Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 22 (1972), No. 3, 361–367

Persistent URL: http://dml.cz/dmlcz/101107

Czechoslovak Mathematical Journal, 22 (97) 1972, Praha

ON IDEALS OF A SEMILATTICE

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(Received May 4, 1970)

A semilattice S can be defined as a commutative idempotent semigroup or as a partially ordered set in which any two elements have a g.l.b. (in the latter case S is also called a lower semilattice, dually one defines an upper semilattice). If S is a semilattice in the former sense, denoting its multiplication by juxtaposition, the relation on S defined by $a \leqq b \Leftrightarrow ab = a$ defines a partial ordering in which ab is the g.l.b. of a and b and makes S a semilattice in the latter sense. Conversely, if S is a semilattice in the latter sense (i.e., a lower semilattice), defining the product of any elements a and b in S to be their g.l.b., we obtain a semilattice in the former sense. It will be clear from the context whether S is regarded as a semigroup or as a poset.

We first consider S as a semigroup giving a characterization of semilattices among all semigroups in terms of bitranslations, and then prove that every bitranslation of a semilattice S is induced by retraction onto an ideal. We then consider S both as a semigroup and a poset in discussing various (lower) subsemilattices of the lattice $\mathscr{I}$ of all ideals of S such as: the idealizer of the subsemilattice of principal ideals, the normal (MacNeille) completion of S, injective hull in the category of semilattices, and present some examples for illustration. For material concerning semigroups consult [6] and for posets [3].

An *ideal* I of a semigroup S is a nonempty subset of S for which sa, $as \in I$ for all $a \in I$, $s \in S$. A homorphism η of S onto its ideal I which leaves I elementwise fixed is a *retraction* and I is a *retract* of S. Note that the usual definition of retract does not include the requirement that it be an ideal. A function λ (resp. ϱ) of S into itself written on the left (resp. right) is a *left* (resp. *right*) *translation* of S if $\lambda(xy) = (\lambda x)y$ (resp. $(xy)\varrho = x(y\varrho)$) for all $x, y \in S$; the pair (λ, ϱ) is a *bitranslation* if $x(\lambda y) = (x\varrho)y$ for all $x, y \in S$. Defining multiplication for left (resp. right) translations by $(\lambda\lambda')\, x = \lambda(\lambda' x)$ (resp. $x(\varrho\varrho') = (x\varrho)\,\varrho'$) for all $x \in S$, the set of all bitranslations of S is a semigroup under multiplication $(\lambda, \varrho)(\lambda', \varrho') = (\lambda\lambda', \varrho\varrho')$, called the *translational hull* of S and denoted by $\Omega(S)$. A semigroup S is *weakly reductive* if $ax = bx$, $xa = xb$ for all $x \in S$ implies $a = b$.

Proposition 1. *For a retraction η of a semigroup S, let $\lambda s = s\varrho = s\eta$ for all $s \in S$; then $(\lambda, \varrho) \in \Omega(S)$. If $S\eta$ is weakly reductive, then η is the only retraction of S onto $S\eta$.*

Proof. The hypothesis on η implies that $S\eta$ is an ideal of S and that $\eta^2 = \eta$. For any $x, y \in S$, we then have $(x\eta)y$, $x(y\eta) \in S\eta$ and thus

$$(x\eta)y = [(x\eta)\, y]\, \eta = (x\eta^2)(y\eta) = (xy)\, \eta = (x\eta)(y\eta^2) = [x(y\eta)]\, \eta = x(y\eta)$$

which implies $(\lambda, \varrho) \in \Omega(S)$. If ξ is another retraction of S with $S\xi = S\eta$, then for any $s \in S$ and $x \in S\eta$, we obtain

$$x(s\xi) = (xs)\, \xi = (xs)\, \eta = x(s\eta)\,, \quad (s\xi)\, x = (sx)\, \xi = (sx)\, \eta = (s\eta)\, x$$

and hence $s\xi = s\eta$.

We say that the bitranslation (λ, ϱ) in Proposition 1 is *induced* by η. For any element s of a semigroup S, the *left* (resp. *right*) *translation induced* by s is given by $\lambda_s x = sx$ (resp. $x\varrho_s = xs$) for all $x \in S$; $\pi_s = (\lambda_s, \varrho_s)$ is the *bitranslation induced* by s.

