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APPLICATION OF ROTHE'S METHOD TO ABSTRACT PARABOLIC EQUATIONS

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(Received August 25, 1973)

Introduction. The method of Rothe, introduced by Rothe in his paper [1], for the investigation of parabolic equations, was used by many authors, see for example O. A. Ladyženskaja [2], A. M. Iljin, A. S. Kalašnikov, O. A. Olejnik [3]. Recently, a paper of K. Rektorys [4] and papers of J. Kačur [5] led the author to apply this method on the initial value problem for abstract parabolic equations $du/dt + A(u(t)) = f(t)$ with A a nonlinear operator. The author followed the paper of F. E. Browder [6], where the theory of semigroups is used and generalised his result for nonvanishing $f(t)$; see also H. Brézis [7]. The method of Rothe consists in replacing the equation in fixed points $t_j = jh$, by $(u(t_j) - u(t_{j-1}))/h + A(u(t_j)) = f(t_j)$. The nonlinear operator A is supposed monotone in its domain $D(A) \subset H$, where H is a real Hilbert space, and such that Range $(A + I) = H$. If $u_0 \in D(A)$ and if f is a continuous function from the interval $\langle 0, T \rangle$ to H and of bounded variation, then a unique solution of the problem exists in the sense precised below and the piecewise linear functions or piecewise constant functions inset in points t_j tend uniformly in the interval $\langle 0, T \rangle$ for $h \to 0$ to the solution.

Assertions and main theorem. Let the operator A be monotone in $D(A)$:

(1) $$(Au - Av, u - v) \geqq 0 \ \forall \ u, v \in D(A)$$

and let us suppose that

(2) $$(A + I)(D(A)) = H .$$

Let u_0 be in $D(A)$ and f in $C(\langle 0, T \rangle, H)$ with bounded variation in the interval $\langle 0, T \rangle$. By a solution of the problem

(3) $$\frac{du}{dt} + A(u(t)) = f(t) , \quad 0 \leqq t \leqq T , \quad u(0) = u_0 ,$$

we mean a strongly continuous function $u : \langle 0, T \rangle \to H$, weakly differentiable to H for $t \in \langle 0, T \rangle$, such that $u(t) \in D(A)$ for $t \in \langle 0, T \rangle$, $A(u(t))$ is weakly continuous,

$u(t)$ satisfies the equation (3) for $t \in \langle 0, T \rangle$ and the initial condition in the obvious sens. (See [6].)

Put for $\lambda > 0 : A_\lambda = A + \lambda I$. We obtain as in [6]:

Assertion 1. *For $\lambda > 0$, Range $(A_\lambda) = H$ and $\|A_\lambda^{-1}x - A_\lambda^{-1}y\| \leq \lambda^{-1}\|x - y\|$.*

Proof. Let $u, v \in D(A)$. Then $(A_\lambda u - A_\lambda v, u - v) \geq \lambda\|u - v\|^2$, hence $\|u - v\| \leq$ $\leq \lambda^{-1}\|A_\lambda u - A_\lambda v\|$. If for some $\lambda_0 > 0$ the Range $(A_{\lambda_0}) = H$, then $A_{\lambda_0}^{-1}$ is defined and is Lipschitzian with Lipschitz constant λ_0^{-1}. To solve the equation $A_\lambda u = y$, put $u = A_{\lambda_0}^{-1}x$. We obtain the equation

$$(4) \qquad x + (\lambda - \lambda_0)A_{\lambda_0}^{-1}x = y,$$

whose solution gives a solution of the original equation. If $|\lambda - \lambda_0|\lambda_0^{-1} < 1$, then by the theorem on contractive mappings, the equation (4) has a unique solution, q.e.d.

Let $h = T/n$, $t_j = jh$, $j = 0, 1, \ldots, n$ and put $z_0 = u_0$, $Az_1 + (z_1 - z_0)/h =$ $= f(t_1)$, $A(z_j) + (z_j - z_{j-1})/h = f(t_j)$, $j = 1, 2, \ldots, n$. It follows from the Assertion 1 that z_j are uniquely determined. We have

$$\left(Az_1 - Az_0, z_1 - z_0\right) + \frac{\|z_1 - z_0\|^2}{h} = (f(t_1), z_1 - z_0) - (Az_0, z_1 - z_0),$$

hence

$$(5) \qquad h^{-1}\|z_1 - z_0\| \leq \|f(t_1)\| + \|Az_0\|$$

and similarly for $j \geq 2$: $(Az_j - Az_{j-1}, z_j - z_{j-1}) + \|z_j - z_{j-1}\|^2/h = (f(t_j),$ $z_j - z_{j-1}) - (f(t_{j-1}), z_j - z_{j-1}) + h^{-1}(z_{j-1} - z_{j-2}, z_j - z_{j-1})$, so

$$(6) \qquad h^{-1}\|z_j - z_{j-1}\| \leq \|f(t_j) - f(t_{j-1})\| + h^{-1}\|z_{j-1} - z_{j-2}\|.$$

