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PERIODIC SOLUTIONS OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION WITH QUADRATIC DISSIPATIVE TERM

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1. INTRODUCTION

The aim of this paper is to investigate the existence of periodic solutions of several boundary-value problems (with homogeneous boundary conditions) for the biharmonic wave equation in some bounded domain in E_N, i.e. for the equation

$$(1,1)\qquad u_{tt}(t, x) + \Delta^2 u(t, x) + cu(t, x) + u_t(t, x) + u_t(t, x)\,|u_t(t, x)| = f(t, x)\,,$$

where $f(t, x)$ is a given function, periodic in the variable t, and $x \in \Omega$, where $\Omega \subset E_N$ is a bounded domain in Euklidean space E_N.

This original problem is solved in a somewhat more general form. The existence of periodic solutions of a homogeneous boundary-value problems for the equation

$$(1,2)\qquad u_{tt}(t, x) + Au(t, x) + u_t(t, x) + u_t(t, x)\,|u_t(t, x)| = f(t, x)$$

is investigated, where A is a elliptic differential operator of the order $2k$, k-natural, of the form

$$(1,3)\qquad A\,u(x) = \sum_{|i|,|j| \leq k} (-1)^{|i|}\, D^i(a_{ij}(x)\, D^j\, u(x))\,.$$

Thus a similar problem for the wave equation (i.e. $A = -\Delta$), which has been solved by G. PROUSE in the paper [8], is included here. The technique of Prouse's work has become the base of the present paper. A similar problem for the wave equation, but with more general non-linear term, is solved also by G. PRODI (in the paper [7]).

We shall consider weak solutions of the equation (1,2) which satisfy the boundary conditions in some generalized sense. The precise formulation of these concepts will be given in the sequel. The periodic solutions of our problems will be constructed with help of the Galerkin approximation procedure often used in such cases.

After necessary mathematical preliminaries (section 2), section 3 contains a precise formulation of the given problem, i.e. the definition of a weak periodic solution of the boundary-value problem for the equation (1,2). The existence theorem is formulated in section 4. In section 5, the demonstration of the existence of the Galerkin approximations is presented. The estimates of these approximations and the proof of the convergence of a certain subsequence of the approximative solutions are contained in sections 6 and 7. In section 8 the uniqueness of the solution is established. Eventually, some special problems for the wave and the biharmonic wave equations are studied as examples.

Because of application of imbedding theorems in the proof of the existence of the solution, the dimension of the space E_N, in which the problem is considered, is limited. We shall be able to prove the existence and the uniqueness of the solution in the space E_3 for the second-order operator and in the space E_5 (maximally) for the operator of the order $2k$, $k \geqq 2$.

2. PRELIMINARIES

First, we shall recall definitions and basic properties of some well-known function spaces. Further, the definition of the spaces of functions which are Bochner integrable, the differentiability in these spaces and some necessary theorems for these spaces will be presented.

We denote by E_N the N-dimensional real Euklidean space. All functions, if not said otherwise, will be considered as real-valued ones. Let Ω be a bounded domain in E_N; $\mathscr{D}(\Omega)$ is a space of functions, which have all derivatives in Ω, with a compact support. As usual, we denote by $L_p(\Omega)$, $1 \leqq p < \infty$, the space of all functions v, Lebesgue measurable in Ω and such, that Lebesgue integral $\int_\Omega |v(x)|^p \, \mathrm{d}x$ is convergent. $L_p(\Omega)$ is a separable Banach space with norm $|v|_{L_p(\Omega)} = (\int_\Omega |v(x)|^p \, \mathrm{d}x)^{1/p}$. The space $L_2(\Omega)$ is a Hilbert space with inner product

$$(u, v)_{L_2(\Omega)} = \int_\Omega u(x) \, v(x) \, \mathrm{d}x \, .$$

Let $\alpha = (\alpha_1, \ldots, \alpha_N)$ be N-dimensional vector with $\alpha_i \geqq 0$ integer for $i = 1, 2, \ldots$ $\ldots, N$. Let us set $|\alpha| = \sum_{i=1}^{N} \alpha_i$; we shall use notation

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}} \, .$$

As usual, the function v is called the generalized α-derivative of u, if the relation

$$\int_\Omega u(x) \, D^\alpha \varphi(x) \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v(x) \, \varphi(x) \, \mathrm{d}x$$

holds for all functions $\varphi \in \mathscr{D}(\Omega)$. We denote $v = D^\alpha u$ again.

The space $W_p^{(k)}(\Omega)$ (k non-negative integer, $p \geqq 1$) is the linear space of all functions $u \in L_p(\Omega)$ having the generalized derivative $D^\alpha u \in L_p(\Omega)$ for $|\alpha| \leqq k$. The space $W_p^{(k)}(\Omega)$ is a reflexive Banach space with norm $|u|_{W_p^{(k)}(\Omega)} = (\sum_{|\alpha| \leqq k} |D^\alpha u|^p_{L_p(\Omega)})^{1/p}$. We shall use notation $W_2^{(k)}(\Omega) = H_k(\Omega)$ (for $p = 2$). The spaces $H_k(\Omega)$ are separable Hilbert spaces with inner product

$$(u, v)_{H_k(\Omega)} = \sum_{|\alpha| \leqq k} \int_\Omega D^\alpha u(x)\, D^\alpha v(x)\, \mathrm{d}x\,.$$

Norm in the space $H_k(\Omega)$ will be denoted by $|.|_{H_k(\Omega)}$.

The space $\mathring{W}_p^{(k)}(\Omega)$ is the closure of $\mathscr{D}(\Omega)$ under norm of the space $W_p^{(k)}(\Omega)$. We shall denote $\mathring{W}_2^{(k)}(\Omega) = \mathring{H}_k$; it holds $\mathring{H}_0(\Omega) = H_0(\Omega) = L_2(\Omega)$, and for $k \geqq 1$ the space $\mathring{H}_k(\Omega)$ is the proper subspace of $H_k(\Omega)$.

Let us recall three of the known imbedding theorems (see, for example, [5]).

Theorem 2.1. *Let $\Omega \subset E_N$ be a bounded domain with lipschitz boundary, $p \geqq 1$, $k . p < N$, $1/q \geqq 1/p - k/N$. Then $W_p^{(k)}(\Omega) \subset L_q(\Omega)$ in algebraic and topological sense. Further, if $1 \geqq 1/q > 1/p - k/N$, the identical transformation from $W_p^{(k)}(\Omega)$ into $L_q(\Omega)$ is compact.*

Theorem 2.2. *Let $\Omega \subset E_N$ be a bounded domain with lipschitz boundary, $p \geqq 1$, $k . p = N$. Then $W_p^{(k)}(\Omega) \subset L_q(\Omega)$ for all $q \geqq 1$ (algebraically and topologically). The identical transformation from $W_p^{(k)}(\Omega)$ into $L_q(\Omega)$ is compact.*

Theorem 2.3. *Let $\Omega \subset E_N$ be a bounded domain with lipschitz boundary, $p \geqq 1$, $kp > N$. Set $\mu = k - N/p$ for $k - N/p < 1$ and let $\mu < 1$ for $k - N/p = 1$ and $\mu = 1$ for $k - N/p > 1$. Then, $W_p^{(k)}(\Omega) \subset C^{(0),\mu}(\bar{\Omega})$ algebraically and topologically. Further, the identical transformation from $W_p^{(k)}(\Omega)$ into $C(\bar{\Omega})$ is compact.*

(Here, $C^{(0),\mu}(\bar{\Omega})$ denotes the space of functions $u(x)$, which are continuous on $\bar{\Omega}$ and such, that there exists a constant $c > 0$ such, that $|u(x) - u(y)| \leqq c|x - y|^\mu$ for all $x, y \in \bar{\Omega}$.)

Now, all linear normed spaces will be considered over the field of real numbers. Let H be a separable Hilbert space with inner product $(., .)_H$ and with norm $|.|_H$. If $M \subset E_1$, we denote by $u : M \to H$ a function which is defined on the set M and values of which are in the space H. Continuity and strong derivative of such function are defined by the same way as in the case of real-valued function (see, for example, [3]). Let I be an interval in E_1. We introduce the following notation:

$$C^{(k)}(I, H) = \left\{u;\ u : I \to H, \frac{\mathrm{d}^{(k)}u}{\mathrm{d}t^k} \text{ is continuous in } I\right\},$$

$$C_0^{(k)}(I, H) = \{u;\ u \in C^{(k)}(I, H) \text{ and supp } u \text{ is compact in } I^0\}\,,$$

$$C(T, H) = \{u : E_1 \to H, u \text{ is continuous in } E_1 \text{ and } T\text{-periodic}\}\,.$$

Now set $I = (a, b)$, $-\infty \leqq a < b \leqq +\infty$. We denote by $L^1(I, H)$ the space of functions $u : I \to H$ which are strongly measurable and Bochner integrable on I; it is a separable Banach space with norm

$$|u|_{L^1(I,H)} = \int_a^b |u(t)|_H \, dt .$$

Further, we denote by $L^2(I, H)$ the space, consisting of strongly measurable functions $u : I \to H$ with $\int_a^b |u(t)|_H^2 \, dt < \infty$; it is a Hilbert space with inner product $(u, v)_{L^2(I,H)} = \int_a^b (u(t), v(t))_H \, dt$. Finally, the space $L^\infty(I, H)$ consists of functions $u : I \to H$ which are strongly measurable and such that $\operatorname{supess}_{t \in I} |u(t)|_H < \infty$. The space $L^\infty(I, H)$ is a Banach space with norm $|u|_{L^\infty(I,H)} = \operatorname{supess}_{t \in I} |u(t)|_H$.

Let $T > 0$. A function u, defined a.e. in E_1, with values in H, is called T-periodic, if the following holds: if u is defined for $t \in E_1$, it is defined also for $t + kT$, k integer, and $u(t) = u(t + kT)$. We denote by $L^p(T, H)$ $(p = 1, 2, \infty)$ the space of T-periodic functions u such, that $u \in L^p((0, T), H)$. (T-periodic function means the equivalence class of functions, which is represented by any T-periodic function (in the sense above).)

Definition 2.1. A function $u \in L^p(I, H)$, $p = 1, 2, \infty$ is called *differentiable in $L^p(I, H)$ with a derivative $v \in L^p(I, H)$*, if the relation

$$\int_I (u(t), \varphi'(t))_H \, dt = - \int_I (v(t), \varphi(t))_H \, dt$$

holds for each function $\varphi \in C_0^{(1)}(I, H)$.

Definition 2.2. A function $u \in L^p(T, H)$, $p = 1, 2, \infty$ is *differentiable in $L^p(T, H)$ with a derivative $v \in L^p(T, H)$*, if the relation

$$\int_{-\infty}^{+\infty} (u(t), \varphi'(t))_H \, dt = - \int_{-\infty}^{+\infty} (v(t), \varphi(t))_H \, dt$$

holds for each function $\varphi \in C_0^{(1)}(E_1, H)$. (We shall denote this derivative by u'.)

Lemma 2.1. *Let $u \in L^2(T, H)$ be differentiable in $L^2(T, H)$. Then there exists a continuous T-periodic function $\tilde{u} : E_1 \to H$ such, that $u(t) = \tilde{u}(t)$ a.e. in E_1, while $\tilde{u}(t)$ may be written in the form*

$$\tilde{u}(t) = \tilde{u}(0) + \int_0^t u'(\tau) \, d\tau$$

(*the integral above is ment in the Bochner sense*).