Proposition 2. *A semigroup S is a semilattice if and only if every bitranslation of S is induced by some retraction.*

Proof. Let S be a semilattice and let $(\lambda, \varrho) \in \Omega(S)$. Then for any $x, y \in S$ we obtain

$$\begin{aligned} &x(y\varrho) = (xy)\, \varrho = y(x\varrho)(x\varrho) = (yx)\, \varrho(x\varrho) = x(y\varrho)(x\varrho) = (x\varrho)(y\varrho) \in S\varrho\,, \\ &(x\varrho)\, \varrho = [(xx)\, \varrho]\, \varrho = [x(x\varrho)]\, \varrho = (x\varrho)(x\varrho) = x\varrho\,, \\ &\lambda x = \lambda(xx) = (\lambda x)\, x = x(\lambda x) = (x\varrho)\, x = x(x\varrho) = (xx)\, \varrho = x\varrho\,, \end{aligned}$$

which proves that ϱ is a retraction inducing (λ, ϱ).

Conversely, let S be a semigroup all of whose bitranslations are induced by retractions. Then for every $s \in S$, π_s is induced by a retraction, which implies that $sx = \lambda_s x = x\varrho_s = xs$ for all $x \in S$ and thus S is commutative. Since every retraction is idempotent, we have $\varrho_s^2 = \varrho_s$ which implies that

$$xs = x\varrho_s = x\varrho_s^2 = x\varrho_{s^2} = xs^2 \tag{1}$$

for all $x \in S$. Define the function ϱ by: $x\varrho = x^2$ for all $x \in S$. Letting ι_S be the identity function on S and using commutativity and (1), we obtain

$$(xy)\varrho = (xy)^2 = x^2y^2 = xy^2 = x(y\varrho)\,, \quad (x\varrho)y = x^2y = xy = x(\iota_S y)\,,$$

which shows that $(\iota_S, \varrho) \in \Omega(S)$. The hypothesis then implies $x = \iota_S x = x\varrho = x^2$ and S is indeed a semilattice.

We now restrict our attention to semilattices and *fix a semilattice S*. Note that in view of the above discussion, we can speak of a *translation* ϱ instead of a bitranslation

(λ, ϱ), multiplication of translations is their composition. As is customary in posets, we now include the empty set $\emptyset$ as an ideal of S; observe that in terms of order, an ideal I of S is a subset of S satisfying: $x \in I$, $y \in S$, $y \leqq x$ implies $y \in I$; the principal ideal generated by $s \in S$ is given by $(s) = \{x \in S \mid x \leqq s\}$. We further fix the following notation: $\mathscr{I}$ is the complete lattice of all ideals of S, $\mathscr{P}$ is the lower subsemilattice of $\mathscr{I}$ consisting of all principal ideals of S, $\mathscr{R}$ is the poset of all retracts of S, where the ordering is always the set theoretic inclusion. The following are consequences of Proposition 2.

Corollary 1 (Szász [12]). *A function ϱ on S is a translation if and only if ϱ is a retraction.*

Corollary 2 (cf. Szász [13], Kolibiar [10]). *The function $\varrho \to S\varrho$ is an isomorphism of $\Omega(S)$ onto $\mathscr{R}$.*

Proof. Let ϱ and ϱ' be translations of S and $x \in S$. Then

$$x\varrho\varrho' = [x(x\varrho)]\,\varrho' = [(x\varrho)x]\,\varrho' = (x\varrho)(x\varrho') = (x\varrho')(x\varrho) = [(x\varrho')\,x]\,\varrho \in S\varrho$$

so that $S\varrho\varrho' \subseteq S\varrho \cap S\varrho'$; conversely, if $x = y\varrho = z\varrho'$, then

$$x = (y\varrho)(z\varrho') = [(y\varrho)z]\,\varrho' = [z(y\varrho)]\,\varrho' = (zy)\,\varrho\varrho' \in S\varrho\varrho'$$

and thus $S\varrho \cap S\varrho' \subseteq S\varrho\varrho'$. Consequently $S\varrho\varrho' = S\varrho \cap S\varrho'$. That $S\varrho = S\varrho'$ implies $\varrho = \varrho'$ follows from the last part of Proposition 1.

This proof shows that $\mathscr{R}$ is a lower subsemilattice of $\mathscr{I}$.

