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ON MANIFOLDS WITH CONNECTION

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This paper presents an exact justification of the so-called invariant method of investigation (by É. CARTAN and G. F. LAPTĚV) for manifolds with connection.

In the present terminology, a manifold with connection is (at least locally) a submanifold of a space with Cartan connection. É. Cartan himself showed in the case of a surface in a 3-dimensional space with projective connection, [1], that his invariant method can be also applied to such submanifolds. Some further development of this method was outlined by G. F. Laptěv, [12]. A great contribution to the theory of manifolds with connection was presented by A. ŠVEC in a large series of papers, see the bibliography in [13]. In particular, he has given an exact explanation of the Cartan's method of specialization of frames for manifolds with connection, [14]. Our attention is concentrated on the "prolongation procedure" in its pure form, i.e. without any specialization of frames. We give the invariant definitions of all concepts based on the theory of jets by Ehresmann and we deduce "a posteriori" an algorithm for finding the corresponding coordinate functions in some natural local coordinates.

Our considerations are in the category C^∞. The standard terminology and notation of the theory of jets is used throughout the paper, see [2], [3], [5].

1. Let $P(B, G, \pi)$ be a principal fibre bundle and let $E(B, F, G, P)$ be a fibre bundle associated with P. We have defined a generalized space with connection as a quadruple $\mathcal{S} = \mathcal{S}(P(B, G), F, C, \sigma)$, where C is a connection (of the first order) on the groupoid PP^{-1} associated with P and σ is a cross section of E, [6]. Such a space will be called a manifold with connection, if it holds, moreover,

a) $m = \dim B < \dim F = n$,

b) $C^{-1}(x)(\sigma)$ is regular for every $x \in B$,

c) G acts on F transitively.

A manifold with connection is locally equivalent to a submanifold of a space with Cartan connection. From now on, $\mathcal{S}$ will denote a manifold with connection.

Let $\Gamma : P \to J^1P$ be the representant of C on P, [5], and let ω be the connection form of Γ. The classical definition of ω can be obviously restated as follows. If $\Gamma(u) = j^1_x\varrho$, where ϱ is a local cross section of P, $u \in P$, $x = \pi(u)$, then the relation

$$v = \varrho(\pi(v))\,\psi(v) \tag{1}$$

determines a local mapping $\psi(v)$ of P into G and $\omega_u = j^1_u\,\psi(v)$, $\omega_u : T_u(P) \to \mathfrak{g}$ is the value of ω at u.

Fix an element $c \in F$ and denote by H the stability group of c. Then the fundamental section σ of $\mathcal{S}$ determines a reduction

$$Q = \{u \in P,\ u^{-1}(\sigma(\pi(u))) = c\}$$

of P to the subgroup $H \subset G$. Since $C^{-1}(x)\,(\sigma)$ is regular for every $x \in B$, the mapping $\omega_u \mid T_u(Q)$ is injective for every $u \in Q$. Fix a basis e_α of $\mathfrak{g}$ such that $e_\lambda \in \mathfrak{h}$,

$$\alpha, \beta, \ldots = 1, \ldots, \dim G\,; \quad i, j, \ldots = 1, \ldots, n\,; \quad \lambda, \mu, \ldots = n + 1, \ldots, \dim G\,.$$

This determines a decomposition $\mathfrak{g} = N \oplus \mathfrak{h}$, where N is the linear span of e_i, as well as a decomposition $N = N_1 \oplus N_2$, where N_1 or N_2 is the linear span of e_p or e_J respectively,

$$p, q, \ldots = 1, \ldots, m\,; \quad J, K \ldots = m + 1, \ldots, n\,.$$

Let $f : N_1 \oplus N_2 \to N_1$ be the canonical projection. Put $N_u = N \cap \omega(T_u(Q))$ for every $u \in Q$. Let $\hat{Q}$ be the subspace of all $u \in Q$ such that m-dimensional subspace $N_u \subset N$ is transversal to N_2, i.e. $f(N_u) = N_1$. For the sake of simplicity, we shall further denote by $\omega = \omega^\alpha e_\alpha$ the restriction of ω to $\hat{Q}$ and by π the restriction of $\pi : P \to B$ to $\hat{Q}$.

Introduce a mapping $\mu : \hat{Q} \to H^1(B)$ as follows. Let $u \in \hat{Q}_x$ and let $X_p \in T_u(\hat{Q})$ be the vectors determined by the relations $\omega(X_p) \in N_u$ and $f(\omega(X_p)) = e_p$. Then the vectors $\pi_*(X_p)$ form a basis of $T_x(B)$ and this basis can be identified with an element $\mu(u) \in H^1(B)$.

Lemma 1. *Let φ be the canonical form of $H^1(B)$ and let φ^p be the components of φ. Then*

$$\omega^p = \mu^*\varphi^p\,. \tag{2}$$

Proof. Let $X \in T_u(\hat{Q})$. By definition of the canonical form of $H^1(B)$, $\varphi^p(\mu_*X)$ are the components of the vector $\beta_*(\mu_*X)$ with respect to the frame $\mu(u)$, i.e. $\beta_*(\mu_*X) = \varphi^p(\mu_*X)\,\pi_*(X_p)$. Since $\beta\mu = \pi$, this can be rewritten as $\pi_*(X) = \varphi^p(\mu_*X)\,\pi_*(X_p)$. On the other hand, if $X_\lambda \in T_u(\hat{Q}_x)$ are the vectors determined by $\omega(X_\lambda) = e_\lambda$, then $\omega(X) = \omega^\alpha(X)\,e_\alpha$ implies $X = \omega^p(X)\,X_p + \omega^\lambda(X)\,X_\lambda$. Hence $\pi_*(X) = \omega^p(X)\,\pi_*(X_p)$, which proves (2).