Proof. According to Theorem 2.2 from [11] it holds: if $I = (a, b)$, $-\infty \leqq a < < b \leqq +\infty$, $u \in L^2(I, H)$ is differentiable in $L^2(I, H)$ with derivative u', then

$$u(\alpha) - u(\beta) = \int_\alpha^\beta u'(\tau) d\tau$$

for almost all (further we use abbreviation a. a.) $\alpha, \beta \in I$ and $u(t)$ is equivalent in $L^2(I, H)$ to an absolutely continuous function

$$\tilde{u}(t) = \tilde{u}(\alpha) + \int_\alpha^t u'(\tau)\, d\tau .$$

Now, let us consider the intervals $I_n = \langle -n, n \rangle$, n natural. Then there exist a set $M_n \subset I_n$, $\mu(M_n) = 0$ and the function $u_n(t) = u_n(0) + \int_0^t u'(\tau)\, d\tau$ such, that $u(t) = u_n(t)$ for all $t \in I_n - M_n$. Evidently, $u_m(0) = u_n(0)$ for all m, n; (namely, if $u_m(0) \neq u_n(0)$ for some $m, n(m > n$ for instance), then there would be $u_m(t) \neq u_n(t)$ for all $t \in I_n$, which is clearly impossible.) We denote $M = \bigcup_{n=1}^{\infty} M_n$, $\tilde{u}(t) = u_1(0) + \int_0^t u'(\tau)\, d\tau$. Obviously, the function $\tilde{u}(t)$ is continuous in E_1 (see, for example, [3]). Further, $\mu(M) = 0$ and for all $t \in E_1 - M$ there is $u(t) = \tilde{u}(t)$. It remains to prove that $\tilde{u}(t)$ is a T-periodic function. But this fact follows easily by continuity of $\tilde{u}(x)$ and by the definition of T-periodicity of $u(x)$.

We mention one another statement ([11], Theorem 2.3), which will be used later.

Lemma 2.2. *Let $I = (a, b)$, $-\infty < a < b < \infty$, and let $u, v \in L^2(I, H)$, respectively, be differentiable in $L^2(I, H)$, with derivatives u', v', respectively. Then*

$$\int_a^b (u'(t), v(t))_H\, dt + \int_a^b (u(t), v'(t))_H\, dt = (u(b), v(b))_H - (u(a), v(a))_H .$$

Proof. (We sketch it for completeness.) According to Theorem 2.2. from [11] the functions u, v can be written as

$$u(t) = u(a) + \int_a^t u'(\tau)\, dt , \quad v(t) = v(a) + \int_a^t v'(\tau)\, d\tau , \quad t \in I .$$

It follows by properties of the Bochner integral, that $u(t)$ or $v(t)$, respectively, have the strong derivative $u'(t)$ or $v'(t)$, respectively, a.e. in I; further, inner product $(u(t), v(t))_H$ is a absolutely continuous real function in I and the derivative

$$\frac{d}{dt} (u(t), v(t))_H = (u'(t), v(t))_H + (u(t), v'(t))_H$$

exists a.e. in I. Whence, the statement of this lemma follows easily (using well-known properties of the Lebesgue integral).

From Lemma 1 and Lemma 2 this consequence results :

Lemma 2.3. *Let $u, v \in L^\infty(T, H)$ be differentiable in $L^\infty(T, H)$. Then*

$$\int_0^T (u'(t), v(t))_H \, dt = - \int_0^T (u(t), v'(t))_H \, dt .$$

3. FORMULATION OF PERIODIC BOUNDARY-VALUE PROBLEM

Let $\Omega \subset E_5$ for $k \geqq 2$ integer and $\Omega \subset E_3$, if $k = 1$, be a bounded domain with the lipschitz boundary. Let a_{ij}, $|i|, |j| \leqq k$ be real-valued functions defined, measurable and bounded on Ω.

We consider the differential operator

$$A = \sum_{|i|, |j| \leqq k} (-1)^{|i|} D^i(a_{ij}(x) D^j) , \tag{3.1}$$

to which we associate the bilinear form

$$((u, v)) = \int_\Omega \sum_{|i|, |j| \leqq k} a_{ij}(x) D^i v(x) D^j u(x) \, dx , \tag{3.2}$$

which is defined and continuous on $H_k(\Omega) \times H_k(\Omega)$.

Let V, $\mathring{H}_k(\Omega) \subset V \subset H_k(\Omega)$, be a linear subspace, closed in $H_k(\Omega)$. (For brevity in notation we shall use only H_k instead of $H_k(\Omega)$.)

Definition 3.1. Let $f \in L^2(T, H_0)$. A function $u : E_1 \to V$ is called *a weak solution of the periodic boundary-value problem corresponding to the space V for the equation*

$$u''(t) + A\,u(t) + u'(t) + u'(t)\,|u'(t)| = f(t) , \tag{3.3}$$

if $u \in L^\infty(T, V)$, u is differentiable in $L^\infty(T, V)$ with the derivative $u' \in L^\infty(T, V)$, u' is differentiable in $L^\infty(T, H_0)$ with the derivative $u'' \in L^\infty(T, H_0)$, and the relation

$$\int_0^T \{(u''(t), \varphi(t)) + ((u(t), \varphi(t))) + (u'(t), \varphi(t)) + \\ + (u''(t)\,|u'(t)|, \varphi(t))\} \, dt = \int_0^T (f(t), \varphi(t)) \, dt \tag{3.4}$$

holds for each function $\varphi \in L^2((0, T), V)$. (We denote, for brevity, $(.,.)$ inner product in H_0, and $|.|_k$ norm of space L_k.)

We shall call this problem *the periodic V boundary-value problem* and we shall use the notation $\mathscr{P}(V, f)$ or briefly $\mathscr{P}(V)$.

Remark 3.1. For a.a. $t \in E_1$ there is $u'(t) \in V$, and therefore, according to Theorem 2.1 (imbedding theorem) $u'(t) \in L_4$ for $N \leqq 4k$; thus, the integral $\int_0^T (u'(t)\,|u'(t)|, \varphi(t))\,\mathrm{d}t$ is meaningful for the dimension, which we consider.

Remark 3.2. (The interpretation of a solution of $\mathscr{P}(V)$.) Let u be a solution of the problem $\mathscr{P}(V)$. Let $\{v_n\}_{n=1}^{\infty}$ be a dense countable subset in V (V is a separable space). Consider functions $\varphi(t)$ of the form $\varphi(t) = \psi(t)\,v_n$, where $\psi(t) \in C_{\langle 0,T\rangle}$; then $\varphi(t) \in L^2((0, T), V)$ and thus

$$\text{(3.5)} \qquad \int_0^T \{(u''(t), v_n) + ((u(t), v_n)) + (u'(t), v_n) + \\ + (u'(t)\,|u'(t)|, v_n)\}\,\psi(t)\,\mathrm{d}t = \int_0^T (f(t), v_n)\,\psi(t)\,\mathrm{d}t\,.$$

As the space $C_{\langle 0,T\rangle}$ is dense in $L_2(0, T)$, there exists a set $M_n \subset \langle 0, T\rangle$, $\mu(M_n) = 0$ such that for all $t \in \langle 0, T\rangle - M_n$ there is

$$\text{(3.6)} \qquad (u''(t), v_n) + ((u(t), v_n)) + (u'(t), v_n) + (u'(t)\,|u'(t)|, v_n) = (f(t), v_n)\,.$$

If we set $M = \bigcup_{n=1}^{\infty} M_n$, then $\mu(M) = 0$ and (3.6) holds for $t \in \langle 0, T\rangle - M$ and $n = 1, 2, \ldots$ It results from the density of the set $\{v_n\}_{n=1}^{\infty}$ in V, that

$$\text{(3.7)} \qquad ((u(t), v)) = (-u''(t) - u'(t) - u'(t)\,|u'(t)| + f(t), v)$$

holds for all $v \in V$ and $t \in \langle 0, T\rangle - M$.

There is a known fact, that we can associate to the form $((u, v))$ the linear, in general unbounded, operator $\tilde{A}$, the domain of definition of which, $\mathbf{D}(\tilde{A})$, consists of $v \in V$ for which the mapping

$$\text{(3.8)} \qquad v \to ((u, v))$$

is continuous in the topology of the space H_0 (see, for example [4]). The space V is dense in H_0 obviously (because $\mathscr{D}(\Omega) \subset V$ and $\mathscr{D}(\Omega)$ is dense in H_0), and so the mapping (3.8) may be extended for $v \in \mathbf{D}(\tilde{A})$ linearly and continuously onto the whole space H_0. Then the mapping (3.8) is continuous linear functional on H_0, and thus there exists an element, which we denote by $\tilde{A}u$, $\tilde{A}u \in H_0$, such that $((u, v)) = (\tilde{A}u, v)$ for all $v \in V$.

Let us show that the operator (3.1) corresponds to the form (3.2) in this sense. Let $v \in \mathscr{D}(\Omega)$. Then

$$\int_\Omega \sum_{|i|,|j| \leq k} a_{ij} D^i v D^j u \, dx = \langle \sum_{|i|,|j| \leq k} (-1)^{|i|} D^i(a_{ij} D^j u), v \rangle ,$$

where the derivatives D^i are derivatives in the sense of the theory of distributions. But for $u \in \mathbf{D}(\tilde{A})$ the mapping $v \to \langle \sum_{|i|,|j| \leq k} (-1)^{|i|} D^i(a_{ij} D^j u), v \rangle$ is continuous in the topology H_0, and therefore

$$\sum_{|i|,|j| \leq k} (-1)^{|i|} D^i(a_{ij} D^j u) \in H_0 .$$

Thus, if $u \in \mathbf{D}(\tilde{A})$, then $\tilde{A}u = \sum_{|i|,|j| \leq k} (-1)^{|i|} D^i(a_{ij} D^j u)$ (in H_0).

So, for $u \in \mathbf{D}(\tilde{A})$ there is $((u, v)) = (\sum_{|i|,|j| \leq k} (-1)^{|i|} D^i(a_{ij} D^j u), v)$ for each $v \in V$.

According to (3.7), the mapping $v \to ((u(t), v))$ is continuous in norm of H_0 for $t \in \langle 0, T \rangle - M$; thus $Au(t) \in H_0$ for these t (the derivatives are considered in the sense of distributions) and the equation (3.3) is satisfied in the sense of H_0 for $t \in \in \langle 0, T \rangle - M$.

The choice of the space V corresponds to the boundary conditions of considered problem. Let us take, for instance, $A = \Delta^2$ and $V = \mathring{H}_2$; by the definition (3.2), there is

$$((u, v)) = \int_\Omega \sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \, \partial x_j} \cdot \frac{\partial^2 v}{\partial x_i \, \partial x_j} \, dx . \tag{3.9}$$

Using the Green theorem in (3.9) formally (n_i denotes the i-th component of the exterior normal in the boundary $\partial\Omega$ of Ω), we have

$$\int_\Omega \sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \, \partial x_j} \cdot \frac{\partial^2 v}{\partial x_i \, \partial x_j} \, dx = \int_{\partial\Omega} \sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \, \partial x_j} \cdot \frac{\partial v}{\partial x_j} n_i \, dS -$$

$$- \int_\Omega \sum_{i,j=1}^{N} \frac{\partial^3 u}{\partial x_i^2 \, \partial x_j} \cdot \frac{\partial v}{\partial x_j} \, dx = \int_{\partial\Omega} \sum_{i,j=1}^{N} \frac{\partial^2 u}{\partial x_i \, \partial x_j} \cdot \frac{\partial v}{\partial x_j} n_i \, dS -$$

$$- \int_{\partial\Omega} \sum_{i,j=1}^{N} \frac{\partial^3 u}{\partial x_i^2 \, \partial x_j} v n_j \, dS + \int_\Omega \sum_{i,j=1}^{N} \frac{\partial^4 u}{\partial x_i^2 \, \partial x_j^2} v \, dx .$$

The two area integrals vanish, because $v \in \mathring{H}_2$. Thus, here the boundary condition is determined only by the condition $u \in \mathring{H}_2$, which means, that u and $\partial u / \partial n$ vanish on $\partial\Omega$ in some generalized sense. The choice $V = \mathring{H}_2$ corresponds to the Dirichlet problem for biharmonic operator.

Other examples will be treated (as application of the general theory) in the last section of the paper.

Let us consider the problem $\mathscr{P}(V)$ for the operator (3.1) with the corresponding form (3.2) again. The existence of the weak solution of this problem is established in the following theorem.