Corollary 3 (Szász-Szendrei [14]). *$\Omega(S)$ is a semilattice.*

The next result shows that a retract can be regarded as a generalization of a principal ideal. Kolibiar [10] proved it for the upper semilattice of a lattice but his proof is valid in any semilattice. The proof below is shorter.

Proposition 3. *An ideal I of a semilattice S is a retract if and only if for every $s \in S$, the ideal $I \cap (s)$ is principal.*

Proof. Let η be a retraction of S onto I and let $s \in S$. If $x \in I \cap (s)$, then $x \in I$ and $x \leqq x$, so $x = x\eta = (xs)\,\eta = (x\eta)(s\eta) = x(s\eta)$ nad thus $x \leqq (s\eta)$. Consequently $I \cap (s) \subseteq (s\eta)$. Conversely, if $x \leqq s\eta$, then $x = x(s\eta) = (xs)\,\eta = s(x\eta)$ so that $x \leqq s$; furthermore

$$x\eta = (s\eta)(x\eta^2) = (s\eta)(x\eta) = (sx)\,\eta = s(x\eta) = x$$

and $x \in I$. Thus $(s\eta) \subseteq I \cap (s)$, which proves $I \cap (s) = (s\eta)$.

Conversely, suppose that for every $s \in S$ there exists $\bar{s} \in S$ such that $I \cap (s) = (\bar{s})$. Since $\bar{s}$ is then the unique maximal element of the set $\{x \in I \mid x \leqq s\}$, the theorem and Proposition 2 of [11] imply that I is a retraction of S. However, one can show directly that the mapping $s \to \bar{s}$ is the desired retraction.

If A is a subsemigroup of a semigroup B, then the *idealizer* of A in B is the largest subsemigroup of B containing A as an ideal and is given by $\{b \in B \mid ba, ab \in A$ for all $a \in A\}$. If I is an ideal of B, then B is an (*ideal*) *extension* of I; B is a *dense extension* of I if the equality relation on B is the only congruence on B whose restriction to I is the equality relation on I; I is a *densely embedded ideal* of B if B is under inclusion a maximal dense extension of I. A subsemigroup A of B is *densely embedded* in B if A is a densely embedded ideal of its idealizer in B. For an extensive study of these concepts consult [7].

Corollary 1. *$\mathscr{R}$ is the idealizer of $\mathscr{P}$ in $\mathscr{I}$ and $\mathscr{P}$ is a densely embedded subsemigroup of $\mathscr{I}$.*

Proof. The first statement follows from Proposition 3. For the second, we note that the function in Corollary 2 to Proposition 2 has the property: $\varrho_s \to (s)$ for all $s \in S$, which by ([8], 1.3.5, see also [9], 3.12) implies that $\mathscr{P}$ is a densely embedded ideal of $\mathscr{R}$.

A poset P is a *meet* (resp. *join*) *dense extension* of its partially ordered subset Q if every element of P is the meet (join) of some subset of Q.

Corollary 2. *Let V be a semilattice and an ideal extension of S. Then V is a dense extension (qua semigroup) if and only if V is a join dense extension (qua poset).*

Proof. This follows from ([8], 1.5) and Corollary 1.

An ideal I of a semilattice S is *normal* if $I = \bigcap_{a \in A} (a)$ for some subset A of S (if $A = \emptyset$, then $I = S$). The set $\mathscr{N}$ of all normal ideals of S is a complete lattice under inclusion called *the normal* (or MacNeille, or Dedekind-MacNeille) *completion* of S. Note that $\mathscr{N}$ is a lower subsemilattice of $\mathscr{I}$ and contains $\mathscr{P}$. A complete lattice L containing S is *a normal* completion of S if there exists a lattice isomorphism of L onto $\mathscr{N}$ which restricted to S coincides with the mapping $s \to (s)$. In particular, $\mathscr{N}$ is a normal completion of $\mathscr{P}$. Let $\mathscr{D}$ denote the set of all ideals which are arbitrary intersections of retracts of S; then $\mathscr{D}$ is a lower subsemilattice of $\mathscr{I}$.

Proposition 4. *$\mathscr{D}$ is a normal completion of $\mathscr{R}$.*

Proof. Since $\mathscr{D}$ is closed under taking intersections and $S \in \mathscr{R}$ implies $S \in \mathscr{D}$, $\mathscr{D}$ is a complete lattice. By its very definition, $\mathscr{D}$ is a meet dense extension of $\mathscr{R}$, and since it is also a join dense extension of $\mathscr{P}$, it is also a join dense extension of $\mathscr{R}$. It now follows from ([1], Kor 3, p. 123) that $\mathscr{D}$ is a normal completion of $\mathscr{R}$.

