a convergence space and $(G, .)$ a group fulfilling the condition* (TG). *Then λ is a topology.*

Proof. Suppose that, on the contrary, there is a set $A \subset G$ and a point $a \in \lambda\lambda A - \lambda A$. Choose a λ-neighbourhood $V(a)$ of a such that $A \cap V(a) = \emptyset$. By (TG), there are λ-neighbourhoods $U(e)$ of e and $U(a)$ of a such that $U(a) . U(e) \subset V(a)$. Since $a \in \lambda\lambda A$, there is a point $b \in \lambda A \cap U(a)$. Consequently, by Lemma 4, there is a point $c \in A \cap (b . U(e))$. Hence, $b . U(e) \subset U(a) . U(e) \subset V(a)$ implies $c \in A \cap V(a)$. This is a contradiction. (Cf. [2].)

Remark. L. Mišík [4] has proved that a convergence group $(L, \mathfrak{L}^*, \lambda, .)$ is a topological space if and only if there is no ϱ-point in L. Since each convergence belonging to the class $[\mathfrak{L}]$ induces the same convergence closure on L and because, by Lemma 3, both conditions (γ) and (non ϱ) are equivalent, it follows

Corollary 3. *The convergence group $(L, \mathfrak{L}, \lambda, .)$ is a topological space if and only if each point of L has a cross-subsequence property.*

Let $(L, \mathfrak{L}, \lambda)$ be a convergence space in which a group operation . is defined. Denote[3]) λ_{12} the convergence product closure for $L \times L$ and by $\lambda \times \lambda$ the T_1-closure for $L \times L$ defined by the product of λ-neighbourhoods. From (D_2) and (D_1) it follows that if $(L, \mathfrak{L}, \lambda)$ fulfills (TG), then (SG) is fulfilled as well. If $\lambda \times \lambda = \lambda_{12}$, then both systems are equivalent: the system of all $\lambda \times \lambda$-neighbourhoods and the system of all λ_{12}-neighbourhoods; in this case both conditions (TG) and (SG) are equivalent[9]).

From this it can be deduced that the theory of convergence groups $(L, \mathfrak{L}, \lambda, .)$ such that $\lambda \times \lambda = \lambda_{12}$ is involved in the theory of topological groups. All such convergence groups fulfil the axiom of closed closure (F), by Lemma 12.

There are, however, convergence groups containing non-closed closures as subsets, for example the convergence group $(R, \mathfrak{R}_a, \varrho_a, +)$. The group operations xy^{-1} in such convergence groups are sequentially continuous in the Cartesian convergence product closure λ_{12} without being continuous in the Cartesian neighbourhood closure $\lambda \times \lambda$. This is possible only in the case when $\lambda \times \lambda \neq \lambda_{12}$. A convergence group $(R, \mathfrak{R}_a, \varrho_a, +)$ mentioned above is the case like this.

The existence of some convergence nontopological groups of quite another kind follows from the fact that the set R of all rational numbers r_n, $n = 1, 2, \ldots$, is not G_δ in the set of reals and consequently

$$R \in \lambda\lambda\mathbf{A} - \lambda\mathbf{A} \quad \text{where} \quad \mathbf{A} = \bigcup_{m,n=1}^{\infty} \left(\bigcup_{k=1}^{m} (r_k - n^{-1}, r_k + n^{-1}) \right).$$

From this it can be deduced that the following groups are convergence nontopological groups: The system of all linear Borel sets, the system 2^X of all subsets of a given set of power $\geqq 2^{\aleph_0}$ with the symmetric difference as a group operation and with the usual convergence of sequences of sets, the class of all Baire functions and the class of all real-valued functions on a given point set of power $\geqq 2^{\aleph_0}$ with the addition as a group-operation and with the convergence at each point.

From Corollary 1 and from what has been noticed above it instantly follows

Corollary 4. *Let $(L, \mathfrak{L}, \lambda, .)$ be a convergence group every point of which has a countable character. Then $(L, \lambda, .)$ is a topological group.*

Now we are going to construct a convergence group $(\mathbf{F}, \mathfrak{L}, \lambda, +)$ such that $(\mathbf{F}, \lambda, +)$ is a topological group every point of which has an uncountable character.

Let X be an infinite point set of power $\aleph_\alpha$. Denote by $\mathbf{F}$ the system of all finite subsets of X including the empty set $\emptyset$. Then $\mathbf{F}$ is a convergence group with the symmetric difference as a group operation. Denote it by $(\mathbf{F}, \mathfrak{L}, \lambda, +)$.

---

[9]) I do not know whether the equivalence of both conditions (TG) and (SG) implies $\lambda \times \lambda = \lambda_{12}$.

Let $\mathbf{X}\{H_x;\ x \in X\}$ be the product of topological groups H_x where each H_x, $x \in X$, consists of two elements 0 and 1 with the addition modulo 2 as a group operation. It is well known that $\mathbf{X}\{H_x;\ x \in X\}$ is homeomorphic to the topological group $(2^X, u, +)$. Then $(\mathbf{F}, v, +)$ is a subgroup and a subspace of 2^X with the topology v such that $\mathbf{G} \subset \mathbf{F}$ implies $v\mathbf{G} = \mathbf{F} \cap u\mathbf{G}$. Hence $(\mathbf{F}, v, +)$ is a topological group.