We obtain from (5) and (6)

$$(7) \qquad h^{-1}\|z_j - z_{j-1}\| \leq \operatorname*{Var}_{\langle 0,T\rangle} f + \operatorname*{Max}_{\langle 0,T\rangle} \|f(t)\| + \|Az_0\|.$$

Put in $[0, T]$, $z^n(t) = z_{j-1} + h^{-1}(t - t_{j-1})(z_j - z_{j-1}) = h^{-1}(h - (t - t_{j-1}))$. $\cdot z_{j-1} + h^{-1}(t - t_{j-1})z_j$ for $t_{j-1} \leq t \leq t_j$. It follows from (5) and (7)

Assertion 2.

$$(8) \qquad \|z^n(\tau) - z^n(\mu)\| \leq |\tau - \mu|\left(\operatorname*{Var}_{\langle 0,T\rangle} f + \operatorname*{Max}_{\langle 0,T\rangle} \|f(t)\| + \|Az_0\|\right).$$

Let us define $x^n(t)$ by $x^n(0) = u_0$, $x^n(t) = z_j$ for $(j - 1)h < t \leq jh$. In the same manner we define $f^n(t)$. It follows from the construction above that

$$(9) \qquad \int_0^t A(x^n)(\tau))\,\mathrm{d}\tau + z^n(t) = \int_0^t f^n(\tau)\,\mathrm{d}\tau + u_0$$

and for the derivatives from the left:

(10)
$$A(x^n(t)) + \frac{d^- z^n}{dt} = f(t).$$

It follows from the definition of the functions $x^n(t)$ and $f^n(t)$ that for all $t \in \langle 0, T \rangle$, $f^n(t) \to f(t)$ (strong convergence), uniformly in $\langle 0, T \rangle$, and

(11)
$$x^n(t) - z^n(t) \to 0 \quad \text{uniformly in} \quad \langle 0, T \rangle.$$

Assertion 3. $x^n(t) \to u(t)$ *uniformly in* $\langle 0, T \rangle$.

Proof. We have

(12)
$$\frac{d^-}{dt} \| z^m(t) - z^n(t) \|^2 = 2 \left(\frac{d^- z^m}{dt}(t) - \frac{d^- z^n}{dt}(t) \right),$$

$$z^m(t) - z^n(t)) = 2 \left(f^m(t) - A(x^m(t)) - f^n(t) + A(x^n(t)), z^m(t) - z^n(t) \right).$$

It follows from (8) and (10) that

(13)
$$\| A(x^n(t)) \| \leqq 2 \operatorname*{Max}_{\langle 0,T \rangle} \| f(t) \| + \operatorname*{Var}_{\langle 0,T \rangle} f + \| Az_0 \|,$$

hence (11), (12) and (13) give

$$\frac{d^-}{dt} \| z^m(t) - z^n(t) \|^2 \leqq \varepsilon_{m,n} + 2 \left(f^m(t) - A(x^m(t)) - f^n(t) + \right.$$

$$+ \left. A(x^n(t)), x^m(t) - x^n(t) \right) \leqq \varepsilon_{m,n} + 2 \| f^m(t) - f^n(t) \| \ \| x^m(t) - x^n(t) \|,$$

where $\varepsilon_{m,n} \to 0$. Because $\| z^m(t) - z^n(t) \| \geqq 0$ and $\| z^m(0) - z^n(0) \| = 0$, it follows that with some $\bar{\varepsilon}_{m,n} \to 0 : \| z^m(t) - z^n(t) \| \leqq \bar{\varepsilon}_{m,n} t$, q.e.d.

We prove as in the paper [6] the following Assertions 4 and 5:

Assertion 4. *A is maximal monotone, i.e. if* $(Av - w, v - u) \geqq 0$ *for all v in* $D(A)$, *then* $u \in D(A)$ *and* $Au = w$.

Proof. We have $(Av - w + v - u, v - u) \geqq 0$. For fixed $z \in H$, $t > 0$, put $v_t = A_1^{-1}(u + w + tz)$. It follows that $Av_t + v_t - u - w = tz$, hence $t(z, v_t - u) \geqq \geqq 0$ so $(z, v_t - u) \geqq 0$. Letting $t \to 0+ : v_t \to A_1^{-1}(u + w)$, hence $(z, A_1^{-1}(u + w) - u) \geqq 0$ so $u = A_1^{-1}(u + w)$, $u \in D(A)$, and $Au = w$, q.e.d.

