Czechoslovak Mathematical Journal

Larry Mansfield

The method of moving frames applied to a space of bilinear forms

Czechoslovak Mathematical Journal, Vol. 21 (1971), No. 3, 453–460

Persistent URL: http://dml.cz/dmlcz/101047

Czechoslovak Mathematical Journal, 21 (96) 1971, Praha

THE METHOD OF MOVING FRAMES APPLIED TO A SPACE OF BILINEAR FORMS

LARRY MANSFIELD, Flushing

(Received June 12, 1970)

1. Introduction. The space of real symmetric binary bilinear forms is identified with R^3 up to a group action. This allows the vector space structure of R^3 to be used to define moving frames compatible with the action induced by a change of basis. Then invariants of families of bilinear forms are found using the information contained in the structure equations of the frame bundle. The construction of the frame bundle parallels the construction in Euclidean space but it is not based on the group of rigid motions.

The author is indebted to ROBERT GARDNER for conversations which were helpful in the present work.

2. Notation and definitions. Let V_2 be a two-dimensional real vector space, $\odot^2 V_2$ the set of real symmetric bilinear forms on V_2, $S(2)$ the set of 2×2 real symmetric matrices and $\mathrm{Gl}(n)$ the set of $n \times n$ real non-singular matrices. If B and B' are bases of V_2 such that $B' = P \cdot B$ with $P \in \mathrm{Gl}(2)$ and φ_B, $\varphi_{B'}$ are the corresponding matrices of a form $\varphi \in \odot^2 V_2$ then $\varphi_B = {}^tP \cdot \varphi_{B'} \cdot P$. Thus when $S(2)$ is studied in place of $\odot^2 V_2$ only those properties which are invariant under conjugation are of interest. Now identify $S(2)$ with R^3 via the map

$$f : S(2) \to R^3 \quad \text{where} \quad f : \begin{bmatrix} r & s \\ s & t \end{bmatrix} \to \begin{bmatrix} r \\ s \\ t \end{bmatrix}.$$

Elements of R^3 will be viewed as 3×1 column matrices throughout and if $X, Y, Z \in R^3$ then (XYZ) will denote the 3×3 matrix with X, Y and Z as columns. The congruence relation on $S(2)$ induces a group action on R^3 by $f({}^tP \cdot A \cdot P) = g(P) f(A)$ where $A \in S(2)$ and if

$$P = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \quad \text{then} \quad g(P) = \begin{bmatrix} x^2 & 2xz & z^2 \\ xy & xw + yz & zw \\ y^2 & 2yw & w^2 \end{bmatrix}.$$

Let $G = \{g(P) : P \in \mathrm{Gl}(2)\}$.

If $\varphi_B = {}^t P \cdot \varphi_{B'} \cdot P$ then $\det \varphi_B = (\det P)^2 \det \varphi_{B'}$ so the sign of a determinant associated with φ is independent of the basis used. Since the determinant function is a quadratic form there is a corresponding symmetric bilinear form Φ given by

$$\Phi(A, B) = \tfrac{1}{2}(\det(A + B) - \det A - \det B)$$

which can be used to define an innerproduct on R^3. For $X, Y \in R^3$ let $A, B \in S(2)$ be such that $f(A) = X$, $f(B) = Y$ and set $(X, Y)_L = 2\Phi(A, B)$. This inner product is Lorentzian in that it has index 2 and signature 1 which accounts for the subscript L. If $X = {}^t(x_1, x_2, x_3)$ and $Y = {}^t(y_1, y_2, y_3)$ then $(X, Y)_L = x_1 y_3 + x_3 y_1 - 2x_2 y_2$. Note that $(X, X)_L = 2 \det \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}$. If $g \in G$ corresponds to $P \in \mathrm{Gl}(2)$ then

$$(gX, gY)_L = \det({}^t P \cdot A \cdot P - {}^t P \cdot B \cdot P) - \det({}^t P \cdot A \cdot P) - \det({}^t P \cdot B \cdot P) =$$
$$= (\det P)^2 (X, Y)_L = (\det g)^{2/3} (X, Y)_L .$$

So the sign of $(\, , \,)_L$ is invariant under the action of G.