It will be convenient to express $\mu(u)$ as 1-jet at 0 of a local diffeomorphism ζ of $\mathbf{R}^m$ into B, but it will suffice to deduce only an "equation" for ζ. Let U be a coordinate neighbourhood of $c \in F$, let $\varkappa$ be a coordinate diffeomorphism $\varkappa : U \to \mathbf{R}^n$, $\varkappa(c) = 0$ with coordinates x^i and let $a : U \to G = G(F, H)$ be a local cross section such that $a_*((\partial/\partial x^i)_0) = e_i$. Considering $\mathbf{R}^n$ as fibered manifold over $\mathbf{R}^m$, U is a fibered manifold $(U, h, \mathbf{R}^m)$ and we have a natural cross section $b : \mathbf{R}^m \to U$. By (1), if $\mu(u) = j_0^1\zeta$, then ζ satisfies the relation

$$(3) \qquad \chi(\zeta(y)) = \varrho(\zeta(y))\, a(b(y))\,, \quad y \in \mathbf{R}^m\,,$$

where χ is a local cross section of Q.

2. Consider a more general situation. Let (A, v, F) be a fibered manifold and let G act from the left on A in such a way that v is an equivariant map. Then we have a well-defined projection $\bar{v}$ of the associated fibre bundle $Z = Z(B, A, G, P)$ into E, $\bar{v}(\{(u, s)\}) = \{(u, v(s))\}$, $u \in P$, $s \in A$. Let $\vartheta : B \to Z$ be a cross section such that $\bar{v}\vartheta = \sigma$. The first development (or the absolute differential) ϑ^1 of ϑ is a cross section of $\bigcup_{x\in B} J_x^1(B, Z_x)$, while the mapping $x \mapsto k(\vartheta^1(x))$ (where $k(\vartheta^1(x))$ means the contact element determined by $\vartheta^1(x)$, cf. [3]) is a cross section of $\bigcup_{x\in B} K_m^1(Z_x) = (B, K_m^1(A), G, P)$. Put $M = K_m^1(A) \mid A_0$, where $A_0 = v^{-1}(c)$. Since M is an H-invariant subspace of $K_m^1(A)$, we have a subbundle $(B, M, H, Q) \subset (B, K_m^1(A), G, P)$ and $\bar{v}\vartheta = \sigma$ implies that the values of $k(\vartheta^1)$ lie in (B, M, H, Q), i.e. $k(\vartheta^1) : B \to (B, M, H, Q)$.

Put $q = hv$ so that $v^{-1}(U) \subset A$ is a fibered manifold $(v^{-1}(U), q, \mathbf{R}^m)$. Let $\hat{M} \subset M$ be the subspace of all elements of M transversal with respect to q. We have a natural imbedding $\varepsilon : \hat{M} \to T_m^1(A)$ which can be introduced as follows. If $X \in \hat{M}$, $X = k(Y)$, where Y is a 1-jet of an m-dimensional manifold V into A with the target in A_0 and transversal with respect to q, then qY is an invertible 1-jet of V into $\mathbf{R}^m$ with target 0 and we define

$$(4) \qquad \varepsilon(k(Y)) = Y(qY)^{-1} \in T_m^1(A)\,.$$

If $k(\Theta^1)$ means the restriction of the indirect form (see [4]) of $k(\vartheta^1)$ to $\hat{Q} \subset Q$, then the values of $k(\Theta^1)$ lie in $\hat{M}$. Further, let Θ^1 be the indirect form of ϑ^1, which is a mapping of $H^1(B) \circledast P$ into $T_m^1(A)$ (in this paper, the symbol $\circledast$ will denote the fibre product over B), cf. [10], and let $i : \hat{Q} \to P$ be the injection.

Lemma 2. *The diagram*

$$(5) \qquad \begin{array}{ccc} \hat{M} & \xleftarrow{k(\Theta^1)} & \hat{Q} \\ \Big\downarrow{\scriptstyle\varepsilon} & & \Big\downarrow{\scriptstyle(\mu, i)} \\ T_m^1(A) & \xleftarrow{\Theta^1} & H^1(B) \circledast P \end{array}$$

is commutative.

