Czechoslovak Mathematical Journal

Ivan Straškraba; Otto Vejvoda

Periodic solutions to a singular abstract differential equation

Czechoslovak Mathematical Journal, Vol. 24 (1974), No. 4, 528–540

Persistent URL: `http://dml.cz/dmlcz/101272`

Czechoslovak Mathematical Journal, 24 (99) 1974, Praha

PERIODIC SOLUTIONS TO A SINGULAR ABSTRACT DIFFERENTIAL EQUATION

IVAN STRAŠKRABA, OTTO VEJVODA, Praha

(Received April 4, 1973)

In this paper the convergence of the ω-periodic solutions $u_\varepsilon(t)$ and $u_\varepsilon(\mu)(t)$ respectively of the equations

$$(1)\quad \varepsilon\, u''(t) + u'(t) + A\, u(t) = f(t)\,, \quad \varepsilon \in R^+ = \langle 0, \infty)\,, \quad t \in R = (-\infty, \infty)\,,$$

$$(2)\quad \varepsilon\, u''(t) + u'(t) + A\, u(t) = \mu\, F(\varepsilon)\,(u)\,(t),\ \varepsilon, \mu \in R^+\,, \quad t \in R$$

respectively for $\varepsilon \to 0_+$ are studied. Here A is any linear selfadjoint strongly positive operator with domain $\mathscr{D}(A)$ and range $\mathscr{R}(A)$ in a Hilbert space H, μ is a small parameter, $f(t)$ is an ω-periodic function with values in H and F is an operator mapping from the space of ω-periodic functions with values in H into itself.

This problem was studied in the concrete form

$$\varepsilon u_{tt} - u_{xx} + 2au_t + cu = g(t, x) + \varepsilon f(t, x, u, u_x, u_t)\,,$$

$(a > 0,\ c > 0,\ t \in R^+,\ x \in R)$, by M. KOPÁČKOVÁ in [2] where the classical solutions were investigated. The present article is intended as a supplement to our preceding paper [1] the notation of which we use frequently.

The text is divided into two paragraphs. The first paragraph deals with the linear equation (1) and in the second one the weakly nonlinear equation (2) is studied. The results are formulated in Theorems 1.1, 2.1 and 2.2.

Let us recall the notation: The space H is equipped by the norm $\|\cdot\|$ induced by the scalar product $(.\,,.)$ and for a non negative integer k, $\nu \in R^+$ and

$$u(\cdot) \in C^k_\omega(R; \mathscr{D}(A^\nu)) \overset{\text{def.}}{=} \{u : R \to H;\ u \in C^k(\langle 0, \omega\rangle; \mathscr{D}(A^\nu)),\ u(t + \omega) = u(t),\ t \in R\}$$

we write

$$\|u\|_{(k,\nu)} = \sup\left\{\left\|\frac{\mathrm{d}^l}{\mathrm{d}t^l} A^\nu u(t)\right\|\,;\quad t \in R\,,\quad l = 0, 1, \ldots, k\right\}.$$

In all paragraphs c_j, $(j = 1, 2, \ldots)$ are some constants.

In this paragraph we will investigate the character of the convergence of $u_\varepsilon(t)$ to $u_0(t)$ in the graph norm of $A^{1/2}$ under the assumption $A^{1/2} f(t)$ is continuous on R. Firstly, we shall introduce some useful facts.

Let us consider the equations

$$u''(t) + 2\alpha\, u'(t) + A\, u(t) = f(t)\,, \quad (\alpha > 0)\,, \tag{1.1}$$

$$u'(t) + A\, u(t) = f(t)\,. \tag{1.2}$$

The function $u_2(t)$ and $u_1(t)$ respectively is called a solution in $\langle 0, \omega\rangle$ of (1.1) and (1.2) respectively if

$$u_2 \in U_2(\langle 0, \omega\rangle) \stackrel{\text{def.}}{=} C^2(\langle 0, \omega\rangle; H) \cap C^1(\langle 0, \omega\rangle; \mathscr{D}(A^{1/2})) \cap C^0(\langle 0, \omega\rangle; \mathscr{D}(A))$$

and

$$u_1 \in U_1(\langle 0, \omega\rangle) \stackrel{\text{def.}}{=} C^1(\langle 0, \omega\rangle; H) \cap C^0(\langle 0, \omega\rangle; \mathscr{D}(A))$$

respectively and u_2 and u_1 satisfy the equation (1.1) and (1.2) respectively in $\langle 0, \omega\rangle$.