Let $L = \{X \in R^3 : (X, X)_L = 0\}$ and call an element of L an *isotropic vector*. If $X \in L$ and $X = {}^t(x_1, x_2, x_3)$ with $x_1 > 0$ or $x_3 > 0$ then X is called a *positive isotropic vector*. $L = \{{}^t(x, y, z) : xz - y^2 = 0\}$ so if x, y, z are viewed as Cartesian coordinates then L is an elliptic cone and the positive isotropic vectors correspond to the points in one half of the cone. For a non-zero isotropic vector X the set $\{Y \in R^3 : (X, Y)_L = 0\}$ is the tangent plane to L containing the vector X.

It can easily be seen that R^3 is partitioned into six orbits by G. They are the two halves of the cone L, its vertex, the region outside L and the two regions inside L. These give rise to the usual six canonical forms for the matrix of a bilinear form namely

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The inner product can be used to obtain a cross product on R^3. For $X, Y \in R^3$ let $X x_L Y$ be the vector which satisfies

$$(X x_L Y, Z)_L = \det(XYZ) \quad \text{for all} \quad Z \in R^3 .$$

With X and Y as above

$$X x_L Y = {}^t(x_1 y_2 - x_2 y_1, \tfrac{1}{2}(x_1 y_3 - x_3 y_1), x_2 y_3 - x_3 y_2) .$$

And if $g \in G$ then

$$gX x_L gY = (\det g)^{1/3} g(X x_L Y) .$$

3. The G-frame space. Let $F = \{(e_1, e_2, e_3) : \{e_1, e_2, e_3\}$ is linearly independent in $R^3\}$ be the set of all frames at the origin in R^3. By our convention on matrices $F = \{(e_1, e_2, e_3) : (e_1 e_2 e_3) \in \mathrm{Gl}(3)\}$. Let $F_G = \{(e_1, e_2, e_3) : (e_1 e_2 e_3) \in G\}$. Since G is a group and columns are sent into columns under matrix multiplication F_G is closed under the action of G. That is if $(e_1, e_2, e_3) \in F_G$ and $g \in G$ then $(ge_1, ge_2, ge_3) \in F_G$. An element of F_G will be called a *G-frame.* The following theorem characterizes a G-frame.

Theorem. $(e_1, e_2, e_3) \in F_G$ *if and only if*

(1) e_1 *and* e_3 *are linearly independent positive isotropic vectors*

(2) $(e_1, e_2)_L = (e_2, e_3)_L = 0$

(3) $(e_2, e_2)_L + 2(e_1, e_3)_L = 0.$

The proof consists of simple computations using the fact that a positive isotropic vector has the form ${}^t(a^2, ab, b^2)$ where $a, b \in R$ and $a^2 + b^2 \neq 0$. The same procedure can be used to prove the following theorem which gives a useful method for the construction of G-frames.

Theorem. *Let* e_1, e_3 *be linearly independent positive isotropic vectors and* $e_2 = 2(e_1, e_3)_L^{-1/2}\, e_1 x_L e_3$. *Then* $(e_1, e_2, e_3) \in F_G$.

The converse of this theorem can be obtained using the relation

$$(Xx_LY, Z)_L = \frac{\det(e_1e_2e_3)}{(e_2, e_2)_L\,(e_1, e_3)_L^2} \det \begin{bmatrix} (X, e_3)_L & (Y, e_3)_L & (Z, e_3)_L \\ (X, e_2)_L & (Y, e_2)_L & (Z, e_2)_L \\ (X, e_1)_L & (Y, e_1)_L & (Z, e_1)_L \end{bmatrix}$$

which results from expanding $\det(XYZ)$ in coordinates with respect to the basis (e_1, e_2, e_3).