Proof. If $u \in \hat{Q}_x$, $\Gamma(u) = j_x^1\, \varrho(y)$, then $\Theta^1(u) = u^{-1}[\vartheta^1(x)] = j_x^1[\varrho^{-1}(y)\,(\vartheta(y))]$ and $\varepsilon(k(\Theta^1(u))) = j_x^1[\varrho^{-1}(y)\,(\vartheta(y))]\,\{q j_x^1[\varrho^{-1}(y)\,(\vartheta(y))]\}^{-1}$. But $q j_x^1[\varrho^{-1}(y)\,(\vartheta(y))] =$

$= hj^1_x[\varrho^{-1}(y)(\bar{v}(\vartheta(y)))] = hj^1_x[\varrho^{-1}(y)(\sigma(y))]$. On the other hand, $\Theta^1(\mu(u), u) = u^{-1}(\vartheta^1(x))\,\mu(u) = j^1_x[\varrho^{-1}(\zeta(y))(\vartheta(\zeta(y)))]$, where $\zeta(y)$ satisfies (3), or, equivalently,

$$a(b(y)) = \varrho^{-1}(\zeta(y))\,\chi(y))\,. \tag{6}$$

Applying to (6) the projection $g \mapsto g(c)$ of G into F and taking into account that χ is a local cross section of Q, we obtain

$$b(y) = \varrho^{-1}(\zeta(y))\,(\sigma(\zeta(y)))\,. \tag{7}$$

Using the projection $h : U \to \mathbf{R}^m$, we further deduce $y = h[\varrho^{-1}(\zeta(y))(\sigma(\zeta(y)))]$, which implies $j^1_0\,\zeta(y) = \{hj^1_x[\varrho^{-1}(y)(\sigma(y))]\}^{-1}$. This proves Lemma 2.

3. We shall first discuss the case $A = F$, $v = id$, so that $\vartheta = \sigma$. Let Σ^1 be the indirect form of the first development σ^1 of σ and let $k(\Sigma^1)$ be the restriction of the indirect form of $k(\sigma^1)$ to $\hat{Q}$. Taking into account the local coordinates x^i on F, we have local identifications $M = K^1_{n,m}$ and $\hat{M} = \hat{K}^1_{n,m}$, see [9]. By (5), we deduce a commutative diagram

$$\begin{array}{ccc} \hat{K}^1_{n,m} & \xleftarrow{\;k(\Sigma^1)\;} & \hat{Q} \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle(\mu,i)} \\ T^1_m(F) & \xleftarrow{\;\Sigma^1\;} & H^1(B) \circledast P \end{array} \tag{7}$$

Let x^i, x^i_p be the local coordinates on $T^1_m(F)$ determined by $\varkappa$ and let y^J_p be the natural coordinates of $\hat{K}^1_{n,m}$, [9]. Then the coordinate form of ε is

$$(y^J_p) \mapsto (0, (\delta^q_p, y^J_p))\,. \tag{8}$$

Proposition 1. *Let $\omega = \omega^\alpha e_\alpha$ be the restriction of the connection form of Γ to $\hat{Q}$ and let $a^J_p : \hat{Q} \to \mathbf{R}$ be the coordinate functions of $k(\sigma^1)$. Then*

$$\omega^J = a^J_p \omega^p\,. \tag{9}$$

Proof. Let $\tilde{\omega}^\alpha$ be the basis of $\mathfrak{g}^*$ dual to e_α and let

$$dx^i + \xi^i_\alpha(x^j)\,\tilde{\omega}^\alpha = 0$$

be the equations of the fundamental distribution on $G \times F$. Obviously, it is

$$\xi^i_j(0) = \delta^i_j\,, \quad \xi^\lambda_j(0) = 0\,. \tag{10}$$

Let $\bar{\omega}$ be the connection form on P, let $\bar{\varphi}_1$ be the canonical form of $H^1(B)$ and let $\hat{\omega} = p_2^*\bar{\omega}$, $\varphi_1 = p_1^*\bar{\varphi}_1$, where p_1 and p_2 are the product projections of $H^1(B) \circledast P$. According to [10], it holds

$$d\hat{a}^i + \xi^i_\alpha(\hat{a}^j)\,\hat{\omega}^\alpha = \hat{a}^i_p \varphi^p\,, \tag{11}$$

where $\hat{a}^i, \hat{a}^i_p : H^1(B) \circledS P \to \mathbf{R}$ are the coordinate functions of σ^1. However, the subspace $H^1(B) \circledS Q \subset H^1(B) \circledS P$ is characterized by $\hat{a}^i = 0$ and if we further restrict all quantities of (11) to $\hat{Q}$ and apply (2), (7), (8), (10), we obtain (9), QED.

Now, we shall treat the general case. Choose a local coordinate system x^i, x^S on A such that it is a prolongation of $\varkappa$ and is compatible with the fibering $\nu : A \to F$,

$$S, T, \ldots = n + 1, \ldots, \dim A .$$

In particular, $x^i = 0$ are the equations of A_0. Let

$$\text{(12)} \qquad dx^i + \xi^i_\alpha(x^j)\,\tilde{\omega}^\alpha = 0\,, \quad dx^S + \xi^S_\alpha(x^i, x^T)\,\tilde{\omega}^\alpha = 0$$

be the equations of the fundamental distribution on $G \times A$. If x^i, x^S, x^i_p, x^S_p are the corresponding local coordinates on $T^1_m(A)$ and y^S, y^J_p, y^S_p are the natural coordinates of $\hat{M}$, then the coordinate form of ε is

$$\text{(13)} \qquad (y^S, y^J_p, y^S_p) \mapsto (0, y^S, (\delta^q_p, y^J_p), y^S_p)\,.$$