Theorem 1. *Let* $\mathring{H}_k \subset V \subset H_k$, *V be a closed subspace of* H_k. *Let for all* $u, v \in V$

$$((u, v)) = ((v, u)), \tag{4.1}$$

and let $\alpha > 0$ *exist such, that the inequality*

$$((u, u)) \geqq \alpha |u|^2_{H_k} \tag{4.2}$$

holds for all $u \in V$. Let $f \in C(T, H_0)$, *f be differentiable in* $L^2(T, H_0)$. *Then there exists a function* $u : E_1 \to V$, $u \in L^\infty(T, V)$, *u is differentiable in* $L^\infty(T, V)$ *with the derivative* u', *where* u' *is differentiable in* $L^\infty(T, H_0)$ *with the derivative* u'' *such, that the relation*

$$\int_0^T \{(u''(t), \varphi(t)) + ((u(t), \varphi(t))) + (u'(t), \varphi(t)) + \\ + (u'(t)\,|u'(t)|, \varphi(t))\}\,dt = \int_0^T (f(t), \varphi(t))\,dt$$

holds for each function $\varphi \in L^1((0, T), V)$, *i.e. a weak solution of the problem* $\mathscr{P}(V, f)$ *exists.*

Remark 4.1. The form $((u, v))$, which is defined in (3.2), defines (under the assumption (4.1)) new inner product in V. If (4.2) holds, then norm $\|\,.\,\|$, which is determined by this new inner product (i.e. $\|u\| = ((u, u))^{1/2}$), is equivalent to origin norm $|\,.\,|_{H_k}$ in V. Hence, the space V with inner product $((.\,,\,.))$ is a separable Hilbert space again.

To prove Theorem 1, we use the Galerkin approximative procedure. Choosing suitably a sequence $\{g_j\}_{j=1}^\infty \subset V$, which spans V, we shall search for the m-th approximation $u_m(t)$ $(m = 1, 2, \ldots)$ in the form $u_m(t) = \sum_{k=1}^m \alpha_{km}(t)\, g_k$, where $\alpha_{km}(t)$ $(k = 1, 2, \ldots, m)$ are T-periodic, twice continuously differentiable real-valued functions, so that $u_m(t)$ may satisfy the system of the equations

$$(u''_m(t), g_j) + ((u_m(t), g_j)) + (u'_m(t), g_j) + \\ + (u'_m(t)\,|u'_m(t)|, g_j) = (f(t), g_j), \quad (j = 1, 2, \ldots, m). \tag{4.3}$$

Some estimates of these approximations $u_m(t)$ enable us to establish the convergence (in a certain sense) of a certain subsequence of sequence $\{u_m(t)\}_{m=1}^\infty$ to a weak solution of our problem $\mathscr{P}(V)$.

We choose the system $\{g_j\}_{j=1}^{\infty}$ as the system of all eigen-functions of the boundary-value problem, which is given by the operator A and the space V. (As usual, a number λ is called the eigen-value of the boundary-value problem, which is given by the operator A and the space V, if there exists an element $u \in V$, $u \neq 0$ such, that $((u, v)) = \lambda(u, v)$ for all $v \in V$. The function u is called eigen-function, corresponding to the eigen-value λ.) Under the assumptions (4.1) and (4.2) of the Theorem 1, the eigen-values of the operator A constitute an infinite non-decreasing sequence $\{\varrho_n\}_{n=1}^{\infty}$, where $\varrho_n \geqq 0$ and $\lim_{n\to\infty} \varrho_n = +\infty$. Further, there exists an orthogonal (with respect to inner product $((.,.))$) complete system of eigen-functions $\{v_n\}_{n=1}^{\infty}$ in V, where v_n corresponds to the eigen-value ϱ_n, such that

$$((v_j, v_k)) = \sqrt{\varrho_j}\sqrt{(\varrho_k)}\,\delta_{jk}\,, \quad (v_j, v_k) = \delta_{jk}$$

(see, for example, [5]). Then, we can put $g_n = v_n$ $(n = 1, 2, \ldots)$.

To define the approximations $u_m(t)$ in the form

$$u_m(t) = \sum_{k=1}^{m} \alpha_{km}(t)\, g_k\,, \tag{4.4}$$

where $\alpha_{km}(t)$ are real-valued, T-periodic functions, so that the relation (4.2) may hold, it is necessary, with regard to the previous choice of the base $\{g_j\}_{j=1}^{\infty}$, to find a T-periodic solution $(\alpha_{1m}(t), \ldots, \alpha_{mm}(t))$ of the system of ordinary differential equations

$$\ddot{\alpha}_{km}(t) + \varrho_k \alpha_{km}(t) + \dot{\alpha}_{km}(t) + \Big(\sum_{j=1}^{m} \dot{\alpha}_{jm}(t)\, g_j \,\Big|\sum_{j=1}^{m} \dot{\alpha}_{jm}(t)\, g_j\Big|, g_k\Big) = f_k(t)\,, \tag{4.5}$$

$(k = 1, 2, \ldots, m)$, where $f_k(t) = (f(t), g_k)$.

Before the investigation of the existence of a solution of the system (4.5), we shall prove two easy assertions, which will be used later.

Lemma 4.1. *Let $f \in C(T, H_0)$, $f' \in L^2(T, H_0)$. Then the functions $f_k(t) = (f(t), g_k)$ $(k = 1, 2, \ldots)$ are continuous, T-periodic and they have the derivative* $\mathrm{d}f_k(t)/\mathrm{d}t = (f'(t), g_k) \in L_2(0, T)$ *for almost all $t \in \langle 0, T \rangle$.*

Proof. Continuity and T-periodicity of $f_k(t)$ follows immediately by the assumption $f \in C(T, H_0)$. Further, from the existence $f' \in L^2(T, H_0)$ it results (by the properties of the Bochner integral) that $f(t)$ is even an absolutely continuous function in $\langle 0, T \rangle$ and that there exists a strong derivative $f'(t)$ a.e. in $\langle 0, T \rangle$. Hence, the function $f_k(t) = (f(t), g_k)$ is also absolutely continuous in $\langle 0, T \rangle$ and it has the derivative $f_k'(t) = (f'(t), g_k)$ a.e. in E_1. Evidently, $(f'(t), g_k) \in L_2(0, T)$.

Lemma 4.2. *Let $f \in C(T, H_0)$, $f' \in L^2(T, H_0)$ and let $\{\alpha_{im}(t)\}_{i=1}^{m}$, where $\alpha_{im}(t)$ $(i = 1, 2, \ldots, m)$ are twice continuously differentiable T-periodic real-valued* *functions, be a solution of the system* (4.5). *Then the functions* $\alpha_{im}(t)$ $(i = 1, 2, \ldots, m)$ *have the third derivative* $\dddot{\alpha}_{im}(t)$ *for almost all* $t \in \langle 0, T\rangle$ *and* $\dddot{\alpha}_{im}(t) \in L_2(0, T)$.

Proof. As

$$\ddot{\alpha}_{km}(t) = f_k(t) - \dot{\alpha}_{km}(t) - \varrho_k \alpha_{km}(t) - \left(\sum_{j=1}^{m} \dot{\alpha}_{jm}(t)\, g_j \left|\sum_{j=1}^{m} \dot{\alpha}_{jm}(t)\, g_j\right|, g_k\right),$$

it will be sufficient to prove that there exists (at least a.e. in $\langle 0, T\rangle$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j=1}^{m} \dot{\alpha}_{jm}(t)\, g_j \left|\sum_{j=1}^{m} \dot{\alpha}_{jm}(t)\, g_j\right|, g_k\right).$$

It can be shown easily, that if there exists $\dot{u}(t)$, continuous in $\langle 0, T\rangle$, then there exists the derivative $(\mathrm{d}/\mathrm{d}t)\,(|u(t)|\, u(t)) = 2\, \dot{u}(t)\, |u(t)|$, continuous on $\langle 0, T\rangle$. Thus, for $x \in \Omega$ fixed, there is

$$\frac{\partial}{\partial t}\left(\sum_{j=1}^{m} \dot{\alpha}_{jm}(t)\, g_j(x) \left|\sum_{j=1}^{m} \dot{\alpha}_{jm}(t)\, g_j(x)\right|\right) = 2 \sum_{j=1}^{m} \ddot{\alpha}_{jm}(t)\, g_j(x) \left|\sum_{j=1}^{m} \dot{\alpha}_{jm}(t)\, g_j(x)\right|.$$

There exists a constant $K > 0$ such, that $|\ddot{\alpha}_{jm}(t)|, |\dot{\alpha}_{jm}(t)| \leqq K$ for $t \in \langle 0, T\rangle$ and $j = 1, 2, \ldots, m$. Then, because $K^2(\sum_{j=1}^{m} |g_j(x)|)^2\, |g_k(x)|$ is integrable in Ω ($g_j \in L_3$ by the imbedding theorems), we have (using the known facts about the derivative of a integral)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{j=1}^{m} \dot{\alpha}_{jm}(t)\, g_j \left|\sum_{j=1}^{m} \dot{\alpha}_{jm}(t)\, g_j\right|, g_k\right) = 2\left(\sum_{j=1}^{m} \ddot{\alpha}_{jm}(t)\, g_j \left|\sum_{j=1}^{m} \dot{\alpha}_{jm}(t)\, g_j\right|, g_k\right).$$

Evidently, $\dddot{\alpha}_{km}(t) \in L_2(0, T)$. This completes the proof.

Remark 4.2. In the proof of Lemma 4.2 we considered the vector-function $u_m(t) : E_1 \to V$ as a function of variables $t \in E_1$ and $x \in \Omega$. Obviously, if $u \in L^\infty(T, V)$, then there exists a function $U = U(t, x)$, defined a.e. in $E_1 \times \Omega$ such, that the vector-function $\tilde{u}(t) : E_1 \to V$, $\tilde{u}(t) = U(t, .)$ for almost all $t \in E_1$, represents the function u. Any use of this point of view will be clear from the context. We shall use the notation u_m for both interpretations of the function u_m.

5. EXISTENCE OF GALERKIN APPROXIMATIONS

Now, let us proceed to the solving of the system (4.5) of the non-linear ordinary differential equations, with help of which the Galerkin approximations are determined. Likewise in the paper of G. Prouse, it will be used the following theorem of L. Amerio ([1]).

Lemma 5.1. *Let*

(i) *a matrix* $\mathbf{A} = \|a_{ik}\|$ *be a* $m \times m$ *matrix, symmetric and such, that the quadratic form* $\sum_{i,k=1}^{m} a_{ik}\xi_i\xi_k$ *is positiv definit;*

(ii) *the vector-valued function* $F(t) = (f_1(t), \ldots, f_m(t))$ *be defined and continuous on* E_1 *and T-periodic, while* $|F(t)|_{E_m} \leqq M$, $M > 0$ *for* $t \in E_1$;

(iii) *the vector-valued function* $\Phi(Y) = (\varphi_1(y_1, \ldots, y_m), \ldots, \varphi_m(y_1, \ldots, y_m))$ *be continuous on the whole space* E_m *and*

$$\liminf_{|Y|_{E_m}\to\infty} \frac{(\Phi(Y), Y)_{E_m}}{|\Phi(Y)|_{E_m} |Y|_{E_m}} = h > 0\,, \quad \liminf_{|Y|_{E_m}\to\infty} |\Phi(Y)|_{E_m} > \frac{M}{h}\,.$$

Then the system of differential equations

$$X''(t) = -\mathbf{A}X(t) - \Phi(X'(t)) + F(t)$$

(*where* $X(t) = (x_1(t), \ldots, x_m(t))$) *has at least one T-periodic (classical) solution.*

In our case the matrix $\mathbf{A}$ has the form

$$\mathbf{A} = \begin{pmatrix} \varrho_1, & 0, & \ldots, & 0 \\ 0, & \varrho_2, & \ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 0, & \ldots, & \varrho_m \end{pmatrix}.$$

Since $\varrho_i \geqq \varrho_1 > 0$ $(i = 1, 2, \ldots, m)$, the matrix $\mathbf{A}$ is positiv definit. The function $F(t) = (f_1(t), \ldots, f_m(t))$ is continuous, T-periodic vector-valued function (by the assumptions of Theorem 1). Further, at the system (4.5) there is $\varphi_i(y_1, \ldots, y_m) = y_i + (\sum_{k=1}^{m} y_k g_k \mid \sum_{k=1}^{m} y_k g_k|, g_i)$ $(i = 1, 2, \ldots, m)$. Let us show that the function $\Phi(Y) = (\varphi_1(y_1, \ldots, y_m), \ldots, \varphi_m(y_1, \ldots, y_m))$, defined in this way, satisfies the assumption (iii) of Lemma 5.1. The continuity of $\Phi(Y)$ on E_m follows from continuity of the mapping $Y \to \sum_{k=1}^{m} y_k g_k |\sum_{k=1}^{m} y_k g_k|$ from E_m to H_0 (for our dimension) immediately. We can show by the same method as in the paper [8] that the two estimates of (iii) are satisfied, too. For completeness, we shall repeat this proof once again.