To obtain the G-frame space let $\gamma_X : R^3 \to T_X(R^3)$ be the canonical vector space isomorphism between R^3 and the tangent space to R^3 at the point X. $\gamma_X(Y)$ is the tangent vector at $c(0)$ to the curve $c : R \to R^3$ given by $c(t) = X + tY$ for $t \in R$. Let $\gamma_X(F_G) = \{(\gamma_X(e_1), \gamma_X(e_2), \gamma_X(e_3)) : (e_1, e_2, e_3) \in F_G\}$ and $F_G(R^3) = \bigcup_{X \in R^3} \{X\} \times \gamma_X(F_G)$. The set $F_G(R^3)$ is the space of G-frames on R^3.

4. The structure equations. A basis for the 1-forms defined on $F_G(R^3)$ is obtained from the left invariant Mauer-Cartan forms defined on $\mathrm{Aff}(R^3)$, the group of affine motions in R^3. $F_G(R^3)$ can be identified up to left translation, with a subgroup of $\mathrm{Aff}(R^3)$ so that the left invariant differential forms on $\mathrm{Aff}(R^3)$ can be pulled back to $F_G(R^3)$. This procedure gives twelve 1-forms ω^i, ω_{ij}, $1 \leqq i, j \leqq 3$, such that if $\omega = {}^t(\omega^1, \omega^2, \omega^3)$ and $\Omega = (\omega_{ij})$ then

$$\omega|_{(X,\gamma_X(e))} = g^{-1}\, \mathrm{d}X|_{(X,\gamma_X(e))},$$

and

$$\Omega|_{(X,\gamma_X(e))} = g^{-1}\, \mathrm{d}g|_{(X,\gamma_X(e))},$$

where $e = (e_1, e_2, e_3)$ and $g = (e_1e_2e_3)$.

This can be written more simply as $\mathrm{d}X = e\omega$ and $\mathrm{d}e = e\Omega$. By pulling the structure equations of $\mathrm{Aff}(R^3)$ back to $F_G(R^3)$ we obtain $\mathrm{d}\omega = -\Omega_\wedge\omega$ and $\mathrm{d}\Omega = -\Omega_\wedge\Omega$.

There are only four independent 1-forms in Ω which are found most easily by using the characterization of a G-frame and pulling $(\ ,\ )_L$ up to $\gamma_X(F_G)$ with γ_X. Since the maps γ_X tend to complicate the notation they will be dropped whenever possible.

Differentiating $(e_1, e_1)_L = 0$ yields $0 = (\mathrm{d}e_1, e_1)_L = (e_1, e_3)_L\,\omega_{31}$. But $(e_1, e_3)_L = -2(e_2, e_2)_L \neq 0$ so $\omega_{31} = 0$. Similarly $e_3 \in L$ implies that $\omega_{13} = 0$.

Differentiation of $(e_1, e_2)_L = 0$ gives

$$\begin{aligned}0 &= (\mathrm{d}e_1, e_2)_L + (e_1, \mathrm{d}e_2)_L = \omega_{i1}(e_i, e_2)_L + \omega_{j2}(e_1, e_j)_L = \\ &= \omega_{21}(e_2, e_2)_L + \omega_{32}(e_1, e_3)_L = (\omega_{23} - 2\omega_{21})(e_1, e_3)_L\,.\end{aligned}$$

Therefore $\omega_{23} = 2\omega_{21}$. Similarly $(e_2, e_3)_L = 0$ gives $\omega_{12} = 2\omega_{23}$.

The condition $(e_2, e_2)_L + 2(e_1, e_3)_L = 0$ gives $\omega_{11} - 2\omega_{22} + \omega_{33} = 0$ so if we set $\omega_{22} = \pi$ then there is a form α such that $\omega_{11} = \pi + \alpha$, and $\omega_{33} = \pi - \alpha$.