Lemma 3. *Let $\omega^\alpha e_\alpha$ be the restriction of the connection form of Γ to $\hat{Q}$ and let $b^S, a^J_p, a^S_p : \hat{Q} \to \mathbf{R}$ be the coordinate functions of $k(\vartheta^1)$. Then*

$$\text{(9)} \qquad \omega^J = a^J_p \omega^p\,,$$

$$\text{(14)} \qquad db^S + \xi^S_\alpha(0, b^T)\,\omega^\alpha = a^S_p \omega^p\,.$$

Proof. According to [10], we have (on $H^1(B) \circledS P$)

$$d\hat{a}^i + \xi^i_\alpha(\hat{a}^j)\,\hat{\omega}^\alpha = \hat{a}^i_p \varphi^p\,, \quad d\hat{b}^S + \xi^S_\alpha(\hat{a}^j, \hat{b}^T)\,\hat{\omega}^\alpha = \hat{a}^S_p \varphi^p\,,$$

where the notation is analogous to that from the proof of Proposition 1. Restricting all quantities of (15) to $\hat{Q}$ and using (2), (5) and (13), we obtain (14), QED.

We shall say that the action of G on A is special with respect to x^i, x^S, if

$$\text{(16)} \qquad \xi^S_j(0, x^T) = 0\,.$$

Set $\xi^S_\lambda(0, x^T) = \eta^S_\lambda(x^T)$ and denote by π^λ the restriction of $\tilde{\omega}^\lambda$ to H so that the equations of the fundamental distribution on $H \times A_0$ are

$$\text{(17)} \qquad dx^S + \eta^S_\lambda(x^T)\,\pi^\lambda = 0\,.$$

Then (14) is equal to

$$\text{(18)} \qquad db^S + \eta^S_\lambda(b^T)\,\omega^\lambda = a^S_p \omega^p\,.$$

Thus, in this special case the equations of the fundamental distribution on $H \times A_0$ only are used in Lemma 3.

4. The development of order r of $\mathscr{S}$ (or the absolute differential of order r of σ with respect to C) is defined as the cross section

$$\sigma^r : B \to \bigcup_{x \in B} \bar{J}^r_x(B, E_x)\,, \quad \sigma^r(x) = [C^{(r-1)}]^{-1}(x)(\sigma)\,,$$

see [5]. We introduce also the weak development of order r of $\mathscr{S}$ as the cross section

$$\lambda^r : B \to \bigcup_{x \in B} \bar{K}^r_{m,\sigma(x)}(E_x)\,, \quad \lambda^r(x) = k(\sigma^r(x))\,,$$

where $k(\sigma^r(x))$ is the contact element determined by $\sigma^r(x)$, [3]. The fibered manifold $\bigcup_{x\in B} \bar{K}^r_{m,\sigma(x)}(E_x)$ has a natural structure of an associated fibre bundle $(B, \bar{K}^r_{m,c}(F), H, Q)$. By means of the local coordinate system $\varkappa$, $\bar{K}^r_{m,c}(F)$ is identified with $\bar{K}^r_{n,m} = \bar{K}^r_{m,0}(\mathbf{R}^n)$. Let ${}^0\bar{K}^r_{n,m} \subset \bar{K}^r_{n,m}$ be the subspace of all elements transversal with respect to the natural projection $\mathbf{R}^n \to \mathbf{R}^m$. Then ${}^0\bar{K}^r_{n,m}$ is identified with $\bar{L}^r_{n-m,m}$ which introduces some coordinates $y^J_p, \ldots, y^J_{p_1\ldots p_r}$ on ${}^0\bar{K}^r_{n,m}$, [1]. The restriction of the indirect form of λ^r to $\hat{Q}$ is a mapping of $\hat{Q}$ into ${}^0\bar{K}^r_{n,m}$.

Proposition 2. *Let $\omega^\alpha e_\alpha$ be the restriction of the connection form of Γ to $\hat{Q}$, let*

$$\begin{aligned} &(19) \qquad & \mathrm{d}x^i + \xi^i_\alpha(x^j)\,\tilde{\omega}^\alpha &= 0\,,\\ && \mathrm{d}y^J_p + \Psi^J_{p\alpha}(x^i, y^k_q)\,\tilde{\omega}^\alpha &= 0\,,\\ && &\vdots\\ && \mathrm{d}y^J_{p_1\ldots p_r} + \Psi^J_{p_1\ldots p_r\alpha}(x^i, y^k_q,\ldots,y^k_{q_1\ldots q_r})\,\tilde{\omega}^\alpha &= 0 \end{aligned}$$

be the equations of the fundamental distribution on $G \times \bar{K}^r_m(F)$ and let $a^J_p, \ldots$ $\ldots, a^J_{p_1\ldots p_r} : \hat{Q} \to \mathbf{R}$ be the coordinate functions of λ^r. Then the coordinate functions $a^J_p, \ldots, a^J_{p_1\ldots p_r}, a^J_{p_1\ldots p_{r+1}} : \hat{Q} \to \mathbf{R}$ of λ^{r+1} satisfy

$$\begin{aligned} &(20) \qquad & \omega^J &= a^J_p\omega^p\,,\\ && \mathrm{d}a^J_p + \Psi^J_{p\alpha}(0, a^K_q)\,\omega^\alpha &= a^J_{pq}\omega^q\,,\\ && \mathrm{d}a^J_{p_1\ldots p_r} + \Psi^J_{p_1\ldots p_r\alpha}(0, a^K_q, \ldots, a^K_{q_1\ldots q_r})\,\omega^\alpha &= a^J_{p_1\ldots p_r q}\omega^q\,. \end{aligned}$$