There is, namely,

$$(5.1) \qquad (\Phi(Y), Y)_{E_m} = \sum_{i=1}^{m} [y_i + (\sum_{k=1}^{m} y_k g_k |\sum_{k=1}^{m} y_k g_k|, g_i)]\; y_i =$$
$$= \sum_{i=1}^{m} y_i^2 + \sum_{i=1}^{m} (\sum_{k=1}^{m} y_k g_k |\sum_{k=1}^{m} y_k g_k|, y_i g_i) =$$
$$= |Y|_{E_m}^2 + \int_\Omega |\sum_{i=1}^{m} y_i g_i|^3 \, dx = |Y|_2^2 + |Y|_3^3\,.$$

(As the mapping $Y \to \sum_{i=1}^{m} y_i g_i$ is isometric and isomorphic, we denote $\sum_{i=1}^{m} y_i g_i$ also by Y.) Further,

$$|\Phi(Y)|_{E_m} \leqq |Y|_{E_m} + \left(\sum_{i=1}^{m} (Y|Y|, g_i)^2\right)^{1/2}.$$

Applying the imbedding theorem 2.1 and the generalized Hölder inequality, we obtain

$$(Y|Y|, g_i)^2 \leqq \left(\int_\Omega |Y|^2 \, |g_i| \, dx\right)^2 \leqq \left(\int_\Omega |Y|^3 \, dx\right)^{4/3} \left(\int_\Omega |g_i|^3 \, dx\right)^{2/3} = |Y|_3^4 \, |g_i|_3^2 .$$

Further (applying again Theorem 2.1), we have (for $i = 1, 2, \ldots, m$)

$$|g_i|_3 \leqq c|g_i|_{H_k} \leqq c\|g_i\| = c\sqrt{\varrho_i} \leqq c\sqrt{\varrho_m},$$

and hence

$$\left(\sum_{i=1}^{m} (Y|Y|, g_i)^2\right)^{1/2} \leqq \left(\sum_{i=1}^{m} |Y|_3^4 \, |g_i|_3^2\right)^{1/2} \leqq |Y|_3^2 \left(\sum_{i=1}^{m} c^2 \varrho_i\right)^{1/2} \leqq |Y|_3^2 \sqrt{(m)} \, . \, c\sqrt{\varrho_m} .$$

Thus,

$$|\Phi(Y)|_{E_m} \leqq |Y|_{E_m} + c\sqrt{(m\varrho_m)}\, |Y|_3^2 = |Y|_2 + c\sqrt{(m\varrho_m)}\, |Y|_3^2 .$$

From here and (5.1) it results

$$\frac{(\Phi(Y), Y)_{E_m}}{|\Phi(Y)|_{E_m} |Y|_{E_m}} \geqq \frac{|Y|_2^2 + |Y|_3^3}{(|Y|_2 + c\sqrt{(m\varrho_m)}\, |Y|_3^2)\, |Y|_2} \geqq$$

$$\geqq \frac{|Y|_2^2 + |Y|_3^3}{|Y|_2^2 + c'\sqrt{(m\varrho_m)}\, |Y|_3^3} \geqq \frac{1}{c'\sqrt{(m\varrho_m)}}$$

for all natural m such that, $c'\sqrt{(m\varrho_m)} \geqq 1$. (Naturally, we consider $|Y|_{E_m} \neq 0$.) Hence, there is

$$\liminf_{|Y|_{E_m} \to \infty} \frac{(\Phi(Y), Y)_{E_m}}{|\Phi(Y)|_{E_m} |Y|_{E_m}} \geqq \frac{1}{c'\sqrt{(m\varrho_m)}} = h > 0$$

for these numbers m. Further,

$$|\Phi(Y)|_{E_m} = \sup_{|Z| \neq 0} \frac{(\Phi(Y), Z)_{E_m}}{|Z|_{E_m}} \geqq \frac{(\Phi(Y), Y)_{E_m}}{|Y|_{E_m}} = \frac{|Y|_{E_m}^2 + |Y|_3^3}{|Y|_{E_m}} \geqq |Y|_{E_m} .$$

(We used (5.1) again.) From here we obtain that

$$\liminf_{|Y|_{E_m} \to \infty} |\Phi(Y)|_{E_m} = +\infty ,$$

i.e. the assumption (iii) is also fulfilled in our case.

Obviously, there exists a natural number m_0 such, that the inequality $c' \sqrt{(m\varrho_m)} \geqq$ $\geqq 1$ holds for all $m \geqq m_0$, m natural, and hence, for each $m \geqq m_0$ there exists (by the lemma 5.1) at least one T-periodic solution $\alpha_m(t) = (\alpha_{1m}(t), \ldots, \alpha_{mm}(t))$ of the system (4.5). Thus, the approximations (4.4) can be defined for all $m \geqq m_0$.

6. ESTIMATES OF GALERKIN APPROXIMATIONS

In this section, all necessary estimates will be established, which enable us to find a subsequence of the sequence of the Galerkin approximations $\{u_m(t)\}_{m=m_0}^{\infty}$, whose limit will define a solution of our problem.

According to the definition, the functions $u_m(t)$ $(m = m_0, m_0 + 1, \ldots)$ fulfil the system

$$(6.1) \qquad (u''_m(t), g_k) + ((u_m(t), g_k)) + (u'_m(t), g_k) + (u'_m(t)\,|u'_m(t)|, g_k) = \\ = (f(t), g_k), \quad (k = 1, 2, \ldots, m).$$

Multiplying the k-th equation of the system (6.1) by the coefficient $\dot{\alpha}_{km}(t)$ and adding up all these equations, we obtain

$$(6.2) \qquad (u''_m(t), u'_m(t)) + ((u_m(t), u'_m(t))) + |u'_m(t)|_2^2 + \\ + (u'_m(t)\,|u'_m(t)|, u'_m(t)) = (f(t), u'_m(t)).$$

According to properties of the function $u_m(t)$, we obtain (from (6.2))

$$(6.3) \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,[|u'_m(t)|_2^2 + \|u_m(t)\|^2] + |u'_m(t)|_2^2 + |u'_m(t)|_3^3 = \\ = (f(t), u'_m(t)) \leqq |f(t)|_2\,|u'_m(t)|_2\,.$$

By the integration of (6.3) from 0 to T, we obtain (with regard to T-periodicity and continuity of $u_m(t)$, $u'_m(t)$ and with help of the Schwarz inequality) the relation

$$(6.4) \qquad \int_0^T |u'_m(t)|_2^2\,\mathrm{d}t + \int_0^T |u'_m(t)|_3^3\,\mathrm{d}t \leqq \left(\int_0^T |f(t)|_2^2\,\mathrm{d}t\right)^{1/2} \left(\int_0^T |u'_m(t)|_2^2\,\mathrm{d}t\right)^{1/2}.$$

From here, it follows

$$(6.5) \qquad \int_0^T |u'_m(t)|_2^2\,\mathrm{d}t \leqq \int_0^T |f(t)|_2^2\,\mathrm{d}t = K_1\,,$$

$$(6.6) \qquad \int_0^T |u'_m(t)|_3^3\,\mathrm{d}t \leqq \int_0^T |f(t)|_2^2\,\mathrm{d}t = K_1\,.$$

Multiplying the k-equation of the system (6.1) by the coefficient $\alpha_{km}(t)$ and adding up all m equations, it results

(6.7)
$$(u''_m(t), u_m(t)) + ((u_m(t), u_m(t))) + (u'_m(t), u_m(t)) + \\ + (u'_m(t)\,|u'_m(t)|, u_m(t)) = (f(t), u_m(t))\,.$$

From (6.7) we obtain (using T-periodicity and continuity of $u_m(t)$, $u'_m(t)$ again)

(6.8)
$$\int_0^T \|u_m(t)\|^2 \,\mathrm{d}t \leqq \left(\int_0^T |f(t)|_2^2 \,\mathrm{d}t\right)^{1/2} \cdot \left(\int_0^T |u_m(t)|_2^2 \,\mathrm{d}t\right)^{1/2} + \\ + \int_0^T |u'_m(t)|_2^2 + \int_0^T |(u'_m(t), |u'_m(t)|, u_m(t))| \,\mathrm{d}t\,.$$

Now, according to Remark 4.2, we shall consider the vector-valued function $u_m(t) = \sum_{k=1}^{m} \alpha_{km}(t)\, g_k$ as a function $u_m = u_m(t, x)$ of the variables $t \in E_1$ and $x \in \Omega$ (we denote by Ω_T the domain $(0, T) \times \Omega$). Obviously, $u_m \in H_1(\Omega_T)$. Then, by Theorem 2.1 $u_m \in L_3(\Omega_T)$ for our dimensions ($N \leqq 5$) and there exists a constant $c > 0$ such, that

(6.9)
$$|u_m|_{L_3(\Omega_T)} \leqq c|u_m|_{H_1(\Omega_T)} \leqq \\ \leqq c\left[\left(\int_0^T |u'_m(t)|_2^2 \,\mathrm{d}t\right)^{1/2} + \left(\int_0^T |u_m(t)|_{H_1}^2 \,\mathrm{d}t\right)^{1/2}\right].$$

As $u_m(t) \in V$ for every $t \in \langle 0, T\rangle$, it holds by (4.2) the inequality

(6.10)
$$|u_m(t)|_{H_1} \leqq c_1 \|u_m(t)\|\,, \quad c_1 > 0\,.$$

Using the Hölder inequality ($q_1 = \frac{3}{2}$, $q_2 = 3$), we get

(6.11)
$$\int_0^T |(u'_m(t), |u'_m(t)|, u_m(t))| \,\mathrm{d}t \leqq \left(\int_0^T |u'_m(t)|_3^3 \,\mathrm{d}t\right)^{2/3} \cdot |u_m|_{L_3(\Omega_T)} \leqq \\ \leqq cK_1^{2/3}\left[K_1^{1/2} + c_1\left(\int_0^T \|u_m(t)\|^2 \,\mathrm{d}t\right)^{1/2}\right],$$

by (6.5), (6.6), (6.9) and (6.10). Now, from (4.2), (6.5), (6.8) and (6.11) it follows easily, that

(6.12)
$$\int_0^T \|u_m(t)\|^2 \,\mathrm{d}t \leqq \left(\int_0^T \|u_m(t)\|^2 \,\mathrm{d}t\right)^{1/2} \left[c_1 c K_1^{2/3} + \left(\int_0^T |f(t)|^2 \,\mathrm{d}t\right)^{1/2}\right] + \\ + K_1 + cK_1^{2/3}K_1^{1/2}\,.$$

Solving the inequality above (that is the inequality of the form $x^2 \leqq Ax + B$, $x, A, B > 0$), we obtain immediately, that there exists a constant $K_2 > 0$ such, that

(6.13)
$$\int_0^T \|u_m(t)\|^2 \, dt \leqq K_2 .$$

We know (see Lemma 4.2) that for almost all $t \in \langle 0, T \rangle$ there exists $\dddot{\alpha}_{km}(t)$ and $\dddot{\alpha}_{km} \in L_2(0, T)$. If we put $u'''_m(t) = \sum_{k=1}^{m} \dddot{\alpha}_{km}(t)\, g_k$, the function $u'''_m(t)$ fulfils the identity (as it was mentioned above)

(6.14)
$$(u'''_m(t), g_k) + ((u'_m(t), g_k)) + (u''_m(t), g_k) + \\ + 2(u''_m(t)\,|u'_m(t)|, g_k) = (f'_k(t), g_k) \quad (k = 1, 2, \ldots, m) .$$