Proposition 1.1. *Let A be selfadjoint with $\inf \sigma(A) = m > 0$, $\sigma(A)$ being the spectrum of A, $\alpha^2 > m$ *), and let $f \in C^0_\omega(R; \mathscr{D}(A^{1/2}))$. Then the function*

$$u_2(t) = -\int_{-\infty}^{0} \exp(\alpha\tau)\, \frac{\sin \tau(A - \alpha^2)^{1/2}}{(A - \alpha^2)^{1/2}}\, f(t + \tau)\, d\tau \tag{1.3}$$

and

$$u_1(t) = \int_{-\infty}^{0} \exp(\tau A) f(t + \tau)\, d\tau \tag{1.4}$$

respectively is a unique ω-periodic solution on R of the equation (1.1) and (1.2) respectively.

Proof. Denote

$$K(t) = \exp(\alpha t)\, \frac{\sin t(A - \alpha^2)^{1/2}}{(A - \alpha^2)^{1/2}}\,, \quad t \in R$$

$$J(t) = \exp(\alpha t) \cos t(A - \alpha^2)^{1/2}\,, \quad t \in R$$

$$T(t) = \exp(tA)\,, \quad t \in R^- = (-\infty, 0\rangle\,.$$

Using theorems on differentiation of Bochner's and spectral integrals (see e.g. [3]

*) $\alpha^2 > m$ is not necessary but the opposite case we need not investigate.

p. 191 and [1]) and integrating by parts we find from (1.3) and (1.4)

$$(1.5)\quad u_2'(t) = \int_{-\infty}^{0} (\alpha K(\tau) + J(\tau)) f(t+\tau)\,\mathrm{d}\tau$$

$$u_2''(t) = -\int_{-\infty}^{0} (\alpha^2 K(\tau) + 2\alpha J(\tau) - (A - \alpha^2) K(\tau)) f(t+\tau)\,\mathrm{d}\tau + f(t)$$

$$u_1'(t) = -\int_{-\infty}^{0} A^{1/2} T(\tau) A^{1/2} f(t+\tau)\,\mathrm{d}\tau + f(t),$$

where the existence of the integrals in (1.5) is ensured by $\sup_{t\in R} \|A^{1/2} f(t)\| < \infty$ and by the estimates

$$(1.6)\qquad \|K(t)\| \leqq (\alpha^2 - m)^{-1/2} \exp t(\alpha - (\alpha^2 - m)^{1/2}) + |t| \exp(\alpha t), \quad t \in R^-,$$

$$\|A^{1/2} K(t)\| \leqq \alpha(\alpha^2 - m)^{-1/2} \exp t(\alpha - (\alpha^2 - m)^{1/2}) + (1 + \alpha|t|) \exp(\alpha t),$$
$$t \in R^-,$$

$$\|J(t)\| \leqq \exp t(\alpha - (\alpha^2 - m)^{1/2}), \quad t \in R^-,$$

$$\|T(t)\| \leqq \exp(tm), \quad t \in R^-,$$

$$(1.6_1)\quad \|A^{1/2} T(t)\| \leqq (2et)^{-1/2}, \quad t \in \left\langle -\frac{1}{2m}, 0\right), \quad \|A^{1/2} T(t)\| \leqq m^{1/2} \exp(tm),$$
$$t \in \left(-\infty, -\frac{1}{2m}\right\rangle.$$

(The norms are taken in the space of continuous linear mappings of H into itself.) Let us prove e.g. (1.6). Let $x \in H$; then

$$\|A^{1/2} K(t)\, x\| = \left(\int_m^\infty \lambda \exp(2\alpha t) \frac{\sin^2 t(\lambda - \alpha^2)^{1/2}}{|\lambda - \alpha^2|} \mathrm{d}\|E(\lambda)\, x\|^2\right)^{1/2} \leqq$$

$$\leqq \left(\int_m^{\alpha^2} \lambda \exp(2\alpha t) \frac{\operatorname{sh}^2 t(\alpha^2 - \lambda)^{1/2}}{\alpha^2 - \lambda} \mathrm{d}\|E(\lambda)\, x\|^2\right)^{1/2} +$$

$$+ \left(\int_{\alpha^2}^\infty \exp(2\alpha t)(\lambda - \alpha^2 + \alpha^2) \frac{\sin^2 t(\lambda - \alpha^2)^{1/2}}{\lambda - \alpha^2} \mathrm{d}\|E(\lambda)\, x\|^2\right)^{1/2} \leqq$$

$$\leqq \alpha(\alpha^2 - m)^{-1/2} \exp t(\alpha - (\alpha^2 - m)^{1/2}) \|(E(\alpha^2) - E(m))\, x\| +$$

$$+ (1 + \alpha|t|) \exp(\alpha t) \|(E(\infty) - E(\alpha^2))\, x\| \leqq$$

$$\leqq [\alpha(\alpha^2 - m)^{-1/2} \exp t(\alpha - (\alpha^2 - m)^{1/2}) + (1 + \alpha|t|) \exp(\alpha t)] \|x\|, \quad t \in R^-,$$

