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CERTAIN FUNDAMENTAL CONGRUENCES ON THE TENSOR PRODUCT OF COMMUTATIVE INVERSE SEMIGROUPS

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1. Two elements of a semigroup X are called to be *$\mathscr{L}$-equivalent* if they generate the same principal left ideal of X. $\mathscr{R}$-equivalence is defined dually. The join of the equivalences $\mathscr{L}$ and $\mathscr{R}$ is denoted by $\mathscr{D}$ and their intersection by $\mathscr{H}$. By an inverse semigroup we mean a semigroup X in which to each element a there corresponds a unique element a^{-1} (the inverse of a) such that

$$aa^{-1}a = a \quad \text{and} \quad a^{-1}aa^{-1} = a^{-1}\,.$$

In this note we shall prove that the tensor product of $\mathscr{L}(\mathscr{R}, \mathscr{D}, \mathscr{H})$-equivalences on commutative inverse semigroups X and Y is also $\mathscr{L}(\mathscr{R}, \mathscr{D}, \mathscr{H})$-equivalence on the tensor product $X \otimes Y$. And we consider the analogous properties for the minimum semilattice congruences and the maximum idempotent-separating congruences on commutative inverse semigroups. MUNN [10] has given that a semigroup X is said to be *fundamental* if the only congruence on X contained in $\mathscr{H}$ is the identity congruence. We also prove that the tensor product of commutative inverse fundamental semigroups is fundamental. For other properties of the tensor product of congruences, see the authors [6], [7] and [8]. The notation and terminology of CLIFFORD and PRESTON [1] will be used throughout.

2. By the *tensor product* $X \otimes Y$ of commutative semigroups X and Y we mean the quotient semigroup $F(X \times Y)/\delta$ where $F(X \times Y)$ is the free commutative semigroup on the set $X \times Y$ and δ is the smallest congruence relation for which:

$$(x_1x_2, y)\,\delta\,(x_1, y)\,(x_2, y)$$

and

$$(x, y_1y_2)\,\delta\,(x, y_1)\,(x, y_2)$$

hold for all $x_1, x_2, x \in X$ and $y, y_1, y_2 \in Y$.

P. A. GRILLET [3] has given the definition of the tensor product of congruences on semigroups: If $\gamma(X)$ and $\gamma(Y)$ are respectively congruences on semigroups X and Y, then the *tensor product* $\gamma(X) \otimes \gamma(Y)$ of $\gamma(X)$ and $\gamma(Y)$ is the smallest congruence relation on the tensor product $X \otimes Y$ containing all pairs

$$(x_1 \otimes y_1, x_2 \otimes y_2)$$

such that

$$(x_1, x_2) \in \gamma(X) \quad \text{and} \quad (y_1, y_2) \in \gamma(Y).$$

3. A *semilattice* is a commutative semigroup of idempotents. By the *minimum semilattice congruence* on a semigroup X we mean the smallest congruence $\alpha(X)$ on X for which $X/\alpha(X)$ is a semilattice. TAMURA and KIMURA has given the minimum semilattice congruence on a commutative semigroup X as follows: For $a, b \in X$, $(a, b) \in \alpha(X)$ if and only if there exist elements $x, y \in X$ and positive integers m, n such that

$$ax = b^m \quad \text{and} \quad by = a^n,$$

([11] Theorem 5 or [1] Theorem 4.12).

A congruence on a semigroup X is called *idempotent-separating* if each congruence class of X contains at most one idempotent. It has been shown by HOWIE ([4] Theorem 2.4) that the maximum idempotent-separating congruence $\beta(X)$ on an inverse semigroup X has been given the following: For $a, b \in X$,

$$(a, b) \in \beta(X) \quad \text{if and only if} \quad a^{-1}ea = b^{-1}eb$$

for all idempotents $e \in X$. And this is the greatest congruence contained in $\mathscr{H}$ ([9]).