Proof. Replacing A by $\bar{K}^r_m(F)$ and ϑ by λ^r, we obtain the situation of item 2. According to [11], Lemma 1, $(\lambda^r)^1(x)$ is identified with an element of $\bar{K}^{r+1}_m(E_x)$ and one deduces easily that this element is equal to $\lambda^{r+1}(x)$, cf. [5]. Hence we have an identification $k((\lambda^r)^1) = \lambda^{r+1}$. On the other hand, by means of the local coordinate system $\varkappa$, $\hat{M}$ is identified with $T^1_m(\bar{L}^r_{n-m,m})$, which introduces some coordinates $y^J_p, \ldots$ $\ldots, y^J_{p_1\ldots p_r}, z^J_p, \ldots, z^J_{p_1\ldots p_{r+1}}$ on $\hat{M}$. By Lemma 3, the coordinate functions $b^J_p, \ldots$

$\ldots, b^J_{p_1\ldots p_r}, a^J_p, \ldots, a^J_{p_1\ldots p_{r+1}}$ of $k((\lambda^r)^1)$ satisfy

(21)
$$\omega^J = a^J_p\omega^p ,$$
$$\mathrm{d}b^J_p + \Psi^J_{p\alpha}(0, b^K_q)\,\omega^\alpha = a^J_{pq}\omega^q ,$$
$$\vdots$$
$$\mathrm{d}b^J_{p_1\ldots p_r} + \Psi^J_{p_1\ldots p_r\alpha}(0, b^K_q, \ldots, b^K_{q_1\ldots q_r})\,\omega^\alpha = a^J_{p_1\ldots p_r q}\omega^q .$$

But the values of λ^{r+1} are semi-holonomic, which is characterized by $b^J_p = a^J_p, \ldots$ $\ldots, b^J_{p_1\ldots p_r} = a^J_{p_1\ldots p_r}$. Hence our assertion is proved.

Proposition 2 gives an algorithm for finding the coordinate functions of λ^{r+1}. In general, this algorithm is rather complicated, but it is essentially simplified for those homogeneous spaces for which one can find such a basis $\tilde{\omega}^\alpha$ of $\mathfrak{g}^*$ that

(22) $$\mathrm{d}\tilde{\omega}^i = c^i_{j\lambda}\tilde{\omega}^j \wedge \tilde{\omega}^\lambda , \quad \mathrm{d}\tilde{\omega}^\lambda = c^\lambda_{j\mu}\tilde{\omega}^j \wedge \tilde{\omega}^\mu + \tfrac{1}{2}c^\lambda_{\mu\nu}\tilde{\omega}^\mu \wedge \tilde{\omega}^\nu$$

holds provided $\tilde{\omega}^i = 0$ are the differential equations of H. (As remarked in [9], a great number of homogeneous spaces investigated in the classical differential geometry are of type (22).) Obviously, $\tilde{\omega}^\lambda = 0$ is an Abelian subgroup $K \subset G$. The canonical coordinates on K determined by the basis e_i of $\mathfrak{k}$ induce some local coordinates on F; these coordinates will be also called the canonical coordinates on F determined by e_i.

5. Let F be a homogeneous space of type (22). The canonical coordinates on F are prolonged to some local coordinates on $\bar{K}^r_m(F)$, [1].

Lemma 4. *The action of G on $\bar{K}^r_m(F)$ is special (in the sense of (16)) with respect to the above-mentioned coordinates.*

Proof. The local coordinates on $\bar{K}^r_m(\mathbf{R}^n)$ are introduced as follows. We have some coordinates $y^J_p, \ldots, y^J_{p_1\ldots p_r}$ on ${}^0\bar{K}^r_{n,m} \subset \bar{K}^r_{m,0}(\mathbf{R}^n)$. Let $\beta : \bar{K}^r_m(\mathbf{R}^n) \to \mathbf{R}^n$ be the jet projection. If $\xi \in \bar{K}^r_m(\mathbf{R}^n)$, $\beta\xi = x$, then $t_x^{-1}\xi \in \bar{K}^r_{n,m}$, where $t_x : \mathbf{R}^n \to \mathbf{R}^n$ means the translation $y \mapsto y + x$. Then the coordinates x^i of $\beta\xi$ and the coordinates of $t_x^{-1}\xi$ are, by definition, the coordinates of ξ. But F is locally identified with $\mathbf{R}^n$ by means of the canonical coordinates and the transformations of K on F are represented by the translations of $\mathbf{R}^n$, which implies easily Lemma 4.

According to (18), the equations of the fundamental distribution on $H \times \bar{K}^r_{n,m}$ only are to be used in (20). In [9], formula (45) and Appendix 2, we have described the homomorphism of H into L^r_n determined by the canonical coordinates on F. Hence it will be sufficient to deduce an algorithm for finding the equations of the fundamental distribution on $L^r_n \times \bar{K}^r_{n,m}$.