where $E(\lambda)$ is the resolution of the identity for A. The continuity in t of the integrals in (1.5) follows from [3] p. 191. The continuity in t of $A\,u_2(t)$ and $A\,u_1(t)$ may be established analogously. The uniqueness of the found solution may be obtained from the relations

$$0 = \int_0^\omega (v_2''(\tau) + 2\alpha\, v_2'(\tau) + A\, v_2(\tau),\ v_2'(\tau))\, d\tau =$$

$$= \int_0^\omega \left[\frac{1}{2}\frac{d}{dt}\|v_2'(\tau)\|^2 + 2\alpha\|v_2'(\tau)\|^2 + \frac{1}{2}\frac{d}{dt}\|A^{1/2}\, v_2(\tau)\|^2\right] d\tau\,,$$

$$0 = \int_0^\omega (v_1'(\tau) + A\, v_1(\tau),\ v_1(\tau))\, d\tau =$$

$$= \int_0^\omega \left[\frac{1}{2}\frac{d}{dt}\|v_1(\tau)\|^2 + (A\, v_1(\tau),\ v_1(\tau))\right] d\tau$$

holding for any solution $v_2(t)$ and $v_1(t)$ respectively of the equation (1.1) and (1.2) (with $f(t) \equiv 0$) respectively.

Now let us investigate the equation

$$\varepsilon\, u''(t) + u'(t) + A\, u(t) = f(t)\,, \quad t \in R\,, \tag{1.7}$$

where A is as above $f \in C_\omega^0(R; \mathscr{D}(A^{1/2}))$ and $\varepsilon \geqq 0$ is a parameter. According to the preceding proposition for any $\varepsilon \geqq 0$ there exists a unique ω-periodic solution $u_\varepsilon(t)$ of (1.7) and it is given by the formula

$$u_\varepsilon(t) = -2\int_{-\infty}^0 K_\varepsilon(\tau)\, f(t+\tau)\, d\tau\,, \quad \varepsilon > 0\,, \tag{1.8}$$

$$u_0(t) = \int_{-\infty}^0 T(\tau)\, f(t+\tau)\, d\tau\,,$$

denoting

$$K_\varepsilon(t) = \exp\left(\frac{t}{2\varepsilon}\right)\frac{\sin \frac{t}{2\varepsilon}(4\varepsilon A - I)^{1/2}}{(4\varepsilon A - I)^{1/2}}\,, \quad t \in R^-\,.$$

Let us formulate the main result of this paragraph:

Theorem 1.1. *Let the assumptions of Proposition* 1.1 *be fulfilled. Then for any* $\nu \in \langle 0, 1)$ *and any* $\varepsilon_0 > 0$ *there exists a constant* $c = c(m, \varepsilon_0)$, $(\lim_{\mu \to 0_+} c(\mu, \varepsilon_0) = \infty)$

such that the inequality

$$(1.9)\qquad \|u_\varepsilon - u_0\|_{(0,1/2)} \leqq \frac{c(m, \varepsilon_0)}{1 - \nu}\, \varepsilon^\nu \|f\|_{(0,1/2)},$$

where u_ε is given by (1.8), holds for $\varepsilon \in \langle 0, \varepsilon_0 \rangle$.

Proof. We may assume $\varepsilon_0 = 3/16m$ without loss of generality. Choose fixed $\varepsilon \in (0, \varepsilon_0\rangle$ and $M \in \langle m, 1/4\varepsilon)$ and denote $P_1 = E(M) - E(m)$, $P_2 = E(1/4\varepsilon) - E(M)$, $P_3 = I - E(1/4\varepsilon)$, $A_i = P_i A$, $i = 1, 2, 3$. Clearly we have $\mathscr{D}(A_1) = \mathscr{D}(A_2) = H$, $\mathscr{D}(A_3) = \mathscr{D}(A)$ and

$$\|A^{1/2}x\| \leqq \|A_1^{1/2}x\| + \|A_2^{1/2}x\| + \|A_3^{1/2}x\|, \quad x \in \mathscr{D}(A^{1/2}).$$

Writing

$$v_\varepsilon(t) = u_\varepsilon(t) - u_0(t), \quad t \in R$$

we get

$$\|A_1^{1/2}\, v_\varepsilon(t)\| \leqq \int_{-\infty}^0 \|2P_1 K_\varepsilon(\tau) + P_1\, T(\tau)\|\, d\tau \sup_{\tau \in R} \|A_1^{1/2} f(\tau)\|, \quad t \in R.$$