Let us multiply the k-th equation of (6.14) by the function $\ddot{\alpha}_{km}(t)$ $(k = 1, 2, \ldots, m)$ and add up all obtained equations. We get

(6.15)
$$(u'''_m(t), u''_m(t)) + ((u'_m(t), u''_m(t))) + |u''_m(t)|_2^2 + \\ + 2(u''_m(t)\,|u'_m(t)|, u''_m(t)) = (f'(t), u''_m(t)) .$$

Using the properties of $u_m(t)$, we get from (6.15)

(6.16)
$$\frac{1}{2}\frac{d}{dt}\left[|u''_m(t)|_2^2 + \|u'_m(t)\|^2\right] + |u''_m(t)|_2^2 + 2(u''_m(t)\,|u'_m(t)|, u''_m(t)) = (f'(t), u''_m(t)).$$

Integrating (6.16) from 0 to T and using T-periodicity of $u'_m(t)$, $u''_m(t)$ again, we obtain

(6.17)
$$\int_0^T |u''_m(t)|_2^2 \, dt + 2\int_0^T (u''_m(t)\,|u'_m(t)|, u''_m(t)) \, dt \leqq \\ \leqq \left(\int_0^T |f'(t)|_2^2 \, dt\right)^{1/2} \left(\int_0^T |u''_m(t)|_2^2 \, dt\right)^{1/2} .$$

Since the two terms on the left-hand side of (6.17) are non-negative, it holds

(6.18)
$$\int_0^T |u''_m(t)|_2^2 \, dt \leqq \int_0^T |f'(t)|_2^2 \, dt = K_3$$

and

(6.19)
$$\int_0^T (u''_m(t)\,|u'_m(t)|, u''_m(t)) \, dt \leqq \frac{1}{2}\int_0^T |f'(t)|_2^2 \, dt = K_4 .$$

Now, let us multiply the k-th equation $(k = 1, 2, \ldots, m)$ of the system (6.14) by $\dot{\alpha}_{km}(t)$ and add up all the equations again. We obtain

$$(6.20) \qquad (u'''_m(t), u'_m(t)) + \|u'_m(t)\|^2 + (u''_m(t), u'_m(t)) + \\ + 2(u''_m(t)\,|u'_m(t)|, u'_m(t)) = (f'(t), u'_m(t)),$$

and hence

$$(6.21) \qquad \frac{\mathrm{d}}{\mathrm{d}t}(u''_m(t), u_m(t)) - |u''_m(t)|_2^2 + \|u'_m(t)\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u'_m(t)|_2^2 + \\ + 2(u''_m(t)\,|u'_m(t)|, u'_m(t)) = (f'(t), u'_m(t)).$$

By the integration of (6.21) from 0 to T we have

$$(6.22) \qquad \int_0^T \|u'_m(t)\|^2\,\mathrm{d}t \leqq \left(\int_0^T |f'(t)|_2^2\,\mathrm{d}t\right)^{1/2} \cdot \left(\int_0^T |u'_m(t)|_2^2\,\mathrm{d}t\right)^{1/2} + \\ + \int_0^T |u''_m(t)|_2^2\,\mathrm{d}t + 2\int_0^T |(u''_m(t), |u'_m(t)|, u'_m(t))|\,\mathrm{d}t.$$

Evidently, to estimate the integral $\int_0^T \|u'_m(t)\|^2\,\mathrm{d}t$, it suffices to estimate only $\int_0^T |(u''_m(t)\,|u'_m(t)|, u'_m(t))|\,\mathrm{d}t$; consider again $u_m(t)$ as a function $u_m = u_m(t, x)$ of the variables $t \in E_1$ and $x \in \Omega$. Then

$$\int_0^T |(u''_m(t)\,|u'_m(t)|, u'_m(t))|\,\mathrm{d}t \leqq \int_0^T\int_\Omega |u''_m(t,\xi)|\,|u'_m(t,\xi)|^{1/2} \cdot |u'_m(t,\xi)|^{3/2}\,\mathrm{d}\xi\,\mathrm{d}t.$$

Using the Hölder inequality $(q_1 = q_2 = 2)$, we have

$$(6.23) \qquad \int_0^T\int_\Omega |u''_m|\,|u'_m|^{1/2} \cdot |u'_m|^{3/2}\,\mathrm{d}\xi\,\mathrm{d}t \leqq \\ \leqq \left(\int_0^T\int_\Omega |u''_m(t,\xi)|^2\,|u'_m(t,\xi)|\,\mathrm{d}\xi\,\mathrm{d}t\right)^{1/2} \cdot \left(\int_0^T\int_\Omega |u'_m(t,\xi)|^3\,\mathrm{d}\xi\,\mathrm{d}t\right)^{1/2} \leqq \sqrt{K_4} \cdot \sqrt{K_1}$$

by (6.6) and (6.19). Then, according to (6.23), (6.22), (6.18) and (6.5) we get

$$(6.24) \qquad \int_0^T \|u'_m(t)\|^2\,\mathrm{d}t \leqq \left(\int_0^T |f'(t)|_2^2\,\mathrm{d}t\right)^{1/2} \sqrt{K_1} + K_3 + 2\sqrt{(K_1K_4)} = K_5.$$

It follows by the inequalities (6.5), (6.13), (6.18) and (6.24), that there exists a constant $K_6 > 0$, not depending on m, such that

$$\int_0^T \{|u''_m(t)|_2^2 + \|u'_m(t)\|^2 + |u'_m(t)|_2^2 + \|u_m(t)\|\}\,\mathrm{d}t < K_6.$$

Obviously, then there exists $t_m \in \langle 0, T \rangle$ for each $m = m_0, m_0 + 1, \dots$ such that the following inequality holds

$$(6.25) \qquad |u_m''(t_m)|_2^2 + \|u_m'(t_m)\|^2 + |u_m'(t_m)|_2^2 + \|u(t_m)\|^2 \leqq \frac{K_6}{T} = K_7 .$$

Now, let $t \in \langle 0, T \rangle$ be arbitrary but fixed for a moment, and let $m \geqq m_0$. If we integrate the identity (6.3) from t_m to t, we obtain (using continuity of the function $\|u_m(t)\|^2 + |u_m'(t)|_2^2$)

$$|u_m'(t)|_2^2 + \|u_m(t)\|^2 - |u_m'(t_m)|_2^2 - \|u_m(t_m)\|_2^2 + \\ + 2\int_{t_m}^{t} \{|u_m'(\tau)|_2^2 + |u_m'(\tau)|_3^3\} \, d\tau = 2\int_{t_m}^{t} (f(\tau), u_m'(\tau)) \, d\tau .$$

Hence, according to the previous results, the following inequality holds

$$(6.26) \qquad |u_m'(t)|_2^2 + \|u_m(t)\|^2 \leqq |u_m'(t_m)|_2^2 + \|u_m(t_m)\|^2 + \\ + 2\int_0^T \{|u_m'(\tau)|_2^2 + |u_m'(\tau)|_3^3\} \, d\tau + 2\left(\int_0^T |f(t)|_2^2 \, dt\right)^{1/2} . \left(\int_0^T |u_m'(t)|_2^2 \, dt\right)^{1/2} \leqq \\ \leqq K_7 + 4K_1 + 2\left(\int_0^T |f(t)|_2^2 \, dt\right)^{1/2} . K_1^{1/2} = K_8 .$$

The integration of the identity (6.16) from t_m to t yields

$$(6.27) \qquad |u_m''(t)|_2^2 + \|u_m'(t)\|^2 - |u_m''(t_m)|_2^2 - \|u_m'(t_m)\| + \\ + 2\int_{t_m}^{t} |u_m''(\tau)|_2^2 \, d\tau + 4\int_{t_m}^{t} (u_m''(\tau) \, |u_m'(\tau)|, u_m''(\tau)) \, d\tau = 2\int_{t_m}^{t} (f'(\tau), u_m''(\tau)) \, d\tau ,$$

and hence, with help of the previous results, we get

$$(6.28) \qquad |u_m''(t)|_2^2 + \|u_m'(t)\|^2 \leqq |u_m''(t_m)|_2^2 + \|u_m'(t_m)\|^2 + \\ + 2\int_0^T |u_m''(t)|_2^2 \, dt + 4\int_0^T (u_m''(t) \, |u_m'(t)|, u_m''(t)) \, dt + \\ + 2\left(\int_0^T |f'(t)|_2^2 \, dt\right)^{1/2} . \left(\int_0^T |u_m''(t)|_2^2 \, dt\right)^{1/2} \leqq \\ \leqq K_7 + 2K_3 + 4K_4 + 2\left(\int_0^T |f'(t)|_2^2 \, dt\right)^{1/2} . K_3^{1/2} = K_9 .$$

Thereby, we have proved

Lemma 6.1. *There exists a constant $M > 0$ such that for $m = m_0, m_0 + 1, \ldots$ and all $t \in \langle 0, T \rangle$*

$$\|u_m(t)\| \leqq M\,, \quad \|u'_m(t)\| \leqq M\,, \quad |u''_m(t)|_2 \leqq M\,, \tag{6.29}$$

i.e. $|u_m|_{L^\infty(T,V)} \leqq M$, $|u'_m|_{L^\infty(T,V)} \leqq M$ *and* $|u''_m|_{L^\infty(T,H_0)} \leqq M$.

7. CONVERGENCE OF GALERKIN APPROXIMATIONS

To prove the existence of a subsequence of the sequence of the Galerkin approximations which converges in the weak* topology in $L^\infty(T, V)$ to a weak solution of the problem $\mathscr{P}(V)$, we use the following well-known theorem (see, e.g., [5]).

Theorem. *Let X be a separable Banach space. Then for each bounded sequence $\{x_n^*\} \subset X^*$ there exists a subsequence $\{x_{n_k}^*\}$ which is convergent in the weak* topology in X^*, i.e. there exists an element $x^* \in X^*$ such that $\lim_{k\to\infty} x_{n_k}^*(x) = x^*(x)$ for all $x \in X$.*

If H is a Hilbert space, then the adjoint space $(L^1((0, T), H))^*$ to the space $L^1((0, T), H)$ is isometric and isomorphic to the space $L^\infty((0, T), H)$ (see [10]). If $f \in (L_1((0, T), H))^*$, then there exists a unique element $F \in L^\infty((0, T), H)$ such that $|f|_{L^1((0,T),H)^*} = |F|_{L^\infty((0,T),H)}$ and $f(u) = \int_0^T (F(t), u(t))_H \, dt$ for each $u \in L^1((0, T), H)$. Further, if H is a separable space, the space $L^1((0, T), H)$ is separable, too.

Applying the theorem above, we prove

Lemma 7.1. *There exist a subsequence $\{u_{m_k}\}_{k=1}^\infty$ of the sequence of the Galerkin approximations $\{u_m\}_{m=m_0}^\infty$ and a vector-valued function $u \in L^\infty(T, V)$ which is differentiable in $L^\infty(T, V)$ with derivative $u' \in L^\infty(T, V)$, where u' is differentiable in $L^\infty(T, H_0)$ with the derivative $u'' \in L^\infty(T, H_0)$ such, that*

$$u_{m_k} \to u \quad \textit{weakly* in} \quad L^\infty((0, T), V)\,,$$

$$u'_{m_k} \to u' \quad \textit{weakly* in} \quad L^\infty((0, T), H_0)\,,$$

and

$$u''_{m_k} \to u'' \quad \textit{weakly* in} \quad L^\infty((0, T), H_0)\,,$$

i.e.

$$\lim_{k\to\infty} \int_0^T ((u_{m_k}(t), \varphi(t))) \, dt = \int_0^T ((u(t), \varphi(t))) \, dt$$

for each function $\varphi \in L^1((0, T), V)$ *and*

$$\lim_{k \to \infty} \int_0^T (u'_{m_k}(t), \psi(t))\, dt = \int_0^T (u'(t), \psi(t))\, dt ,$$

$$\lim_{k \to \infty} \int_0^T (u''_{m_k}(t), \psi(t))\, dt = \int_0^T (u''(t), \psi(t))\, dt$$

for each function $\psi \in L^1((0, T), H_0)$.