Thus

$$\Omega = \begin{bmatrix} \pi + \alpha & 2\omega_{23} & 0 \\ \omega_{21} & \pi & \omega_{23} \\ 0 & 2\omega_{21} & \pi - \alpha \end{bmatrix}.$$

And the structure equations for $F_G(R^3)$ are

$$\begin{aligned}
\mathrm{d}X &= \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3 \\
\mathrm{d}e_1 &= (\pi + \alpha)e_1 + \omega_{21}e_2 \\
\mathrm{d}e_2 &= 2\omega_{23}e_1 + \pi e_2 + 2\omega_{21}e_3 \\
\mathrm{d}e_3 &= \omega_{23}e_2 + (\pi - \alpha)\,e_3 \\
\mathrm{d}\omega^1 &= \omega^1_\wedge(\pi + \alpha) + 2\omega^2_\wedge\omega_{23} \\
\mathrm{d}\omega^2 &= \omega^1_\wedge\omega_{21} + \omega^2_\wedge\pi + \omega^3_\wedge\omega_{23} \\
\mathrm{d}\omega^3 &= 2\omega^2_\wedge\omega_{21} + \omega^3_\wedge(\pi - \alpha) \\
\mathrm{d}\omega_{21} &= \alpha_\wedge\omega_{21} \\
\mathrm{d}\omega_{23} &= \omega_{23\wedge}\alpha \\
\mathrm{d}\alpha &= 2\omega_{21\wedge}\omega_{23} \\
\mathrm{d}\pi &= 0\,.
\end{aligned}$$

These equations were obtained by E. Cartan [1] in a paper on differential equations.

5. Surfaces in R^3. A surface in R^3 will be given as a pair (M^2, h) where M^2 is a two-dimensional manifold and $h: M^2 \to R^3$ is an immersion. In Euclidean geometry frames on a surface are chosen to be equivariant under rigid motions whereas here they must be equivariant under the action of G. Thus before defining a moving frame on a surface it is necessary to select a collection of "admissable" frames at each point which is equivariant under G. The following definition provides a criterion for choosing such collections.

Definition. An *equivariant framing* of (M^2, h) is a map $h^{\#}$ satisfying

(1) $h^{\#} : M^2 \to \{S : S \subset F_G(R^3)\}$

(2) $h^{\#}(p) \subset \{h(p)\} \times \gamma_{h(p)}(F_G)$ for $p \in M^2$

(3) If $k : M^2 \to R^3$ is another immersion with $k(p) = g\, h(p)$ for some $g \in G$ then $k^{\#}(p) = g\, h^{\#}(p)$. This means that if $(h(p), \gamma_{h(p)}(e)) \in h^{\#}(p)$ then $g(h(p), \gamma_{h(p)}(e)) = (g\, h(p), L_{g*}\, \gamma_{h(p)}(e)) \in k^{\#}(p)$ where $L_g : R^3 \to R^3$ is given by $L_g(X) = gX$.

Definition. Let U be an open subset of M^2 and $h^{\#}$ an equivariant framing. A *moving G-frame* is a map $e : U \to F_G(R^3)$ such that for each point $p \in U$, $e(p) \in h^{\#}(p)$.

Every point of a surface (M^2, h) can be put into one of three classes by the way in which $h_*(T_p(M^2))$ intersects the isotropic cone $\gamma_{h(p)}(L)$ in $T_{h(p)}(R^3)$. A point p will be called *hyperbolic, parabolic* or *elliptic* according as to whether there are two, one or zero linearly independent isotropic vectors in $T_{h(p)}(R^3)$. In the hyperbolic and parabolic cases there is a natural way of choosing an equivariant framing.

6. The hyperbolic case. Suppose each point of (M^2, h) is hyperbolic. For each $p \in M^2$ let

$$h^{\#}(p) = \{h(p)\} \times \{(e_1, e_2, e_3) \in \gamma_{h(p)}(F_G) : e_1, e_3 \text{ span } h_*(T_p(M^2))\}\,.$$

Then $h^{\#}$ is an equivariant framing of (M^2, h). So on a sufficiently small open set $U \subset M^2$ a moving G-frame e can be defined by requiring that $e_1(p)$, $e_3(p)$ be positive isotropic vectors which span $h_*(T_p(M^2))$ and then letting $e_2(p) = 2(e_1(p), e_3(p))_L^{-1/2}$ . . $e_1(p) \times_L e_3(p)$, for each $p \in U$. Now use e to pull the 1-forms back to U. These new 1-forms will be denoted by the same symbols since no confusion should occur. For $p \in U$, $\mathrm{d}p = \omega^1 e_1 + \omega^3 e_3$ so $\omega^2 = 0$. Therefore $0 = \mathrm{d}\omega^2 = \omega^1_{\wedge}\omega_{21} + \omega^2_{\wedge}\omega_{23}$ and since ω^1 and ω^3 are independent there are three functions a, b, c defined on U such that

$$\omega_{21} = a\omega^1 + b\omega^3\,, \quad \omega_{23} = b\omega^1 + c\omega^3\,.$$