Further, for $x \in H$ it is

$$P_1(2K_\varepsilon(\tau) + T(\tau))\, x = \int_m^M \sum_{i=1}^3 q_i(\varepsilon, \tau, \lambda)\, dE(\lambda)\, x,$$

where

$$q_1(\varepsilon, \tau, \lambda) = \exp(\tau\lambda) - \exp\frac{\tau}{2\varepsilon}(1 - (1 - 4\varepsilon\lambda)^{1/2}),$$

$$q_2(\varepsilon, \tau, \lambda) = \frac{-4\varepsilon\lambda(1 - 4\varepsilon\lambda)^{-1/2}}{1 + (1 - 4\varepsilon\lambda)^{1/2}} \exp\left(\frac{2\tau\lambda}{1 + (1 - 4\varepsilon\lambda)^{1/2}}\right)$$

and

$$q_3(\varepsilon, \tau, \lambda) = (1 - 4\varepsilon\lambda)^{-1/2} \exp\frac{\tau}{2\varepsilon}(1 + (1 - 4\varepsilon\lambda)^{1/2}), \quad \tau \in R^-, \quad \lambda \in \left\langle m, \frac{1}{4\varepsilon}\right).$$

Obviously

$$q_3(\varepsilon, \tau, \lambda) \leqq q_3(\varepsilon, \tau, M) \leqq (1 - 4\varepsilon M)^{-1/2} \exp\left(\frac{\tau(1 - 4\varepsilon M)^{1/2}}{2\varepsilon}\right),$$

$$\tau \in R^-, \quad \lambda \in \langle m, M \rangle$$

and hence setting

$$\varphi_i(\varepsilon, \tau) = \max_{\lambda \in \langle m, M \rangle} |q_i(\varepsilon, \tau, \lambda)|, \quad (i = 1, 2),$$

we find

$$(1.10) \qquad \int_{-\infty}^{0} \|P_1 K_\varepsilon(\tau) + P_1 T(\tau)\| \, d\tau \leqq \int_{-\infty}^{0} (\varphi_1(\varepsilon, \tau) + \varphi_2(\varepsilon, \tau)) \, d\tau + \\ + 2\varepsilon(1 - 4\varepsilon M)^{-1} .$$

Making use of the monotonicity of sh ξ/ξ we can derive easily the estimate

$$\|P_2 K_\varepsilon(\tau)\| \leqq (1 - 4\varepsilon M)^{-1/2} \exp \frac{\tau}{2\varepsilon} (1 - (1 - 4\varepsilon M)^{1/2}) .$$

Also the estimates

$$\|P_2 T(\tau)\| \leqq \exp (M\tau) , \quad \|P_3 K_\varepsilon(\tau)\| \leqq \frac{|\tau|}{2\varepsilon} \exp \left(\frac{\tau}{2\varepsilon}\right), \quad \|P_3 T(\tau)\| \leqq \exp \left(\frac{\tau}{4\varepsilon}\right),$$

holding for $\tau \in R^-$ may be obtained in an elementary way. Thus

$$(1.11) \qquad \int_{-\infty}^{0} (2\|P_2 K_\varepsilon(\tau)\| + \|P_2 T(\tau)\|) \, d\tau \leqq \frac{4\varepsilon(1 - 4\varepsilon M)^{-1/2}}{1 - (1 - 4\varepsilon M)^{1/2}} + \frac{1}{M} \leqq \\ \leqq 3M^{-1}(1 - 4\varepsilon M)^{-1/2} ,$$

$$(1.12) \qquad \int_{-\infty}^{0} (2\|P_3 K_\varepsilon(\tau)\| + \|P_3 T(\tau)\|) \, d\tau \leqq 6\varepsilon .$$

Since

$$\|A_i^{1/2} v_\varepsilon(t)\| \leqq \int_{-\infty}^{0} (2\|P_i K_\varepsilon(\tau)\| + \|P_i T(\tau)\|) \, d\tau \max_{\tau \in R} \|A_i^{1/2} f(\tau)\| , \quad i = 2, 3 ,$$

we find from (1.10), (1.11) and (1.12)

$$(1.13) \qquad \|A^{1/2} v_\varepsilon(t)\| \leqq \left[\int_{-\infty}^{0} (\varphi_1(\varepsilon, \tau) + \varphi_2(\varepsilon, \tau)) \, d\tau + \frac{2\varepsilon}{1 - 4\varepsilon M} + \right. \\ \left. + 3M^{-1}(1 - 4\varepsilon M)^{-1/2} + 6\varepsilon \right] \|f\|_{(0,1/2)} , \quad t \in R .$$