Proof. The described subsequence is obtained after several refinements of the original sequence; for brevity in notation, all subsequences are still denoted by $\{u_m\}$. The sequence $\{u''_m\}$ is bounded in the space $L^\infty(T, H_0)$ (Lemma 6.1) and thus (all assumptions of the previous theorem are fulfilled for $\{u''_m\}$ and $L^\infty((0, T), H_0)$) there exist a subsequence $\{u_m\}$ and an element $\bar{w} \in L^\infty((0, T), H_0)$ such, that

(7.1) $$u''_m \to \bar{w} \quad \text{weakly* in} \quad L^\infty((0, T), H_0) .$$

Obviously, we may extend this function $\bar{w}$ from the interval $(0, T)$ onto E_1 to be T-periodic (in our sense). We denote the extension of $\bar{w}$ by w. (If $\bar{v} \in L^\infty((0, T), H)$, H is a Hilbert space, we choose some representative $\bar{v}_*(t)$ of the class $\bar{v}(t)$; the function $\bar{v}_*(t)$ is defined on $\langle 0, T) - M$, $\mu(M) = 0$. We set $v_*(t + kT) = \bar{v}_*(t)$ for each k integer and $t \in \langle 0, T) - M$. The class v, which is represented by v_*, will be called a T-periodic extension of the function $\bar{v}$ onto E_1. Obviously, $v \in L^\infty(T, H)$.)

Further, it is evident, that there exist the derivatives $(d/dt)(u_m(t), v)$ and (d^2/dt^2) . . $(u_m(t), v)$ for all $v \in H$ and that $(d/dt)(u_m(t), v) = (u'_m(t), v)$ and $(d^2/dt^2)(u_m(t), v) = (u''_m(t), v)$. This both derivatives are real-valued functions, continuous on E_1. Hence, we may express $(u_m(t), v)$ and $(u'_m(t), v)$ $(v \in H_0,\ t \in E_1)$ as follows:

$$(u_m(t), v) = (u_m(0), v) + (u'_m(0), v) \cdot t + \int_0^t \int_0^\tau (u''_m(\eta), v)\, d\eta\, d\tau ,$$

$$(u'_m(t), v) = (u'_m(0), v) + \int_0^t (u''_m(\eta), v)\, d\eta .$$

As the sequences $\{u_m(0)\}$ and $\{u'_m(0)\}$ are bounded in H_0, there exist a subsequence of $\{u_m\}$ and elements $u_0, u_1 \in H_0$ such that

(7.2) $$u_m(0) \to u_0 , \quad u'_m(0) \to u_1 \quad \text{weakly in } H_0 .$$

Now, let $t \in E_1$; then (according to (7.1))

(7.3) $$\lim_{m \to \infty} \int_0^t (u''_m(\tau), v)\, d\tau = \int_0^t (w(\tau), v)\, d\tau .$$

Further, we can prove without any difficulty (using (7.1) and boundedness of $\{u''_m\}$ in $L^\infty(T, H_0)$) that

(7.4) $$\lim_{m\to\infty} \int_0^t \int_0^\tau (u''_m(\eta), v)\, d\eta\, d\tau = \int_0^t \int_0^\tau (w(\eta), v)\, d\eta\, d\tau .$$

Hence, according to (7.1) ÷ (7.3), there exist the limits

$$\lim_{m\to\infty} (u_m(t), v) = (u_0, v) + t(u_1, v) + \int_0^t \int_0^\tau (w(\eta), v)\, d\eta\, d\tau ,$$

$$\lim_{m\to\infty} (u'_m(t), v) = (u_1, v) + \int_0^t (w(\tau), v)\, d\tau ,$$

and they are continuous functionals on the space H_0. So, there exist elements $u(t)$, $v(t) \in H_0$ such that

(7.5) $$\lim_{m\to\infty} (u_m(t), v) = (u(t), v)$$

and

(7.6) $$\lim_{m\to\infty} (u'_m(t), v) = (v(t), v)$$

for each $v \in H_0$, i.e. the sequences $\{u_m(t)\}$ and $\{u'_m(t)\}$ are weakly convergent in H_0.

To prove a weak convergence of the sequences $\{u_m(t)\}$ and $\{u'_m(t)\}$ in the space V, we use the following lemma (see [6], Lemma 1).

Lemma 7.2. *Let $u_n \in H_k$, $|u_n|_{H_k} \leqq c$ $(n = 1, 2, \ldots)$ and let $u_n \to u$ weakly in H_0. Then, $u \in H_k$ and also $D^j u_n \to D^j u$ weakly in $H_0(|j| \leqq k)$.*

Lemma 7.2 is analogous to Lemma 1 of [6] and it may be proved easily from this lemma by mathematical induction with respect to k. (The space H_k is taken instead of the space H_1 in the Nierenberg lemma.)

As the sequences $\{u_m(t)\}$, $\{u'_m(t)\}$ are bounded in the space H_k and they are weakly convergent in H_0, it holds even $u(t)$, $v(t) \in H_k$ and for $|j| \leqq k$

(7.7) $$D^j\, u_m(t) \to D^j\, u(t)\, , \quad D^j\, u'_m(t) \to D^j\, v(t)$$

weakly in H_0. Hence, $u_m(t) \to u(t)$ weakly in H_k, and $u'_m(t) \to v(t)$ weakly in H_k as well. But $u(t)$, $v(t) \in V$, because V is a closed subspace of H_k.

Further, using (7.7), we obtain immediately, that the relations

(7.8) $$\lim_{m\to\infty} ((u_m(t), v)) = ((u(t), v))\, , \quad \lim_{m\to\infty} ((u'_m(t), v)) = ((v(t), v))$$

hold for each $v \in V$, i.e. $u_m(t) \to u(t)$ weakly in V, and also $u'_m(t) \to v(t)$ weakly in V. Then,

$$(7.9) \qquad \|u(t)\| \leqq \liminf_{m\to\infty} \|u_m(t)\| \leqq M\,, \quad \|v(t)\| \leqq \liminf_{m\to\infty} \|u'_m(t)\| \leqq M\,.$$

We get from here (using the T-periodicity of $u_m(t)$ and $u'_m(t)$), that $u, v \in L^\infty(T, V)$. Now it may be shown easily, that $u_m \to u$, $u'_m \to v$ weakly* in $L^\infty((0, T), H_0)$, and $u_m \to u$, $u'_m \to v$ weakly* also in $L^\infty((0, T), V)$. Let $\varphi \in L^1((0, T), V)$. Then, $\varphi(t) \in V$ for almost all $t \in \langle 0, T \rangle$ and thus, for these t, there is

$$\lim_{m\to\infty} ((u_m(t), \varphi(t))) = ((u(t), \varphi(t)))\,.$$

Further, $|((u_m(t), \varphi(t)))| \leqq \|u_m(t)\|\,\|\varphi(t)\| \leqq M\|\varphi(t)\|$. Hence (using well-known properties of Lebesgue integral)

$$\lim_{m\to\infty} \int_0^T ((u_m(t), \varphi(t)))\,\mathrm{d}t = \int_0^T ((u(t), \varphi(t)))\,\mathrm{d}t\,,$$

which is one part of the statement; remaining one can be proved quite analogously.

To complete the proof of Lemma 7.1, it remains to prove the function u to be differentiable in $L^\infty(T, V)$, the function v to be differentiable in $L^\infty(T, H_0)$ and the following relations to be valid: $u' = v$ and $u'' = w$. Let us prove, for instance, the differentiability of the function u.

Let $\varphi \in C_0^{(1)}(E_1, V)$. Then (using the periodicity of $u_m(t)$ and results proved relative to the convergence of $\{u_m\}$) it may be easily shown that

$$\int_{-\infty}^{+\infty} ((u(t), \varphi'(t)))\,\mathrm{d}t = \lim_{m\to\infty} \int_{-\infty}^{+\infty} ((u_m(t), \varphi'(t)))\,\mathrm{d}t\,.$$

Since the functions $u_m(t)$, $\varphi(t)$ are differentiable in $L^2(\operatorname{supp}\varphi, V)$, it holds (according to Lemma 2.2)

$$\int_{-\infty}^{+\infty} ((u_m(t), \varphi'(t)))\,\mathrm{d}t = -\int_{-\infty}^{+\infty} ((u'_m(t), \varphi(t)))\,\mathrm{d}t$$

and thus

$$\int_{-\infty}^{+\infty} ((u(t), \varphi'(t)))\,\mathrm{d}t = -\lim_{m\to\infty} \int_{-\infty}^{+\infty} ((u'_m(t), \varphi(t)))\,\mathrm{d}t = -\int_{-\infty}^{+\infty} ((v(t), \varphi(t)))\,\mathrm{d}t\,.$$

So we proved, that $u' = v$ in $L^\infty(T, V)$.

Evidently, we may obtain the second part of the statement quite analogously. This completes the proof of Lemma 7.1.

Lemma 7.3. *Let $\varphi : \langle 0, T \rangle \to V$ be a vector-valued function of the form $\varphi(t) = \sum_{j=1}^{r} \psi_j(t)\, g_j$, $\psi_j(t) \in C_{\langle 0, T\rangle}$ $(j = 1, 2, \ldots, r)$, and let $\{u_m\}$ be a subsequence, which we obtained in Lemma 7.1. Then,*

$$\lim_{m\to\infty} \int_0^T (u'_m(t)\, |u'_m(t)|, \varphi(t))\, dt = \int_0^T (u'(t)\, |u'(t)|, \varphi(t))\, dt\,.$$

Proof. Let us consider the functions u'_m and φ as the functions of $N + 1$ variables $t \in \langle 0, T\rangle$ and $x \in \Omega$. It can be shown easily, that $u'_m(t, x)$, $u'(t, x) \in H_1(\Omega_T)$ and $u'_m \to u'$ weakly in $H_1(\Omega_T)$. As the imbedding of $H_1(\Omega_T)$ into $L_2(\Omega_T)$ is completely continuous, there is

$$\lim_{m\to\infty} u'_m = u' \quad \text{in} \quad L_2(\Omega_T)\,. \tag{7.10}$$

Further,

$$\left|\int_0^T (u'_m(t)\, |u'_m(t)|, \varphi(t))\, dt - \int_0^T (u'(t)\, |u'(t)|, \varphi(t))\, dt\right| \leqq \tag{7.11}$$
$$\leqq \int_0^T \int_\Omega |u'_m(t, x) - u'(t, x)|\, (|u'_m(t, x)| + |u'(t, x)|)\, |\varphi(t, x)|\, dx\, dt\,.$$

The functions $u'_m(t, x)$ $(m = 1, 2, \ldots)$ and $u'(t, x)$ are elements of $H_1(\Omega_T)$; hence (by Theorem 2.1) $u'_m, u' \in L_3(\Omega_T)$ and

$$|u'_m|_{L_3(\Omega_T)} \leqq c|u'_m|_{H_1(\Omega_T)} \leqq 3cT^{1/2}M = K\,,$$

and, analogously, $|u|_{L_3(\Omega_T)} \leqq K$. Further, if $k = 1$ (it means, that the operator A is of the order 2), there is $g_j \in L_6(\Omega)$ (for $N \leqq 3$) and if $k \geqq 2$, then there is $g_j \in L_6(\Omega)$ (for $N \leqq 5$); then

$$\left(\int_0^T \int_\Omega |\varphi(t, x)|^6\, dx\, dt\right)^{1/6} \leqq cT\left(\int_\Omega |\sum_{i=1}^{r} g_i(x)|^6\, dx\right)^{1/6} < \infty$$

(because $|\varphi_j(t)| \leqq c$, $j = 1, 2, \ldots, r$ for some $c > 0$). Applying the generalized Hölder inequality (with $q_1 = 2$, $q_2 = 3$, $q_3 = 6$) in (7.11), we obtain

$$\left|\int_0^T (u'_m(t), |u'_m(t)|, \varphi(t))\, dt - \int_0^T (u(t)\, |u'(t)|, \varphi(t))\, dt\right| \leqq$$
$$\leqq \left(\int_0^T \int_\Omega |u'_m(t, x) - u'(t, x)|^2\, dx\, dt\right)^{1/2} \left[\left(\int_0^T \int_\Omega |u'_m(t, x)|^3\, dx\, dt\right)^{1/3} + \right.$$
$$\left. + \left(\int_0^T \int_\Omega |u'(t, x)|^3\, dx\, dt\right)^{1/3}\right] \cdot \left(\int_0^T \int_\Omega |\varphi(t, x)|^6\, dx\, dt\right)^{1/6} \leqq$$
$$\leqq |u'_m - u'|_{L_2(\Omega_T)} \cdot 2K|\varphi|_{L_6(\Omega_T)}\,,$$

which (by 7.10) proves the statement of Lemma 7.3.