Theorem. *Let a, b and c be associated with a moving G-frame e as above. Then the vanishing of b, $b^2 - ac$ or ac is independent of the choice of moving G-frame e. And if $b \neq 0$ then ac/b^2 is independent of e.*

Proof. Note that

$$\omega^1 = (\mathrm{d}X, e_3/(e_1, e_3)_L)_L\,, \quad \omega_{23} = (\tfrac{1}{2}\,\mathrm{d}e_2, e_3/(e_1, e_3)_L)_L\,.$$

And

$$\omega^3 = (\mathrm{d}X, e_1/(e_1, e_3)_L)_L\,, \quad \omega_{21} = (\tfrac{1}{2}\,\mathrm{d}e_2, e_1/(e_1, e_3)_L)_L\,.$$

So if an admissable change of e_1 and e_3 is made leaving e_2 fixed then the forms ω_{23} and ω_{21} are transformed by the same linear transformation that transforms ω^1 and ω^3. Thus the 2-forms $\omega^1_\wedge\omega^3$ and $(\omega^1 + y\omega_{23})_\wedge (\omega^3 + y\omega_{21})$, for $y \in R$, are transformed by multiplication by the determinant associated with the linear transformation. And since

$$(\omega^1 + y\omega_{23})_\wedge (\omega^3 + y\omega_{21}) = (1 + 2by + (b^2 - ac)\, y^2)\, \omega^1_\wedge\omega^3$$

the polynomial $1 + 2by + (b^2 - ac)\, y^2$ is invariant under admissable changes of e_1 and e_3. However, it is acted upon by an admissable change of e_2. Such a change takes the form $2se_2$, where s is a non-zero function. This changes the indeterminant y by multiplication by s. Hence the vanishing of the coefficients b and $b^2 - ac$ is invariant and if $b \neq 0$ the ratio $(b^2 - ac)/b^2$ is invariant under admissable changes of the moving G-frame.

Theorem. *Suppose $a = b = c = 0$ for a moving G-frame e defined on U then $h(U)$ is contained in a two-dimensional linear subspace.*

Proof. We can assume ${}^t(0, 0, 0) \in h(U)$. The idea is to construct a linear map which contains $h(U)$ in its kernel.

For $Y \in R^3$ fixed define

$$\zeta : U \to R^3$$

by

$$\zeta(p) = \frac{(e_2(p), Y)_L}{(e_2(p), e_2(p))_L}\, e_2(p)\,.$$

$\zeta(p)$ is the projection of Y on $e_2(p)$. Note that $\gamma_{h(p)}$ has again been left out. Since $\omega_{21} = \omega_{23} = 0$

$$\mathrm{d}(e_2(p)) = \pi|_{(h(p),e)}\, e_2(p)$$

therefore $\mathrm{d}\zeta(p) = 0$ and ζ is constant on U. Now using any $q \in U$ we can define

$$\eta : R^3 \to R^3$$

by

$$\eta(X) = \frac{(e_2(q), X)_L}{(e_2(q), e_2(q))_L}\, e_2(q)\,.$$

Now for $X = h(p)$, $p \in U$

$$\mathrm{d}\eta(h(p)) = \frac{(e_2(q), \mathrm{d}h(p))_L}{(e_2(q), e_2(q))_L}\, e_2(q) = \omega^2|_{(h(p),e)}\, e_2(p) = 0\,.$$

Hence η is constant on $h(U)$ and since η is linear $h(U) \subset \ker \eta$. Since $\dim U = 2$, $\dim(\ker \eta) \geqq 2$. But $\eta(e_2) \neq 0$ so $\dim(\ker \eta) = 2$ and $h(U)$ is contained in a two-dimensional linear subspace.