Let $M = M(\varepsilon) = 3/16\varepsilon$. The function $q_1(\varepsilon, \tau, \lambda)$ reaches its maximum at the point $\lambda = m$ for any $\tau \in R^-$ and $\varepsilon > 0$. Indeed, it is

$$\frac{\partial q_1}{\partial \lambda} (\varepsilon, \tau, \lambda) = \tau \left[\exp (\tau\lambda) - (1 - 4\varepsilon\lambda)^{-1/2} \exp \left(\frac{2\tau\lambda}{1 + (1 - 4\varepsilon\lambda)^{1/2}}\right)\right] \leqq \\ \leqq \tau \left[\exp (\tau\lambda) - (1 - 4\varepsilon \, M(\varepsilon))^{-1/2} \exp \left(\frac{2\tau\lambda}{1 + (1 - 4\varepsilon m)^{1/2}}\right)\right] \leqq \\ \leqq \tau[\exp (\tau\lambda) - \tfrac{1}{2} \exp (\tau\lambda)] \leqq 0$$

for $\tau \leqq 0$, $\lambda \in \langle m, M(\varepsilon)\rangle$ from where it follows

$$\int_{-\infty}^{0} \varphi_1(\varepsilon, \tau)\, \mathrm{d}\tau = \int_{-\infty}^{0} q_1(\varepsilon, \tau, m)\, \mathrm{d}\tau = \frac{1}{m} - \frac{2\varepsilon}{1 - (1 - 4\varepsilon m)^{1/2}} \leqq 2\varepsilon .$$

For the function $q_2(\varepsilon, \tau, \lambda)$ we have

$$|q_2(\varepsilon, \tau, \lambda)| \leqq \frac{8}{3}\varepsilon(1 - 4\varepsilon\, M(\varepsilon))^{-1/2}\, \lambda^{1-\nu}\lambda^{\nu} \exp\left(\frac{\tau\lambda}{2}\right) \leqq$$

$$\leqq \frac{2^{4\nu-1}}{3} \varepsilon^{\nu} (1 - \varepsilon\, M(\varepsilon))^{-1/2} \max_{\mu\in\langle m, 1/4\varepsilon\rangle} \left(\mu^{\nu} \exp\left(\frac{\tau\mu}{2}\right)\right),$$

where $\tau \in R^-$, $\lambda < m, M(\varepsilon)\rangle$ and $\nu \in \langle 0, 1)$ is arbitrary. As

$$\max_{\mu\in\langle m, M(\varepsilon)\rangle} \left(\mu^{\nu} \exp \frac{4}{3}\tau\mu\right) = \begin{cases} m^{\nu} \exp\left(\frac{4}{3}\tau m\right), & \tau \in \left(-\infty, -\frac{3\nu}{4m}\right\rangle \\ \left(\frac{-3\nu}{4\tau}\right)^{\nu} \exp(-\nu), & \tau \in \left\langle -\frac{3\nu}{4m}, -4\varepsilon\nu\right\rangle \\ \left(\frac{3}{16\varepsilon}\right)^{\nu} \exp\left(\frac{\tau}{4\varepsilon}\right), & \tau \in \langle -4\varepsilon\nu, 0\rangle \end{cases}$$

and $(1 - 4\varepsilon\, M(\varepsilon))^{-1/2} = 2$, it is

$$\int_{-\infty}^{0} \varphi_2(\varepsilon, \tau)\, \mathrm{d}\tau \leqq \frac{2^{4\nu}}{3} \varepsilon^{\nu} \left[m^{\nu} \int_{-\infty}^{-3\nu/4m} \exp\left(\frac{\tau m}{2}\right) \mathrm{d}\tau + \right.$$

$$\left. + \left(\frac{3\nu}{4}\right)^{\nu} \exp(-\nu) \int_{-3\nu/4m}^{-4\varepsilon\nu} \frac{\mathrm{d}\tau}{(-\tau)^{\nu}} + \left(\frac{3}{16\varepsilon}\right)^{\nu} \int_{-4\varepsilon\nu}^{0} \exp\left(\frac{\tau}{4\varepsilon}\right) \mathrm{d}\tau \right] \leqq \frac{c_1(m)}{1 - \nu} \varepsilon^{\nu},$$

where $\lim_{\mu\to 0+} c_1(\mu) = \infty$. So we have from (1.13)

$$\|v_\varepsilon\|_{(0,1/2)} \leqq \left[2\varepsilon + \frac{c_1(m)}{1 - \nu} \varepsilon^{\nu} + 8\varepsilon + 32\varepsilon + 6\varepsilon\right] \|f\|_{(0,1/2)} \leqq$$

$$\leqq \varepsilon^{\nu} \left(2^{4\nu} 3^{2-\nu} m^{\nu-1} + \frac{c_1(m)}{1 - \nu}\right) \|f\|_{(0,1/2)}, \quad \left(0 < \varepsilon \leqq \frac{3}{16m}\right).$$

q.e.d.