Now it is easy to complete the proof of Theorem 1. Consider functions

$$\varphi_r(t) = \sum_{j=1}^{r} \psi_j(t)\, g_j \quad (r = 1, 2, \ldots), \tag{7.12}$$

where $\psi_j \in C_{\langle 0,T\rangle}$ $(j = 1, 2, \ldots, r)$. It results from the construction of the Galerkin approximations that

$$\begin{aligned}(u''_m(t), \varphi_r(t)) + ((u_m(t), \varphi_r(t))) + (u'_m(t), \varphi_r(t)) + \\ + (u'_m(t)\,|u'_m(t)|, \varphi_r(t)) = (f(t), \varphi_r(t))\end{aligned} \tag{7.13}$$

for $m \geqq r$ and hence also the relation

$$\begin{aligned}\int_0^T \{(u''_m(t), \varphi_r(t)) + ((u_m(t), \varphi_r(t))) + (u'_m(t), \varphi_r(t)) + \\ + (u'_m(t)\,|u'_m(t)|, \varphi_r(t))\}\, dt = \int_0^T (f(t), \varphi_r(t))\, dt\end{aligned} \tag{7.14}$$

holds (all inner products in (7.13) are continuous on $\langle 0, T\rangle$). According to Lemma 7.1 and Lemma 7.2, there exists the limit of (7.14) for $m \to \infty$, and so we get

$$\begin{aligned}\int_0^T \{(u''(t), \varphi_r(t)) + ((u(t), \varphi_r(t))) + (u'(t), \varphi_r(t)) + \\ + (u'(t)\,|u'(t)|, \varphi_r(t))\}\, dt = \int_0^T (f(t), \varphi_r(t))\, dt\,.\end{aligned} \tag{7.15}$$

Since the system of functions of the form (7.12) $(r = 1, 2, \ldots)$ is dense in the space $L^1((0, T), H_k)$, the validity of the equality (7.15) may be established for each function $\varphi \in L^1((0, T), V)$. Further, according to Lemma 7.1, the function $u = u(t)$ has all properties from Theorem 1. So the proof of this theorem is completed.

8. UNIQUENESS OF SOLUTION OF PROBLEM $\mathscr{P}(V)$

The results, concerning the uniqueness of the solution of the problem $\mathscr{P}(V)$, are contained in the following theorem.

Theorem 2. *Let u, v be solutions of the problem $\mathscr{P}(V, f)$, $f \in L^2(T, H_0)$, in the sense of Definition* 3.1. *Let the assumptions* (4.1) *and* (4.2) *be fulfilled. Then* $|u - v|_{L^\infty(T,V)} = 0$, *i.e.* $u = v$ *in the space* $L^\infty(T, V)$.

Proof. Denote $w = u - v$. Then, according to Definition 3.1, $w, w' \in L^\infty(T, V)$, $w'' \in L^\infty(T, H_0)$ and

$$(8.1) \qquad \int_0^T \{(w''(t), \varphi(t)) + ((w(t), \varphi(t))) + (w'(t), \varphi(t)) + \\ + (u'(t) |u'(t)| - v'(t) |v'(t)|, \varphi(t))\} \, dt = 0$$

for each function $\varphi \in L^2((0, T), V)$. First, let us set $\varphi(t) \equiv w'(t)$. We get

$$(8.2) \qquad \int_0^T \{(w''(t), w'(t)) + ((w(t), w'(t))) + |w'(t)|_2^2 + \\ + (u'(t) |u'(t)| - v'(t) |v'(t)|, u'(t) - v'(t))\} \, dt = 0 .$$

Since the function $w'(t)$ is differentiable in $L^\infty(T, H_0)$ and $w(t)$ is differentiable in $L^\infty(T, V)$, we may consider the function $w(t)$ or $w'(t)$, respectively, as a function T-periodic in E_1 and absolutely continuous on $\langle 0, T \rangle$ in norm of the space V or H_0, respectively (Lemma 2.1). Using Lemma 2.3, we obtain immediately, that

$$\int_0^T (w''(t), w'(t)) \, dt = 0 \quad \text{and} \quad \int_0^T ((w(t), w'(t))) \, dt = 0$$

and thus

$$(8.3) \qquad \int_0^T \{|w'(t)|_2^2 + (u'(t) |u'(t)| - v'(t) |v'(t)|, u'(t) - v'(t))\} \, dt = 0 .$$

The second term of (8.3) is non-negative. (There is $(a|a| - b|b|)(a - b) \geqq 0$ for arbitrary real numbers a, b, because (see [8])

$$(a|a| - b|b|)(a - b) = a^2|a| + b^2|b| - ab(|a| + |b|) \geqq \\ \geqq a^2|a| + b^2|b| - \tfrac{1}{2}(|a| + |b|)(a^2 + b^2) = a^2 \left(|a| - \frac{|a| + |b|}{2} \right) + \\ + b^2 \left(|b| - \frac{|a| + |b|}{2} \right) = \tfrac{1}{2}(|a| - |b|)(a^2 - b^2) \geqq 0) .$$

Thus, it follows from (8.3)

$$(8.4) \qquad \int_0^T |w'(t)|_2^2 \, dt = 0 .$$

Now, let us set $\varphi(t) \equiv w(t)$ in (8.1) (which is allowed again). Then

$$(8.5) \qquad \int_0^T \{(w''(t), w(t)) + \|w(t)\|^2 + (w'(t), w(t)) + \\ + (u'(t) |u'(t)| - v'(t) |v'(t)|, u(t) - v(t)\} \, dt = 0 ,$$

and using Lemma 2.3 again, we have (by (8.4))

$$\int_0^T (w''(t), w(t))\, \mathrm{d}t = -\int_0^T |w'(t)|_2^2\, \mathrm{d}t = 0$$

and

$$\int_0^T (w'(t), w(t))\, \mathrm{d}t = 0\,.$$

The equality (8.5) takes the form

$$\int_0^T \|w(t)\|^2\, \mathrm{d}t + \int_0^T (u'(t)\,|u'(t)| - v'(t)\,|v'(t)|, u(t) - v(t))\, \mathrm{d}t = 0\,. \tag{8.6}$$

Let us examine the second term of (8.6). There is

$$\left|\int_0^T (u'(t)\,|u'(t)| - v'(t)\,|v'(t)|, u(t) - v(t))\, \mathrm{d}t\right| \leqq \tag{8.7}$$
$$\leqq \int_0^T (|u'(t)| + |v'(t)|)\,|u'(t) - v'(t)|\,.\,|u(t) - v(t)|\, \mathrm{d}t\,.$$

Let us consider the functions u, u', v and v' as functions of the variables $t \in \langle 0, T\rangle$ and $x \in \Omega$ again. As $u'(t, x), v'(t, x) \in H_1(\Omega_T)$, there is $u'(t, x), v'(t, x) \in L_3(\Omega_T)$ for our dimensions (Theorem 2.1); furthermore, it follows from the definition of the solution easily (with help of the similar considerations as above), that

$$|u'|_{L_3(\Omega_T)} \leqq |f|_{L^2(T,H_0)}\,, \quad |v'|_{L_3(\Omega_T)} \leqq |f|_{L^2(T,H_0)}\,. \tag{8.8}$$

Further, we shall prove, that $w \in L_6(\Omega_T)$ for our dimensions (i.e. $N \leqq 5$, if $k \geqq 2$ and $N \leqq 3$ for $k = 1$). If $k \geqq 2$, there is $w \in H_2(\Omega_T)$ and we have $w \in L_6(\Omega_T)$ for $N \leqq 5$ immediately (by Theorem 2.1). If $k = 1$, then $w(t) \in H_2(\Omega)$ for all $t \in \langle 0, T\rangle$, and hence $w(t) \in L_6(\Omega)$ for $N \leqq 3$ (Theorem 2.1) and there exists a constant $c > 0$ such, that

$$|w(t)|_6 \leqq c|w(t)|_{H_1} \leqq c'\|w(t)\|\,.$$

Then, it holds

$$\left(\int_0^T \int_\Omega |w(t, x)|^6\, \mathrm{d}x\, \mathrm{d}t\right)^{1/6} \leqq c\left(\int_0^T \|w(t)\|^6\, \mathrm{d}t\right)^{1/6} < \infty\,, \tag{8.9}$$

because of $w \in L^\infty(T, V)$. Now, it may be used the generalized Hölder inequality (with $q_1 = 3$, $q_2 = 2$, $q_3 = 6$) in the relation (8.7). This yields (with regard to the previous results)

$$(8.10) \qquad \int_0^T \int_\Omega (|u'(t, x)| + |v'(t, x)|) \, |w'(t, x)| \, |w(t, x)| \, dx \, dt \leqq$$

$$\leqq \left(\int_0^T \int_\Omega |u'(t, x)|^3 \, dx \, dt + \int_0^T \int_\Omega |v'(t, x)|^3 \, dx \, dt \right)^{1/3} .$$

$$. \left(\int_0^T \int_\Omega |w'(t, x)|^2 \, dx \, dt \right)^{1/2} . \left(\int_0^T \int_\Omega |w(t, x)|^6 \, dx \, dt \right)^{1/6} .$$

Hence, in accordance with (8.4), it is

$$\int_0^T (u'(t) \, |u'(t)| - v'(t) \, |v'(t)|, w(t)) \, dt = 0 .$$

We obtain from (8.6) that

$$(8.11) \qquad \int_0^T \|w(t)\|^2 \, dt = 0 ,$$

and so $\|w(t)\| = 0$ for almost all $t \in (0, T)$; i.e. $|u - v|_{L^\infty(T,V)} = 0$, which completes the proof.

Corollary. *Let all assumptions of Theorem* 1 *be fulfilled. Then there exists exactly one weak solution of the problem* $\mathscr{P}(V)$.

Remark 8.1. For $N = 1$ we can prove somewhat stronger result, namely the continuous dependence of the solutions on the right-hand side of the equation in the form

$$|u - v|_{L^\infty(T,V)} \leqq c(|f|_{L^2(T,H_0)}, |g|_{L^2(T,H_0)}) \, |f - g|_{L^2(T,H_0)} .$$

(We denote by u or v, respectively, the solution of the problem $\mathscr{P}(V, f)$ or $\mathscr{P}(V, g)$, respectively.)

9. SOME EXAMPLES AND APPLICATIONS

We shall use previous results to examine several mechanical problems. We shall prove the existence of a weak periodic solution of some boundary-value problems for the wave equation (in the case $N = \dim E_N = 3$) and for the biharmonic wave equation (which describes the transverse vibrations of a plate approximately) for dimension $N = 2$.

Let $\Omega \subset E_3$ be a bounded domain with lipschitz boundary. Let $f = f(t, x)$,

$t \in E_1$, $x \in \Omega$ be a T-periodic in t, and let $c \geqq 0$. We shall investigate the existence of T-periodic solutions of the equation

$$(9.1) \qquad u_{tt}(t, x) - \Delta u(t, x) + cu(t, x) + u_t(t, x) + u_t(t, x) \left|u_t(t, x)\right| = f(t, x),$$

(where $\Delta u = \sum_{i=1}^{3} (\partial^2 u / \partial x_i^2)$) on the domain Ω, with homogeneous boundary conditions.