It is easy to see that the topology v for $\mathbf{F}$ can be defined by means of Cartesian neighbourhoods $\mathbf{U}(A; B) \subset \mathbf{F}$ as follows:

The v-neighbourhood $\mathbf{U}(A; B)$ of an element $A \in \mathbf{F}$ where B is any element of $\mathbf{F}$, $B \cap A = \emptyset$, is the set of all elements $Z \in \mathbf{F}$ such that $A \subset Z \subset X - B$.

If $\mathbf{U}(A; B)$ is any v-neighbourhood of A and $\{A_n\}$ is a sequence of elements $A_n \in \mathbf{F}$ such that $\operatorname{Lim} A_n = A$ then, evidently, $A \subset A_n \subset X - B$ for nearly all n. It follows, by (D_2), that λ is finer than v. Now we shall prove that

(i) $$\lambda = v\,.$$

Proof[10]). Let $\mathbf{A} \subset \mathbf{F}$ and $A \in v\mathbf{A} - \mathbf{A}$. If we have chosen v-neighbourhoods $\mathbf{U}(A; \bigcup_{j=1}^{i} B_j - A)$, $i = 1, 2, \ldots, k$, such that $B_j - A$ are disjoint sets and B_j belong to $\mathbf{A}$ then there is an element $B_{k+1} \in \mathbf{A} \cap \mathbf{U}(A; \bigcup_{j=1}^{k} B_j)$. In such a way we have a sequence of elements $B_n \in \mathbf{A}$ such that $\operatorname{Lim}(B_n - A) = \emptyset$, i.e. $\operatorname{Lim} B_n = A$. Consequently $v\mathbf{A} \subset \subset \lambda\mathbf{A}$ and v is finer than λ.

(ii) *The character of the zero-element $\emptyset$ in $\mathbf{F}$ is $\aleph_\alpha$.*

Proof. First notice that the character of $\emptyset$ in $\mathbf{F}$ cannot exceed $\aleph_\alpha$, the collection of all Cartesian neighbourhoods $\mathbf{U}(\emptyset; B)$ having the power $\aleph_\alpha$. On the other hand, from Example 3 it follows that $\aleph_\alpha$ cannot exceed the character of $\emptyset$ in $\mathbf{F}$.

(iii) *$(\mathbf{F}, \mathfrak{L}, \lambda, +)$ is a σ-countably compact space.*

Proof. Let p be a natural. It suffices to prove that the system of all sets of the power $\leqq p$ is countably compact in $\mathbf{F}$. As a matter of fact, suppose that the assertion is true for all $q \leqq r$ where $r < p$. Let $\{A_n\}$ be a sequence of sets $A_n \subset X$ of powers $\leqq r + 1$. If $\operatorname{Lim\,sup} A_n = \emptyset$ then $\operatorname{Lim} A_n \in \mathbf{F}$. If there is a point $x_0 \in \operatorname{Lim\,sup} A_n$, choose a subsequence $\{A_{n_i}\}$ of all A_n which contain x_0. By our supposition there is a subsequence $\{A_{n_{i_k}} - (x_0)\}_{k=1}^{\infty}$ of the sequence $\{A_{n_i} - (x_0)\}_{i=1}^{\infty}$ and a set $B \subset X$ of power $\leqq r$ such that $\operatorname{Lim}(A_{n_{i_k}} - (x_0)) = B$. Consequently, $\operatorname{Lim} A_{n_{i_k}} = B \cup (x_0)$ and the power of $B \cup (x_0)$ is $\leqq r + 1$.

From (i), (ii) and (iii) it follows that the convergence group $(\mathbf{F}, \mathfrak{L}, \lambda, +)$ is a topological group which is σ-countably compact and each element has the character $\aleph_\alpha$.

---

[10]) For the shortening of the proof I am indebted to V. Koutník.

R e m a r k. The notion of convergence ring is defined in an analogous way as the notion of convergence group. A convergence ring $(L, \mathfrak{L}, \lambda. +, .)$ is a convergence space and a ring such that both maps $\varphi(x, y) = x - y$ and $\psi(x, y) = x \,.\, y$ are sequentially continuous on the convergence product $L \times L$, i.e. if $\lim x_n = x$ and $\lim y_n = y$ then there is an increasing sequence of naturals n_i such that $\lim (x_{n_i} - y_{n_i}) = x - y$ and $\lim (x_{n_i} \,.\, y_{n_i}) = xy$.

References

[1] *P. Alexandroff* et *P. Urysohn:* Une condition nécessaire et suffisante pour qu'une classe ($\mathfrak{L}$) soit une classe ($\mathfrak{D}$). C. R. Acad. Sci. Paris, *177* (1923), 1274.

[2] *E. Čech:* Topological spaces, Prague 1966.

[3] *J. Novák:* On convergence spaces and their sequential envelopes. Czech. Math. Journ. *15 (90)* 1965, p. 74—100.

[4] *J. Novák* and *L. Mišík:* O *L*-priestoroch spojitých funkcií. Matematicko-fyzikálný sborník *1* (1951), p. 1—17.

[5] *J. Novák:* Eine Bemerkung zum Begriff der topologischen Konvergenzgruppen. Celebrazioni Archimedee del secolo XX, Simposio di topologia, 1964, p. 71—74.

[6] *O. Schreier:* L-Gruppen. Abhandlungen aus dem math. Seminar 1926, p. 15.

[7] *O. Schreier:* Verwandtschaft stetiger Gruppen im grossen. Ibidem 1927, p. 236.

Author's address: Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV v Praze).