2. THE NONLINEAR CASE

In this paragraph we will investigate the periodic problem

$$\varepsilon\, u''(t) + u'(t) + A\, u(t) = \mu\, F(\varepsilon)\,(u)\,(t) \tag{2.1}$$

$$u(t + \omega) = u(t), \quad t \in R, \quad \varepsilon \in R^+, \quad \mu \in R, \tag{2.2}$$

with the operator A fulfilling the assumptions of Proposition 1.1. Here the nonlinear perturbation $F(\varepsilon)(u)(t)$ is supposed to be of the form

$$F(\varepsilon)(u)(t) = F_1(\varepsilon)(u)(t) + F_2(\varepsilon)(u)(t) + F_3(\varepsilon)(u)(t), \tag{2.3}$$

where $F_i(\varepsilon)(u)(t)$ are some nonlinear operators mapping from the space of ω-periodic functions with values in H into itself. The behaviour in ε of each of $F_i(\varepsilon)(u)(t)$ in (2.3) is in connection with character of its domain and range. The precise form of this dependence is given in the following assumptions:

(A_0) The function $\|F(\varepsilon)(0)(\cdot)\|_{(0,1/2)}$ is bounded in ε for $\varepsilon \in \langle 0, \varepsilon_0 \rangle$.

(A_1) The operator $F_1(\varepsilon)(u)(t)$ maps $U_2(R)$ into $C^0_\omega(R; \mathscr{D}(A))$ and $U_1(R)$ into $C^0_\omega(R; \mathscr{D}(A^{1/2}))$ respectively for $\varepsilon > 0$ and $\varepsilon = 0$ respectively. It fulfils the Lipschitz continuity conditions

$$\|F_1(\varepsilon)(u_1)(\cdot) - F_1(\varepsilon)(u_2)(\cdot)\|_{(0,1)} \leqq L\|u_1 - u_2\|_{(0,1)}, \quad \varepsilon \in (0, \varepsilon_0\rangle,$$
$$u_1, u_2 \in U_2(R),$$
$$\|F_1(0)(u_1)(\cdot) - F_1(0)(u_2)(\cdot)\|_{(0,1/2)} \leqq L\|u_1 - u_2\|_{(0,1)}, \quad u_1, u_2 \in U_1(R).$$

(A_2) The operator $F_2(\varepsilon)(u)(t)$ maps $U_2(R)$ and $U_1(R)$ respectively into $C^0_\omega(R; \mathscr{D}(A^{1/2}))$ for $\varepsilon > 0$ and $\varepsilon = 0$ respectively. It fulfils the Lipschitz continuity conditions

$$\|F_2(\varepsilon)(u_1)(\cdot) - F_2(\varepsilon)(u_2)(\cdot)\|_{(0,1/2)} \leqq L\varepsilon^{1/2}\|u_1 - u_2\|_{(0,1)}, \quad \varepsilon \in (0, \varepsilon_0\rangle,$$
$$u_1, u_2 \in U_2(R),$$
$$\|F_2(0)(u_1)(\cdot) - F_2(0)(u_2)(\cdot)\|_{(0,1/2)} \leqq L\|u_1 - u_2\|_{(0,1)}, \quad u_1, u_2 \in U_1(R).$$

(A_3) The operator $F_3(\varepsilon)(u)(t)$ maps $U_2(R)$ into $C^0_\omega(R; \mathscr{D}(A^{1/2}))$. It fulfils the Lipschitz continuity condition

$$\|F_3(\varepsilon)(u_1)(\cdot) - F_3(\varepsilon)(u_2)(\cdot)\|_{(0,1/2)} \leqq L\varepsilon^{3/2}\|u_1 - u_2\|_{(1,1/2)}, \quad \varepsilon \in (0, \varepsilon_0\rangle,$$
$$u_1, u_2 \in U_2(R)$$

and $F_3(0)(u)(t) \equiv 0$.

Let us introduce for $\varepsilon > 0$ the Banach space $B_\varepsilon = \{u \in U_2(R);\ u(t) \text{ is } \omega\text{-periodic on } R\}$ with the norm $\|u\|_\varepsilon = \|u\|_{(0,1)} + \varepsilon\|u\|_{(1,1/2)} + \varepsilon^2\|u\|_{(2,0)}$ and the Banach space $B_0 = \{u \in U_1(R);\ u(t) \text{ is } \omega\text{-periodic on } R\}$ with the norm $\|u\|_0 = \|u\|_{(1,0)} + \|u\|_{(0,1)}$.

Theorem 2.1. *Let A satisfy the assumptions of Proposition* 1.1 *and let F satisfy* (A_0), (A_1), (A_2) *and* (A_3). *Then there exists a $\mu_0 > 0$ such that for any $\varepsilon \in \langle 0, \varepsilon_0 \rangle$*

($\varepsilon_0 > 0$ *being arbitrary*) *and any* μ *in* $\langle 0, \mu_0 \rangle$ *there exists a unique solution* $u_\varepsilon(\mu)$ *in* B_ε *of* (2.1) *and* (2.2).