First, let us take $V_1 = \mathring{H}_1$. If u is a solution of the problem $\mathscr{P}(V_1)$ for the equation (9.1), then $u \in L^\infty(T, \mathring{H}_1)$ and therefore $u(t) \in \mathring{H}_1$ for almost all $t \in E_1$, i.e. for almost all $t \in E_1$ $u(t) = 0$ on $\partial\Omega$ in the sense of traces. So, this choice of the space V corresponds to the Dirichlet boundary conditions $u(t, x) = 0$ for $x \in \partial\Omega$.

Second, let be $V_2 = H_1$. If $u \in L^\infty(T, H_1)$ is a solution of the problem $\mathscr{P}(V_2)$ for the equation (9.1), then (by Remark 3.2) there is

$$(9.2) \qquad \int_\Omega \sum_{i=1}^{3} \frac{\partial^2 u(t, x)}{\partial x_i^2} \cdot v(x) \, dx = - \int_\Omega \sum_{i=1}^{3} \frac{\partial u(t, x)}{\partial x_i} \cdot \frac{\partial v(x)}{\partial x_i} \, dx$$

for a.a. $t \in E_1$ and each $v \in H_1$. Using the Green theorem on the left-hand side of (9.2) (formally), we have ($n = (n_1, n_2, n_3)$ denotes the vector of the exterior normal, which exists for a.a. $x \in \partial\Omega$)

$$(9.3) \qquad \int_\Omega \sum_{i=1}^{3} \frac{\partial^2 u(t, x)}{\partial x_i^2} v(x) \, dx = \int_{\partial\Omega} \left(\sum_{i=1}^{3} \frac{\partial u(t, x)}{\partial x_i} n_i(x) \right) v(x) \, dS - $$
$$ - \int_\Omega \sum_{i=1}^{3} \frac{\partial u(t, x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} \, dx .$$

So, we find, that the space V_2 corresponds to the Neumann boundary condition $(\partial u / \partial n)(t, x) = 0$ for $x \in \partial\Omega$.

Third, let $\Gamma = \partial\Omega$, $\mu(\Gamma) \neq 0$, $\mu(\partial\Omega - \Gamma) \neq 0$. Let us set $V_3 = \overline{\{u \in \mathscr{E}(\bar{\Omega}), u(x) = 0 \text{ for } x \in \Gamma\}}^{H_1}$. If $u \in L^\infty(T, V_3)$ is a solution of the problem $\mathscr{P}(V_3)$, then (by (9.2) and (9.3)), we obtain (quite analogously as in the previous case), that this choice of the space V describes the boundary conditions $u(t, x) = 0$ for $x \in \Gamma$ and $(\partial u / \partial n)(t, x) = 0$ for $x \in \partial\Omega - \Gamma$.

Now, let $f : E_1 \to H_0$ fulfils the assumptions of Theorem 1. Evidently, the form

$$((u, v)) = \int_\Omega \sum_{i=1}^{3} \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} \, dx + c \int_\Omega u(x) \, v(x) \, dx ,$$

corresponding to the operator $A = -\Delta + c$ (by (3.2)), satisfies (4.1). In the case $V_1 = \mathring{H}_1$, the condition (4.2) is fulfilled according to the Friedrichs inequality

$$\int_\Omega |u(x)|^2 \, dx \leqq c \int_\Omega \sum_{i=1}^{3} \left| \frac{\partial u(x)}{\partial x_i} \right|^2 dx ,$$

which holds for $u \in \mathring{H}_1$. In the case $V_2 = H_1$, the inequality (4.2) is satisfied obviously only for $c > 0$. And, in the case $V = V_3$, the assumption (4.2) is satisfied for all $c \geqq 0$ again, because of validity of the following theorem (see [5]): If Ω is a bounded domain in E_N with lipschitz boundary and $\Gamma \subset \partial\Omega$, $\mu(\Gamma) \neq 0$, then there exists a constant $c > 0$ such, that the inequality

$$|u|_{H_1} \leqq c \left(\int_\Gamma |u(x)|^2 \, \mathrm{d}S + \int_\Omega \sum_{i=1}^{3} \left| \frac{\partial u}{\partial x_i}(x) \right|^2 \mathrm{d}x \right)^{1/2}$$

holds for all $u \in H_1$. But, if $u \in V_2$, then $\int_\Gamma |u(x)|^2 \, \mathrm{d}S = 0$; our statement follows from this immediately.

The previous results may be summed up as follows:

Theorem 9.1. *Let $f \in C(T, H_0)$ be differentiable in $L^2(T, H_0)$. Then there exists a unique weak solution of the problem $\mathscr{P}(V_1, f)$ and $\mathscr{P}(V_3, f)$ for each $c \geqq 0$, and if $c > 0$, then there exists a unique weak solution of the problem $\mathscr{P}(V_2, f)$.*

Further, we shall study several boundary-value problems for the biharmonic wave equation

$$(9.4) \qquad u_{tt}(t, x) + \Delta^2 u(t, x) + cu(t, x) + u_t(t, x) + u_t(t, x)\,|u_t(t, x)| = f(t, x)$$

for $t \in E_1$ and $x \in \Omega$, where Ω is a bounded domain in E_2 with lipschitz boundary and f is T-periodic in t.

The operator Δ^2 may be also expressed as follows

$$(9.5) \qquad \Delta^2 = \frac{\partial^2}{\partial x_1^2}\left(\frac{\partial^2}{\partial x_1^2}\right) + \frac{\partial^2}{\partial x_1 \, \partial x_2}\left(2(1-\alpha)\frac{\partial^2}{\partial x_1 \, \partial x_2}\right) + \frac{\partial^2}{\partial x_1^2}\left(\alpha \frac{\partial^2}{\partial x_2^2}\right) + $$
$$+ \frac{\partial^2}{\partial x_2^2}\left(\alpha \frac{\partial^2}{\partial x_1^2}\right) + \frac{\partial^2}{\partial x_2^2}\left(\frac{\partial^2}{\partial x_2^2}\right),$$

α-real number. Then, the bilinear form associated to the operator $A = \Delta^2 + c$ by (3.2), has the form

$$(9.6) \qquad ((u, v)) = \int_\Omega \left(\frac{\partial^2 u}{\partial x_1^2}(x) \cdot \frac{\partial^2 v}{\partial x_1^2}(x) + \right.$$
$$+ 2(1-\alpha)\frac{\partial^2 u}{\partial x_1 \, \partial x_2}(x)\frac{\partial^2 v}{\partial x_1 \, \partial x_2}(x) + \alpha \frac{\partial^2 u}{\partial x_1^2}(x)\frac{\partial^2 v}{\partial x_2^2}(x) + \alpha \frac{\partial^2 u}{\partial x_2^2}(x)\frac{\partial^2 v}{\partial x_1^2}(x) +$$
$$\left. + \frac{\partial^2 u}{\partial x_2^2}(x) \cdot \frac{\partial^2 v}{\partial x_2^2}(x) \right) \mathrm{d}x + c \int_\Omega u(x)\, v(x) \, \mathrm{d}x \, .$$

Here, we choose $\alpha = \sigma$, σ being the Poisson coefficient ($0 < \sigma < 1$ for most of materials; this $0 < \sigma < 1$ we shall consider further). Let $\mathring{H}_2 \subset V \subset H_2$, V be closed in H_2. Again by Remark 3.2, if $u \in L^\infty(T, V)$ is a solution of the problem $\mathscr{P}(V)$, then for a.a. $t \in E_1$

(9.7) $$(Au(t), v) = ((u(t), v)) \quad \text{for} \quad v \in V.$$

Now, using the Green theorem for $\int_\Omega Au(t, x)\, v(x)\, dx$ formally, we obtain

$$\int_\Omega Au(t, x)\, v(x)\, dx = ((u(t), v)) - \int_{\partial\Omega} (Tu)(t, x)\, v(x)\, dS -$$
$$- \int_{\partial\Omega} (Mu)(t, x) \frac{\partial v}{\partial n}(x)\, dS,$$

where

$$Mu = \sigma \Delta u + (1 - \sigma)\left(\frac{\partial^2 u}{\partial x_1^2} n_1^2 + 2 \frac{\partial^2 u}{\partial x_1\, \partial x_2} n_1 n_2 + \frac{\partial^2 u}{\partial x_2^2} n_2^2\right)$$

and

$$Tu = -\frac{\partial}{\partial n}(\Delta u) + (1 - \sigma) \frac{\partial}{\partial s}\left(\frac{\partial^2 u}{\partial x_1^2} n_1 n_2 - \frac{\partial^2 u}{\partial x_1\, \partial x_2}(n_1^2 - n_2^2) - \frac{\partial^2 u}{\partial x_2^2} n_1 n_2\right)$$

($n = (n_1, n_2)$ denotes the vector of exterior normal, which is defined a.a. in $\partial\Omega$ again, and $\partial/\partial s$ denotes the derivative by the curve of the boundary). The expression Mu corresponds to the moment of deflection on the boundary of a plate; the term $-(\partial/\partial n)(\Delta u)$ describes the transverse force on the boundary of a plate and the remaining terms of Tu corresponds to the moment of torsion on the boundary.

We shall mention here three boundary-value problems for the equation (9.4).

First, let $V_1 = \overline{\{u \in \mathscr{E}(\overline{\Omega}), u(x) = (\partial u/\partial n)(x) = 0 \text{ for } x \in \partial\Omega\}}^{H_2}$ $(=\mathring{H}_2)$. This choice of the space V corresponds to the cramped boundary of a vibrating plate.

Second, let us set $V_2 = \overline{\{u \in \mathscr{E}(\overline{\Omega}), u(x) = 0 \text{ for } x \in \partial\Omega\}}^{H_2}$. This space V_2 describes free supporting of the boundary of a plate: if $u \in L^\infty(T, V_2)$ is a solution of the problem $\mathscr{P}(V_2)$, we get from the definition of the space V_2, that $u(t) = 0$ on $\partial\Omega$ for a.a. $t \in E_1$ in the sense of traces. Further, in order (9.7) may hold, it has to be $Mu(t) = 0$ on $\partial\Omega$, i.e. the moment of deflection is vanished on the boundary.

Third, let us take $V_3 = H_2$, then, any geometric conditions do not follow from the definition of V_3, but two dynamics conditions $Mu(t) = 0$ and $Tu(t) = 0$ must hold for a.a. $t \in E_1$, in order (9.7) to be satisfied for a solution $u \in L^\infty(T, V_3)$ of the problem $\mathscr{P}(V_3)$. So, this choice of V corresponds to a plate with the free boundary.

Let us examine now, under which conditions the form (9.6) satisfies the assumptions of Theorem 1. Evidently, as above, the form (9.6) fulfils (4.1). So, it remains to study the validity of (4.2). It is known (see [5]), that

(9.8) $$\left(\int_{\partial\Omega} |u(x)|^2\, dS + \int_\Omega \sum_{|i|=2} |D^i u(x)|^2\, dx\right)^{1/2}$$

defines norm in H_2, which is equivalent to origin one. Obviously, there is

$$((u, u)) \geqq (1 - \sigma) \int_\Omega \left[\left(\frac{\partial^2 u(x)}{\partial x_1^2}\right)^2 + 2\left(\frac{\partial^2 u(x)}{\partial x_1 \partial x_2}\right)^2 + \left(\frac{\partial^2 u(x)}{\partial x_2^2}\right)^2\right] dx + c \int_\Omega |u(x)|^2 \, dx$$

using the relations (9.9) and (9.8), the validity of (4.2) m ay be established for the form (9.6) for all $c \geqq 0$ in the case $V = V_1$ and $V = V_2$. As

$$\left(\int_\Omega |u(x)|^2 \, dx + \int_\Omega \sum_{|i|=2} |D^i u(x)|^2 \, dx\right)^{1/2}$$

is also norm, which is equivalent to origin one in H_2, (4.2) holds in the case $V = V_3$ for $c > 0$.

So, we obtained:

Theorem 9.2. *Let $f \in C(T, H_0)$ be differentiable in $L^2(T, H_0)$. Then there exists a unique weak solution of the problems $\mathscr{P}(V_1, f)$ and $\mathscr{P}(V_2, f)$ for $c \geqq 0$. If $c > 0$, then there exist a unique weak solution also for the problem $\mathscr{P}(V_3, f)$.*

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