6. The equations of the fundamental distribution on $L^1_n \times K^1_{n,m}$ can be found in [9], formula (10). Assume by induction that we have deduced the equations of the fundamental distribution on $L_n^{r-1} \times \bar{K}_{n,m}^{r-1}$ in the form

$$(23) \qquad d\bar{y}_p^J + \Psi_p^J(\bar{y}_q^K, \bar{\omega}_j^i) = 0 ,$$
$$d\bar{y}_{p_1 \dots p_{r-1}}^J + \Psi_{p_1 \dots p_{r-1}}^J(\bar{y}_q^K, \dots, \bar{y}_{q_1 \dots q_{r-1}}^K, \bar{\omega}_j^i, \dots, \bar{\omega}_{j_1 \dots j_{r-1}}^i) = 0 ,$$

where $\bar{y}_p^J, \dots, \bar{y}_{p_1 \dots p_{r-1}}^J$ are the above-mentioned coordinates on ${}^0\bar{K}_{n,m}^{r-1}$ and $\bar{\omega}_j^i, \dots$ $\dots, \bar{\omega}_{j_1 \dots j_{r-1}}^i$ is the natural basis of $\mathfrak{l}_n^{r-1*}$.

Proposition 3. *Using* (23), *write formally the relations*

$$(24) \qquad dy_p^J + \Psi_p^J(y_q^K, \varphi_j^i) = y_{pq}^J \varphi^q ,$$
$$\vdots$$
$$dy_{p_1 \dots p_{r-1}}^J + \Psi_{p_1 \dots p_{r-1}}^J(y_q^K, \dots, y_{q_1 \dots q_{r-1}}^K, \varphi_j^i, \dots, \varphi_{j_1 \dots j_{r-1}}^i) = y_{p_1 \dots p_r}^J \varphi^{p_r} .$$

Applying the exterior differentiation to the last row of (24) *and replacing*: a) $d\varphi^p$, $d\varphi_j^i, \dots, d\varphi_{j_1 \dots j_{r-1}}^i$ *according to the structure equations of the canonical form* φ_r *of* $H^r(B)$, [7], b) $dy_p^J, \dots, dy_{p_1 \dots p_{r-1}}^J$ *according to* (24), c) φ^J *according to the formal relation*

$$(25) \qquad \varphi^J = y_p^J \varphi^p ,$$

we obtain an expression of the form

$$[dy_{p_1 \dots p_r}^J + \Psi_{p_1 \dots p_r}^J(y_q^K, \dots, y_{q_1 \dots q_r}^K, \varphi_j^i, \dots, \varphi_{j_1 \dots j_r}^i)] \wedge \varphi^{p_r} = 0 .$$

Then the equations of the fundamental distribution on $L_n^r \times \bar{K}_{n,m}^r$ *are*

$$(26) \qquad dy_p^J + \Psi_p^J(y_q^K, \omega_j^i) = 0 ,$$
$$\vdots$$
$$dy_{p_1 \dots p_r}^J + \Psi_{p_1 \dots p_r}^J(y_q^K, \dots, y_{q_1 \dots q_r}^K, \omega_j^i, \dots, \omega_{j_1 \dots j_r}^i) = 0 ,$$

where $\omega_j^i, \dots, \omega_{j_1 \dots j_r}^i$ *is the natural basis of* $\mathfrak{l}_n^{r*}$.

Proof. Consider an n-dimensional manifold M and an m-dimensional submanifold $V \subset M$. Put $Q^r(V) = H^r(M) \mid V$ and define $\hat{Q}^r(V)$ in the same way as in [9]. Then $\bar{K}_m^{r-1}(M) \mid V = \bar{K}^{r-1}(V)$ is an associated bundle of the symbol $(V, \bar{K}_{n,m}^{r-1}, L_n^{r-1}, Q^{r-1}(V))$. Let $\tilde{K}_m^{r,r-1}(M)$ be the space of all non-holonomic contact m^r-elements on M tangent to $\bar{K}^{r-1}(M)$. Let $\tilde{K}^{r,r-1}(V) = \tilde{K}^{r,r-1}(M) \mid V$, $\tilde{K}_{n,m}^{r,r-1} = (\tilde{K}_m^{r,r-1}(\mathbf{R}^n))_0$ and let ${}^0\tilde{K}_{n,m}^{r,r-1}$ be the subspace of all elements of $\tilde{K}_{n,m}^{r,r-1}$ transversal with respect to the canonical projection $\mathbf{R}^n \to \mathbf{R}^m$. On ${}^0\tilde{K}_{n,m}^{r,r-1}$, there are natural coordinates $z_p^J, \dots, z_{p_1 \dots p_{r-1}}^J, t_p^J, \dots, t_{p_1 \dots p_r}^J$. If σ is a local cross section of $\bar{K}^{r-1}(V)$, then $j_x^1\sigma$, $x \in V$, is identified with an element of $\tilde{K}^{r,r-1}(V)$, see [11], Lemma 1. Consequently, $J^1(\bar{K}^{r-1}(V))$ can be considered as an associated fibre bundle $(B, \tilde{K}_{n,m}^{r,r-1}, L_n^r, Q^r(V))$. We first deduce