Proof. According to the results of the preceding paragraph we are justified to define for $u \in B_\varepsilon$, ($\varepsilon > 0$), the operators

$$G_\varepsilon^i(\mu)(u) = -2\mu \int_{-\infty}^0 K_\varepsilon(\tau) F_i(\varepsilon)(u)(t + \tau) \, d\tau , \quad i = 1, 2, 3 ,$$

$$G_\varepsilon(\mu)(u) = \sum_{i=1}^3 G_\varepsilon^i(\mu)(u) ,$$

$$G_0(\mu)(u) = \mu \int_{-\infty}^0 T(\tau) F(0)(u)(t + \tau) \, d\tau .$$

According to the proposition 1.1 it is sufficient to show that there exists a $\mu_0 > 0$ such that for any $\varepsilon \in \langle 0, \varepsilon_0 \rangle$ and any $\mu \in \langle 0, \mu_0 \rangle$ there exists a unique fixed point $u_\varepsilon(\mu)$ of the operator $G_\varepsilon(\mu)(u)$ in B_ε. So let us apply the Banach fixed point theorem to the operator $G_\varepsilon(\mu)(u)$ in B_ε ($\varepsilon \geqq 0$). First, let us note that as

$$\|K_\varepsilon(\tau)\| \leqq (1 - 4\varepsilon m)^{-1/2} \exp\left[\frac{\tau}{2\varepsilon}(1 - (1 - 4\varepsilon m)^{1/2})\right] + \frac{|\tau|}{2\varepsilon} \exp\left(\frac{\tau}{2\varepsilon}\right),$$

$$\|A^{1/2} K_\varepsilon(\tau)\| \leqq (4\varepsilon)^{-1/2} (1 - 4\varepsilon m)^{-1/2} \exp\left[\frac{\tau}{2\varepsilon}(1 - (1 - 4\varepsilon m)^{1/2}\right] +$$

$$+ [(4\varepsilon)^{-1/2} + (2\varepsilon)^{-3/2} |\tau|] \exp\left(\frac{\tau}{2\varepsilon}\right)$$

and

$$\|J_\varepsilon(\tau)\| \leqq \exp\left[\frac{\tau}{2\varepsilon}(1 - (1 - 4\varepsilon m)^{1/2})\right],$$

where $\varepsilon > 0$, $\tau \in R^-$ and $J_\varepsilon(\tau) = \exp(\tau/2\varepsilon) \cos(\tau/2\varepsilon)(4\varepsilon A - I)^{1/2}$, we have

$$\text{(2.4)} \qquad \int_{-\infty}^0 \|K_\varepsilon(\tau)\| \, d\tau \leqq c_2 < \infty , \quad \int_{-\infty}^0 \|A^{1/2} K_\varepsilon(\tau)\| \, d\tau \leqq c_3 \varepsilon^{-1/2} ,$$

$$\int_{-\infty}^0 \|J_\varepsilon(\tau)\| \, d\tau \leqq c_4 .$$

Besides, from (1.6_1) we have

$$\text{(2.5)} \qquad \int_{-\infty}^0 \|A^{1/2} T(\tau)\| \, d\tau = c_5 < \infty .$$

The operator $G_\varepsilon(\mu)(u)$, $(\varepsilon \geqq 0)$ maps B_ε into itself (Proposition 1.1). Let us prove that the operators $G_\varepsilon^i(\mu)(u)$, $(i = 1, 2, 3)$, are in B_ε Lipschitz continuous with a constant $L(\mu) = L_1\mu$.

If $\varepsilon > 0$ and $u_1, u_2 \in U_2(R)$ then

$$\|A(G_\varepsilon^1(\mu)(u_1)(t) - G_\varepsilon^1(\mu)(u_2)(t))\| =$$

$$= 2\mu \left\| \int_{-\infty}^0 K_\varepsilon(\tau) A(F_1(\varepsilon)(u_1)(t+\tau) - F_1(\varepsilon)(u_2)(t+\tau)) \, d\tau \right\| \leqq$$

$$\leqq 2\mu \int_{-\infty}^0 \|K_\varepsilon(\tau)\| \, d\tau \|F_1^{-\infty}(\varepsilon)(u_1)(\cdot) - F_1(\varepsilon)(u_2)(\cdot)\|_{(0,1)} \leqq 2\mu c_2 L \|u_1 - u_2\|_{(0,1)},$$

$$\|A(G_\varepsilon^2(\mu)(u_1)(t) - G_\varepsilon^2(\mu)(u_2)(t))\| =$$

$$= 2\mu \left\| \int_{-\infty}^0 A^{1/2} K_\varepsilon(\tau) A^{1/2}(F_2(\varepsilon)(u_1)(t+\tau) - F_2(\varepsilon)(u_2)(t+\tau)) \, d\tau \right\| \leqq$$