It is clear that, for a commutative semigroup X, the equivalences $\mathscr{L}$, $\mathscr{R}$, $\mathscr{D}$ and $\mathscr{H}$ are congruence relations on X. In this case we obtain that $\beta(X) = \mathscr{H}$. In the case when X is a commutative inverse semigroup, from these and by Theorem 1.6 of [5], we have the following lemma:

Lemma 1. *Let X be any commutative inverse semigroup, and $\alpha(X)$ and $\beta(X)$ be respectively the minimum semilattice congruence and the maximum idempotent-separating congruence on X. Then*

$$\alpha(X) = \beta(X) = \mathscr{L} = \mathscr{R} = \mathscr{D} = \mathscr{H}.$$

4. The following property is well-known:

Lemma 2. ([3] Corollary 3.5). *Let $\gamma(X)$ and $\gamma(Y)$ be congruences on semigroups X and Y, respectively. Then the tensor product $X/\gamma(X) \otimes Y/\gamma(Y)$ is isomorphic to $(X \otimes Y)/(\gamma(X) \otimes \gamma(Y))$.*

The following property is an immediate consequence of Proposition 4 of [2]. We shall give a proof for completeness according to a point of view of congruences.

Lemma 3. *Let $\alpha(X)$ and $\alpha(Y)$ be the minimum semilattice congruences on commutative semigroups X and Y, respectively. Then the tensor product $\alpha(X) \otimes \alpha(Y)$*

of $\alpha(X)$ and $\alpha(Y)$ is the minimum semilattice congruence on the tensor product $X \otimes Y$ of X and Y.

Proof. As is easily seen, the tensor product of semilattices is also a semilattice. Thus it follows from Lemma 2 that $\alpha(X) \otimes \alpha(Y)$ is a semilattice congruence on $X \otimes Y$. Let $\alpha(X \otimes Y)$ be the minimum semilattice congruence on $X \otimes Y$. Then it is clear that

$$\alpha(X \otimes Y) \subseteq \alpha(X) \otimes \alpha(Y).$$

To prove the converse inclusion, let x_1 and x_2 be any elements of X such that

$$(x_1, x_2) \in \alpha(X).$$

Then it follows from the definition of $\alpha(X)$ that there exist elements u and v and positive integers m, n such that

$$x_1 u = x_2^m \quad \text{and} \quad x_2 v = x_1^n.$$

Then for any element $y \in Y$, we have

$$(x_1 \otimes y)(u \otimes y) = (x_1 u) \otimes y = x_2^m \otimes y = (x_2 \otimes y)^m$$

and

$$(x_2 \otimes y)(v \otimes y) = (x_2 v) \otimes y = x_1^n \otimes y = (x_1 \otimes y)^n.$$

Since $u \otimes y$ and $v \otimes y$ are elements of $X \otimes Y$, it follows from the definition of $\alpha(X \otimes Y)$ that

$$(x_1 \otimes y, x_2 \otimes y) \in \alpha(X \otimes Y).$$

Similarly,

$$(y_1, y_2) \in \alpha(Y)$$

implies

$$(x \otimes y_1, x \otimes y_2) \in \alpha(X \otimes Y)$$

for any element $x \in X$. Therefore it follows that

$$(x_1, x_2) \in \alpha(X) \quad \text{and} \quad (y_1, y_2) \in \alpha(Y)$$

imply

$$(x_1 \otimes y_1, x_2 \otimes y_1) \in \alpha(X \otimes Y)$$

and

$$(x_2 \otimes y_1, x_2 \otimes y_2) \in \alpha(X \otimes Y),$$

and eventually

$$(x_1 \otimes y_1, x_2 \otimes y_2) \in \alpha(X \otimes Y).$$

Therefore we obtain that

$$\alpha(X) \otimes \alpha(Y) \subseteq \alpha(X \otimes Y),$$

which completes the proof of the lemma.