Lemma 5. *Let* $\bar{b}_p^J, \dots, \bar{b}_{p_1 \dots p_{r-1}}^J$ *be the coordinate functions of a geometric object field* $\sigma : V \to \bar{K}^{r-1}(V)$. *Then the coordinate functions* $b_p^J, \dots, b_{p_1 \dots p_{r-1}}^J, a_p^J, \dots, a_{p_1 \dots p_r}^J$

of the section $j^1\sigma : V \to \tilde{K}^{r,r-1}(V)$ *satisfy* $b^J_p = (j^{r-1}_r)^* \bar{b}^J_p, \ldots, b^J_{p_1 \ldots p_{r-1}} = (j^{r-1}_r)^* \cdot \bar{b}^J_{p_1 \ldots p_{r-1}}$ *and*

(27)
$$\tilde{\varphi}^J = a^J_p \tilde{\varphi}^p ,$$
$$\mathrm{d}b^J_p + \Psi^J_p(b^K_q, \tilde{\varphi}^i_j) = a^J_{pq} \tilde{\varphi}^q ,$$
$$\vdots$$
$$\mathrm{d}b^J_{p_1 \ldots p_{r-1}} + \Psi^J_{p_1 \ldots p_{r-1}}(b^K_q, \ldots, b^K_{q_1 \ldots q_{r-1}}, \tilde{\varphi}^i_j, \ldots, \tilde{\varphi}^i_{j_1 \ldots j_{r-1}}) = a^J_{p_1 \ldots p_{r-1}} \tilde{\varphi}^q ,$$

where $\tilde{\varphi}^i, \ldots, \tilde{\varphi}^i_{j_1 \ldots j_{r-1}}$ *are the components of the restriction of the canonical form of* $H^r(B)$ *to* $\hat{Q}^r(V)$.

Proof of Lemma 5 is quite similar to that of Proposition 2 of [9] and we shall use freely the notation introduced there. For every $X \in T_u(\hat{Q}^r(V))$, we have a decomposition $\bar{X} = \bar{X}_1 + \bar{X}_2$, where $\bar{X}_2 \in T_{\bar{u}}(Q^{r-1}_x(V))$. By Lemma 1 of [8], we get

(28)
$$\mathrm{d}\bar{b}^J_p(\bar{X}_2) = - \Psi^J_p(\bar{b}^K_q(\bar{u}), \xi^i_j) ,$$
$$\vdots$$
$$\mathrm{d}\bar{b}^J_{p_1 \ldots p_{r-1}}(\bar{X}_2) = - \Psi^J_{p_1 \ldots p_{r-1}}(\bar{b}^K_q(\bar{u}), \ldots, \bar{b}^K_{q_1 \ldots q_{r-1}}(\bar{u}), \xi^i_j, \ldots, \xi^i_{j_1 \ldots j_{r-1}}) .$$

By Proposition 1 of [9], we obtain

(29)
$$\tilde{\varphi}^J = a^J_p \tilde{\varphi}^p .$$

Finally, in the same way as in [9], we deduce

(30)
$$\mathrm{d}\bar{b}^J_p(\bar{X}_1) = a^J_{pq}(u)\, \xi^q$$
$$\vdots$$
$$\mathrm{d}\bar{b}^J_{p_1 \ldots p_{r-1}}(\bar{X}_1) = a^J_{p_1 \ldots p_{r-1} q}(u)\, \xi^q .$$

However, (28), (29) and (30) are equivalent to (27), QED.

We are now in position to prove Proposition 3. By means of the exterior differentiation of (27) and by the standard procedure based on Lemma 2 of [8], we obtain the equations of the fundamental distribution on $L^r_n \times \tilde{K}^{r,r-1}_{n,m}$. But the subspace $\bar{K}^r_{n,m} \subset \tilde{K}^{r,r-1}_{n,m}$ is characterized by $z^J_p = t^J_p, \ldots, z^J_{p_1 \ldots p_{r-1}} = t^J_{p_1 \ldots p_{r-1}}$. Summarizing all these results in a direct algorithm, we deduce Proposition 3.

7. Using the above-mentioned homomorphism of H into L^r_n, we deduce the equations of the fundamental distribution on $H \times \bar{K}^r_{n,m}$. By Proposition 2, Lemma 4 and (18), we obtain directly the following

Proposition 4. (Cartan-Laptěv algorithm for manifolds with connection of type (22).) *Let* $\mathscr{S}(P(B, G), F, C, \sigma)$ *be a manifold with connection and let* F *be of type* (22). *Let* $\omega = \omega^\alpha e_\alpha$ *the restriction of the connection form of* Γ *to* $\hat{Q}$. *Then the coordinate functions of the successive weak developments of* $\mathscr{S}$ *can be treated by a recurrent algorithm which starts from the relation*

$$\omega^J = a^J_p \omega^p .$$

Assume that after $r - 1$ steps of this algorithm we have deduced

(31)
$$\omega^J = a_p^J \omega^p ,$$
$$\mathrm{d}a_p^J + \Psi_{p\lambda}^J(a_q^K)\,\omega^\lambda = a_{pq}^J \omega^q ,$$
$$\vdots$$
$$\mathrm{d}a_{p_1\ldots p_{r-1}}^J + \Psi_{p_1\ldots p_{r-1}\lambda}^J(a_q^K, \ldots, a_{q_1\ldots q_{r-1}}^K)\,\omega^\lambda = a_{p_1\ldots p_r}^J \omega^{p_r} ,$$

where $a_p^J, \ldots, a_{p_1\ldots p_r}^J$ are the coordinate functions of the r-th weak development of $\mathscr{S}$ and