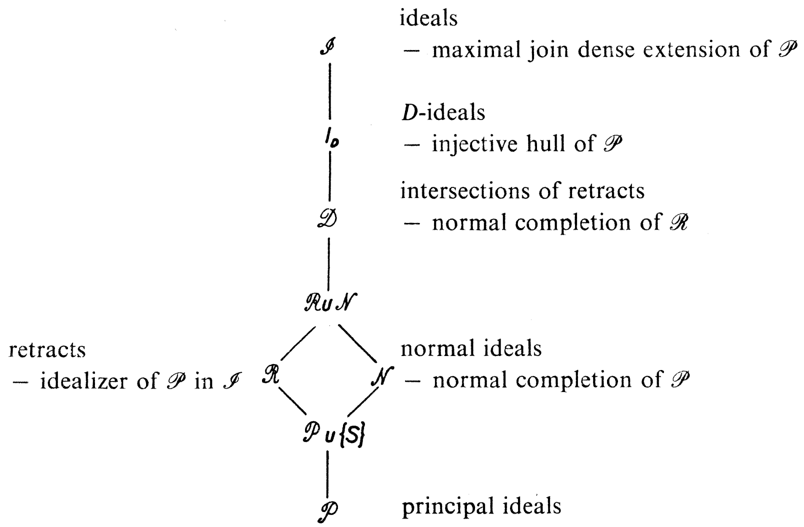
If a subset M of S has a join denote it by $\bigvee M$. Bruns and Lakser [5] call a subset M of S *admissible* if (i) $\bigvee M$ exists, (ii) for any $s \in S$, $\bigvee\{sm \mid m \in M\}$ exists and $s(\bigvee M) = \bigvee\{sm \mid m \in M\}$. They call an ideal I of S a D-ideal if with every admissible subset I also contains its join. They prove that the lower subsemilattice I_D of $\mathscr{I}$, consisting of all D-ideals of S, is an injective hull of S in the category of semilattices and their homomorphisms, and that I_D is a complete lattice.

Proposition 5. $\mathscr{N} \subseteq \mathscr{D} \subseteq I_D$, $\mathscr{R} \cap \mathscr{N} = \mathscr{P} \cup \{S\}$.

Proof. Since normal ideals are arbitrary intersections of principal ideals and the latter are retracts, we have $\mathscr{N} \subseteq \mathscr{D}$. Let $I \in \mathscr{R}$ and M be an admissible subset of I, $m = \bigvee M$. By Proposition 3, there exists $n \in S$ such that $I \cap (m) = (n)$. For every $x \in M$, we obtain $x \in I \cap (m) = (n)$ so that $x \leqq n$ and n is an upper bound for M. But then $n \leqq m$ implies that $m = n \in I$. Consequently $I \in I_D$ and so $\mathscr{R} \subseteq I_D$. Since I_D is a complete lattice, it must contain arbitrary intersections of elements of $\mathscr{R}$ which shows that $\mathscr{D} \subseteq I_D$.

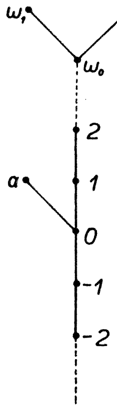
If $I \in \mathscr{R}$ and I has an upper bound m, then $I = I \cap (m) = (n)$ for some $n \in S$ so that $I \in \mathscr{P}$. If $I \in \mathscr{N}$ and I has no upper bound, then $I = S$. Consequently $\mathscr{R} \cap \mathscr{N} \subseteq \mathscr{P} \cup \{S\}$, the converse inclusion is obvious.

Using the terminology and certain results in [1], [4], and [5], we can illustrate a portion of the discussion above by the following diagram.



With the usual identification of S and $\mathscr{P}$, we can write S instead of $\mathscr{P}$ in the diagram. The following examples show that each inclusion in the diagram is in general strict.