5. Now we give our main result.

Theorem 4. *Let α and β be respectively the minimum semilattice congruence and the maximum idempotent-separating congruence. Then the following congruences (1) $\sim$ (12) on the tensor product $X \otimes Y$ of commutative inverse semigroups X and Y are equal to each other*:

(1) $\alpha(X) \otimes \alpha(Y)$,
(2) $\alpha(X \otimes Y)$,
(3) $\beta(X) \otimes \beta(Y)$,
(4) $\beta(X \otimes Y)$,
(5) $\mathscr{L}(X) \otimes \mathscr{L}(Y)$,
(6) $\mathscr{L}(X \otimes Y)$,
(7) $\mathscr{R}(X) \otimes \mathscr{R}(Y)$,
(8) $\mathscr{R}(X \otimes Y)$,
(9) $\mathscr{D}(X) \otimes \mathscr{D}(Y)$,
(10) $\mathscr{D}(X \otimes Y)$,
(11) $\mathscr{H}(X) \otimes \mathscr{H}(Y)$,
(12) $\mathscr{H}(X \otimes Y)$,

where we denote by $\gamma(X)$ a congruence γ on a semigroup X.

Proof. It is well-known ([2] proposition 6) that the tensor product of commutative inverse semigroups X and Y is also a commutative inverse semigroup. Then it follows from Lemma 1 that

$$\alpha(X \otimes Y) = \beta(X \otimes Y) = \mathscr{L}(X \otimes Y) = \mathscr{R}(X \otimes Y) = \mathscr{D}(X \otimes Y) = \mathscr{H}(X \otimes Y).$$

We have also by Lemma 2 that

$$\alpha(X) \otimes \alpha(Y) = \beta(X) \otimes \beta(Y) = \mathscr{L}(X) \otimes \mathscr{L}(Y) = \mathscr{R}(X) \otimes \mathscr{R}(Y) = \\ = \mathscr{D}(X) \otimes \mathscr{D}(Y) = \mathscr{H}(X) \otimes \mathscr{H}(Y).$$

Since by Lemma 3

$$\alpha(X \otimes Y) = \alpha(X) \otimes \alpha(Y),$$

we obtain that these congruences (1)$\sim$ (12) on $X \otimes Y$ are equal to each other. This completes the proof of the theorem.

6. Munn [10] has given the following: A semigroup X is said to be *fundamental* if the only congruence on X contained in $\mathscr{H}$ is the identity congruence $i(X)$. Thus an inverse semigroup X is fundamental if and only if the maximum idempotent-separating congruence $\beta(X)$ is equal to the identity congruence $i(X)$, ([10] p. 160). Moreover he has given that if X is an inverse semigroup then $X/\beta(X)$ is fundamental, ([10] Theorem 2.4). From these we have the following properties:

Theorem 5. *Let X and Y be commutative inverse semigroups. If X and Y are fundamental, then the tensor product $X \otimes Y$ is fundamental.*

Proof. By the definition of the tensor product of congruences, the tensor product $i(X) \otimes i(Y)$ of the identity congruences $i(X)$ on X and $i(Y)$ on Y is also the identity congruence $i(X \otimes Y)$ on $X \otimes Y$. Then it follows from Theorem 4 that

$$\beta(X \otimes Y) = \beta(X) \otimes \beta(Y) = i(X) \otimes i(Y) = i(X \otimes Y).$$

Therefore $X \otimes Y$ is fundamental. This completes the proof.

Corollary 6. *Let $\beta(X)$ and $\beta(Y)$ be the maximum idempotent-separating congruences on commutative inverse semigroups X and Y, respectively. Then the tensor product $X/\beta(X) \otimes Y/\beta(Y)$ is fundamental.*

Proof. From Theorem 2.4 of [10], $X/\beta(X)$ and $Y/\beta(Y)$ are fundamental. Then it follows from Theorem 5 that $X/\beta(X) \otimes Y/\beta(Y)$ is fundamental.

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