7. The parabolic case. Suppose each point of (M^2, h) is parabolic then $h^\#(p) = \{(e_1, e_2, e_3) \in \gamma_{h(p)}(F_G) : e_1 \in h_*(T_p(M^2))\}$, for $p \in M^2$, is an equivariant framing of M^2. On a sufficiently small open set U define a moving G-frame e so that $e(p) \in h^\#(p)$, for $p \in U$. Since $e_*(T_p(M^2))$ is tangent to $\gamma_{h(p)}(L)$, e_1 and e_2 span $T_p(M^2)$ and $\mathrm{d}p = \omega^1 e_1 + \omega^2 e_2$. Therefore $\omega^3 = 0$. So $0 = \mathrm{d}\omega^3 = 2\omega^2_\wedge\omega_{21}$ and there exists a function k such that $\omega_{21} = k\omega^2$. Although k is not invariantly determined we do have the following theorem.

Theorem. *The vanishing of k is independent of the choice of the moving G-frame.*

Proof. Since

$$\omega^2 = (\mathrm{d}X, e_2/(e_2, e_2)_L)_L, \quad \omega_{21} = (\mathrm{d}e_1, e_2/(e_2, e_2)_L)_L$$

and

$$\omega^1 = (\mathrm{d}X, e_3/(e_1, e_3)_L)_L$$

an admissable change of e_2 and e_3 holding e_1 fixed changes ω_{21} and ω^2 by the same linear transformation. Thus $\omega^1_\wedge\omega^2$ and $\omega^1_\wedge\omega_{21}$ change by multiplication by a determinant. And since $\omega^1_\wedge\omega_{21} = k\omega^1_\wedge\omega^2$, k is invariant under changes of e_2 and e_3. But an admissable change in e_1 takes the form re_1, where r is a positive function, thus a change to a new frame changes k to rk. Hence the vanishing of k is independent of admissable choices of G-frames.

8. Pencils of symmetric bilinear forms. A plane π through the origin in R^3 corresponds to a pencil P of bilinear forms given by

$$P = \{u\varphi + v\tau : u, \ v \in R\}$$

where φ and τ are independent elements in $\odot^2 V_2$. Call π *hyperbolic* if it intersects L in two lines, *parabolic* if it is tangent to L and *elliptic* if it contains no non-zero isotropic vectors. Then the following classical theorem is easily proved.

Theorem. *If π_1 and π_2 are planes of the same type through the origin in R^3 then there is a $g \in G$ such that $\pi_1 = g\pi_2$.*

Or equivalently if P is a pencil of forms generated by two independent bilinear forms then there exits a basis B of V_2 such that one of the following holds:

(1) $P_B = \left\{\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} : u, \ v \in R\right\}$, *the hyperbolic case,*

(2) $P_B = \left\{\begin{bmatrix} u & v \\ v & 0 \end{bmatrix} : u, \ v \in R\right\}$, *the parabolic case,*

(3) $P_B = \left\{\begin{bmatrix} u & v \\ v & -u \end{bmatrix} : u, \ v \in R\right\}$, *the elliptic case.*

References

[1] *E. Cartan,* "La Geometria da los Ecuaciones Diferenciales de Tracer Orden", Rev. Mat. Hispano-Amer. t. *4* (1941), 1—31.

[2] *H. Flanders,* "A Method of General Linear Frames in Riemannian Geometry I", Pacific J. Math. *3* (1953), 551—565.

[3] *H. Flanders,* "Development of an Extended Exterior Differential Calculus", Trans. Amer. Math. Soc. *75* (1953), 311—326.

[4] *H. Flanders,* "Differential Forms with Applications to the Physical Sciences", Academic Press, New York—London, 1963.

[5] *R. Gardner,* "Differential Geometric Methods in Partial Differential Equations", Thesis, University of California at Berkely, Unpublished, 1965.

[6] *R. Gardner,* "Invariants of Pfaffian Systems", Trans. Amer. Math. Soc. *126* (1967), 514—533.

Author's address: Flushing, New York 11367, U.S.A. (Queens College of the City University of New York).