Example 1. Let Z be the set of integers and S be the semilattice as in the diagram:



$$\mathscr{P} = \{(\omega_i) \mid i = 0, 1, 2\} \cup \{(n) \mid n \in Z\} \cup \{(a)\},$$

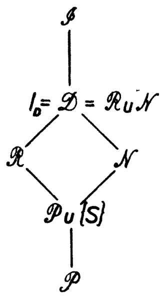
$$\mathscr{N} = \mathscr{P} \cup \{S, \emptyset\},$$

$$\mathscr{R} = \mathscr{P} \cup \{(a) \cup (n) \mid n > 0\} \cup \{(a) \cup (\omega_i) \mid i = 0, 1, 2\} \cup \{S\},$$

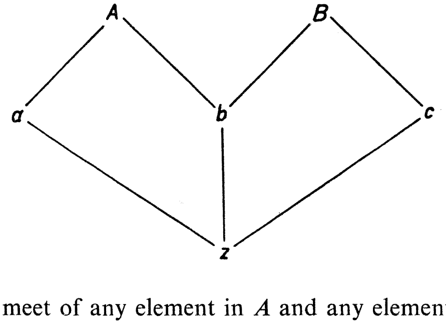
$$I_D = \mathscr{D} = \mathscr{R} \cup \mathscr{N},$$

$$\mathscr{I} = I_D \cup \{Z, Z \cup \{a\}\},$$

and hence we get the following strict inclusion diagram:



Example 2. Let $A = \{1 > 2 > 3 > \ldots\}$, $B = \{1' > 2' > 3' > \ldots\}$, $S = A \cup B \cup \{a, b, c, z\}$ with the diagram:



where, e.g., b is the meet of any element in A and any element in B, etc. Then $I = \{a, b, c, z\}$ has the property $I \in \mathscr{D}$, $I \notin \mathscr{R} \cup \mathscr{N}$.

Example 3. Adding a greatest element to S in Example 2, we get $I \in I_D$, $I \notin \mathscr{D}$.

If S is a chain, it is easy to verify that $I_D = \mathscr{N}$ and hence the normal completion of S is its injective hull in the category of semilattices. Berthiaume [2] has proved that the same holds in the category of S-systems. It is not known what the injective hull of an arbitrary semilattice S in the latter category looks like.

References

[1] *B. Banaschewski,* Hüllensysteme und Erweiterung von Quasi-Ordnungen, Zeitschr. math. Logik Grundl. Math. *2* (1956), 117—130.

[2] *P. Berthiaume,* The injective envelope of S-sets, Canad. Math. Bull. *10* (1967), 261—273.

[3] *G. Birkhoff,* Lattice theory, Amer. Math. Soc. Colloq. Publ. Vol. 25, Amer. Math. Soc., Providence, 1948.

[4] *G. Bruns,* Darstellungen und Erweiterungen geordneter Mengen. I. J. reine angew. Math. *209* (1962), 167—200.

[5] *G. Bruns* and *H. Lakser,* Injective hulls of semilattices, Canad. Math. Bull. *13*(1970), 115—118.

[6] *A. H. Clifford* and *G. B. Preston,* The algebraic theory of semigroups, Vol. I, Math. Surveys No. 7, Amer. Math. Soc., Providence, 1961.

[7] *L. M. Gluskin,* Ideals of semigroups of transformations, Mat. Sbornik *47* (1959), 111—130 (in Russian).

[8] *L. M. Gluskin,* Ideals of semigroups, Mat. Sbornik *55* (1961), 421—448; Correction: ibid. *73* (1967), 303 (in Russian).

[9] *P. A. Grillet* and *M. Petrich,* Ideal extensions of semigroups, Pacific J. Math. *26* (1968), 493—508.

[10] *M. Kolibiar,* Bemerkungen über Translationen der Verbände, Acta Fac. Rer. Nat. Univ. Comenianae Math. *5* (1961), 455—458.

[11] *M. Petrich,* On extensions of semigroups determined by partial homomorphisms, Nederl. Akad. Wetensch. Indag. *28* (1966), 49—51.

[12] *G. Szász,* Die Translationen der Halbverbände, Acta Sci. Math. Szeged *17* (1956), 165—169.

[13] *G. Szász,* Translationen der Verbände, Acta Fac. Rer. Nat. Univ. Comenianae Math. *5* (1961), 449—453.

[14] *G. Szász* und *J. Szendrei,* Über die Translationen der Halbverbände, Acta Sci. Math. Szeged *18* (1957), 44—47.

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