Assertion 5. *If* $u_j \to u$ *and* $Au_j \to w$ (weak convergence) *then* $u \in D(A)$ *and* $Au = w$.

Proof. For $v \in D(A) : (Av - Au_j, v - u_j) \geqq 0$. From this follows taking the limit as $j \to \infty$ that $(Av - w, v - u) \geqq 0$, hence by the Assertion 4, $u \in D(A)$ and $Au = w$, q.e.d.

Assertion 6.

(14) $$\|u(t) - u(\tau)\| \leqq |t - \tau| \left(\operatorname*{Var}_{\langle 0,T \rangle} f + \operatorname*{Max}_{\langle 0,T \rangle} \|f(t)\| + \|Au_0\| \right).$$

Proof. (14) follows from (8), (11) and the Assertion 3.

Assertion 7. $A(x^n(t)) \rightharpoonup A(u(t))$ *and* $A(u(t))$ *is weakly continuous.*

Proof. We use estimations (13), (14) and Assertion 3 and 5.

Theorem. *Let A be a nonlinear operator from the domain $D(A) \subset H$ to H. Let (1) and (2) be satisfied. Let $u_0 \in D(A)$, $f(t)$ be continuous on $\langle 0, T \rangle$ with values in H and of bounded variation on $\langle 0, T \rangle$. Then there exists a unique solution $u(t)$ of the initial problem (3) and $u(t)$ is a Lipschitz continuous function satisfying (14). For the functions x^n, z^n constructed by the Rothe's method (8), (9), (10), (13) are valid. Also*

(15) $$z^n(t) \to u(t) \quad \text{uniformly in} \quad \langle 0, T \rangle,$$

(16) $$A x^n(t) \rightharpoonup A(u(t)).$$

If u, v are two solutions corresponding to the f, u_0, g, v_0 respectively, then

(17) $$\operatorname*{Max}_{\langle 0,T \rangle} \|u(t) - v(t)\| \leqq 2 \int_0^T \|f(t) - g(t)\| \, \mathrm{d}t + \|u(0) - v(0)\|.$$

If $f(t) \equiv 0$, then $\|A(u(t))\|$ is non-increasing.

Proof. First we prove (17) from which follows also uniqueness. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(t) - v(t)\|^2 = 2(f(t) - g(t) - A(u(t)) + A(v(t)),$$

$$u(t) - v(t)) \leqq 2\|f(t) - g(t)\| \, \|u(t) - v(t)\|,$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(t) - v(t)\|^2 \leqq 2 \Big(\operatorname*{Max}_{\langle 0,T \rangle} \|u(\tau) - v(\tau)\| \Big) \|f(t) - g(t)\|,$$

hence

$$\|u(t) - v(t)\|^2 \leqq \|u(0) - v(0)\|^2 + 2 \operatorname*{Max}_{\langle 0,T \rangle} \|u(\tau) - v(\tau)\| \cdot \int_0^T \|f(\tau) - g(\tau)\| \, \mathrm{d}\tau,$$

from which (17) follows. Let us consider first sequences $x^n(t)$, $z^n(t)$ introduced above.

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It follows from (9) that for every $v \in H$

(18) $$\int_0^t (A(x^n(\tau)), v)\, \mathrm{d}\tau + (z^n(t), v) = \int_0^t (f^n(\tau), v)\, \mathrm{d}\tau + (u_0, v)\,.$$

Hence from the Assertions 3,7 and from (11), it follows for $n \to \infty$:

(19) $$\int_0^t (A(u(\tau)), v)\, \mathrm{d}\tau + (u(t), v) = \int_0^t (f(\tau), v)\, \mathrm{d}\tau + (u_0, v)\,,$$

hence it follows from (19) that $u(t)$ is weakly differentiable, because of the Assertions 7, and that the equation (3) is satisfied. (15) and (16) are evident.

From (13) and the Assertion 7 we obtain for $f(t) \equiv 0$

(20) $$\|A(u(t))\| \leqq \|Au_0\|\,,$$

hence from the uniqueness

(21) $$\|A(u(t))\| \leqq \|A(u(s))\|\,, \quad t \geqq s\,,$$

q.e.d.

Remark. If $f(t) \equiv 0$, then clearly for two solutions corresponding to u_0, v_0:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|u(t) - v(t)\|^2 = 2(u'(t) - v'(t), u(t) - v(t)) =$$

$$= 2(A(u(t)) - A(v(t)), u(t) - v(t)) \leqq 0\,,$$

hence the transition operator $U(t)$ defined by $U(t)\, u(0) = u(t)$ is nonexpansive.

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