(32)
$$\mathrm{d}y_p^J + \Psi_{p\lambda}^J(y_q^K)\,\pi^\lambda = 0 ,$$
$$\vdots$$
$$\mathrm{d}y_{p_1\ldots p_{r-1}}^J + \Psi_{p_1\ldots p_{r-1}\lambda}^J(y_q^K, \ldots, y_{q_1\ldots q_{r-1}}^K)\,\pi^\lambda = 0$$

are the equations of the fundamental distribution on $H \times \bar{K}_{n,m}^{r-1}$. Applying the exterior differentiation to the last row of (31) *and replacing*: a) $\mathrm{d}a_p^J, \ldots, \mathrm{d}a_{p_1\ldots p_{r-1}}^J$ *according to* (31), b) $\mathrm{d}\omega^\alpha$ *according to* (22), c) ω^J *by* $a_p^J\omega^p$, *we obtain a relation of the form*

$$[\mathrm{d}a_{p_1\ldots p_r}^J + \Psi_{p_1\ldots p_r\lambda}^J(a_q^K, \ldots, a_{q_1\ldots q_r}^K)\,\omega^\lambda] \wedge \omega^{p_r} = 0 .$$

Then the further coordinate functions $a_{p_1\ldots p_{r+1}}^J$ of the $(r + 1)$-st weak development of $\mathscr{S}$ satisfy

$$\mathrm{d}a_{p_1\ldots p_r}^J + \Psi_{p_1\ldots p_r\lambda}^J(a_q^K, \ldots, a_{q_1\ldots q_r}^K)\,\omega^\lambda = a_{p_1\ldots p_r q}^J \omega^q$$

and the equations of the fundamental distribution on $H \times \bar{K}_{n,m}^r$ are (32) *and*

$$\mathrm{d}y_{p_1\ldots p_r}^J + \Psi_{p_1\ldots p_r\lambda}^J(y_q^K, \ldots, y_{q_1\ldots q_r}^K)\,\pi^\lambda = 0 .$$

Remark 1. It should be underlined that we have to use the structure equations $\mathrm{d}\omega = -\frac{1}{2}[\omega, \omega]$ of G and not the structure equations $\mathrm{d}\omega = -\frac{1}{2}[\omega, \omega] + D\omega$ of the connection form of Γ.

Remark 2. We shall evaluate the condition for a_{pq}^J to be symmetric in both subscripts. Using the equations of the fundamental distribution on $H \times K_{n,m}^1$, [9], we obtain

(33)
$$\omega^J = a_p^J \omega^p ,$$
$$\mathrm{d}a_p^J - (a_q^J c_{p\lambda}^q + a_q^J a_p^K c_{K\lambda}^q - a_p^K c_{K\lambda}^J - c_{p\lambda}^J)\,\omega^\lambda = a_{pq}^J \omega^q .$$

On the other hand, it is

(34)
$$\mathrm{d}\omega^i = c_{j\lambda}^i \omega^j \wedge \omega^\lambda + R_{pq}^i \omega^p \wedge \omega^q .$$

Applying (33) and (34) to

$$\mathrm{d}\omega^J = \mathrm{d}a_p^J \wedge \omega^p + a_p^J\,\mathrm{d}\omega^p ,$$

we deduce that a^J_{pq} are symmetric in both subscripts if and only if

$$R^J_{pq}\omega^p \wedge \omega^q = a^J_r R^r_{pq}\omega^p \wedge \omega^q . \tag{35}$$

Explained geometrically, (35) asserts that the second weak development of $\mathcal{S}$ is holonomic if and only if the reduced torsion form of $\mathcal{S}$ vanishes. We have established this result in a quite different way in [5], Theorem 2.

Remark 3. Modifying in an unessential way our Definition 3 of [3] in the sense of [4] and [9], we can define a semi-holonomic geometric m^r-object on F as an equivariant mapping of H-space $\bar{K}^r_{m,c}(F)$ into another H-space A. If $\mathcal{S}$ is a manifold with connection of type F satisfying $\dim B = m$, then μ is extended to a mapping $\bar{\mu}$: $: (B, \bar{K}^r_{m,c}(F), H, Q) \to (B, A, H, Q)$ and the section $\mu\lambda^r : B \to (B, A, H, Q)$ can be called the value of μ on $\mathcal{S}$. Since the algorithm of Proposition 4 gives the equations of the fundamental distribution on $H \times \bar{K}^r_{n,m}$, the method of G. F. Laptěv, [12], p. 301 (as explained in Appendix 1, [9]) can be used for (at least local) analytic constructions of the above-mentioned equivariant mappings.

Remark 4. If the connection C is integrable, then $\mathcal{S}$ is locally isomorphic to a submanifold of the homogeneous space F. In this case we have deduced again our results of [9] for submanifolds of homogeneous spaces of type (22).

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