$$\leqq 2\mu \int_{-\infty}^0 \|A^{1/2} K_\varepsilon(\tau)\| \, d\tau \|F_2(\varepsilon)(u_1)(\cdot) - F_2(\varepsilon)(u_2)(\cdot)\|_{(0,1/2)} \leqq$$

$$\leqq 2\mu c_3 \|u_1 - u_2\|_{(0,1)},$$

$$\|A(G_\varepsilon^3(\mu)(u_1) - G_\varepsilon^3(\mu)(u_2))\| \leqq 2\mu c_3 L \varepsilon \|u_1 - u_2\|_{(1,1/2)},$$

$$\varepsilon \left\| A^{1/2} \frac{d}{dt} (G_\varepsilon^1(\mu)(u_1)(t) - G_\varepsilon^1(\mu)(u_2)(t)) \right\| =$$

$$= 2\mu\varepsilon \frac{1}{2\varepsilon} \left\| \int_{-\infty}^0 (K_\varepsilon(\tau) + J_\varepsilon(\tau)) A^{-1/2} A(F_1(\varepsilon)(u_1)(t+\tau) - F_1(\varepsilon)(u_2)(t+\tau)) \, d\tau \right\| \leqq$$

$$\leqq \mu(c_2 + c_4) m^{-1/2} L \|u_1 - u_2\|_{(0,1)},$$

$$\varepsilon \left\| A^{1/2} \frac{d}{dt} (G_\varepsilon^2(\mu)(u_1)(t) - G_\varepsilon^2(\mu)(u_2)(t)) \right\| \leqq \mu(c_2 + c_4) L \|u_1 - u_2\|_{(0,1)},$$

$$\varepsilon \left\| A^{1/2} \frac{d}{dt} (G_\varepsilon^3(\mu)(u_1)(t) - G_\varepsilon^3(\mu)(u_2)(t)) \right\| \leqq \mu(c_2 + c_4) L \varepsilon^{3/2} \|u_1 - u_2\|_{(1,1/2)},$$

$$\varepsilon^2 \left\| \frac{d^2}{dt^2} (G_\varepsilon^1(\mu)(u_1)(t) - G_\varepsilon^1(\mu)(u_2)(t)) \right\| .$$

$$. \, 2\mu\varepsilon^2 \frac{1}{4\varepsilon^2} \left\| \int_{-\infty}^0 (K_\varepsilon(\tau) + J_\varepsilon(\tau) - (4_\varepsilon A - I) K_\varepsilon(\tau)) \, . \right.$$

$$\left. . (F_1(\varepsilon)(u_1)(t+\tau) - F_1(\varepsilon)(u_2)(t+\tau)) \, d\tau \right\| \leqq \frac{\mu}{2} \int_{-\infty}^0 \left[\left(\frac{2}{m} + 4\varepsilon \right) \|K_\varepsilon(\tau)\| + \|J_\varepsilon(\tau)\| \right] d\tau \, .$$

$$\|F_1(\varepsilon)(u_1)(\cdot) - F_1(\varepsilon)(u_2)(\cdot)\|_{(0,1)} \leqq \frac{\mu}{2}(3m^{-1}c_2 + c_4) L\|u_1 - u_2\|_{(0,1)},$$

$$\varepsilon^2 \left\| \frac{d^2}{dt^2}(G_\varepsilon^2(\mu)(u_1)(t) - G_\varepsilon^2(\mu)(u_2)(t)) \right\| \leqq$$

$$\leqq \frac{\mu}{2}[(2c_2 + c_4) m^{-1/2} + 4\varepsilon^{1/2}c_3] L\varepsilon^{1/2}\|u_1 - u_2\|_{(0,1)},$$

$$\varepsilon^2 \left\| \frac{d^2}{dt^2}(G_\varepsilon^3(\mu)(u_1)(t) - G_\varepsilon^3(\mu)(u_2)(t)) \right\| \leqq$$

$$\leqq \frac{\mu}{2}[(2c_2 + c_4) m^{-1/2} + 4\varepsilon^{1/2}c_3] L\varepsilon^{3/2}\|u_1 - u_2\|_{(1,1/2)}$$

for $t \in R$ wherefrom the last assertion follows immediately. If $\varepsilon = 0$ and $u_1, u_2 \in \in U_1(R)$ then using (2.5) we can prove similarly as above

$$\|A(G_0(\mu)(u_1)(t) - G_0(\mu)(u_2)(t))\| \leqq 2\mu c_5 L\|u_1 - u_2\|_{(0,1)}$$

and

$$\left\| \frac{d}{dt}(G_0(\mu)(u_1)(t) - G_0(\mu)(u_2)(t)) \right\| \leqq 2\mu L(c_5 + m^{-1/2}) \|u_1 - u_2\|_{(0,1)}$$

for $t \in R$. Now having proved the Lipschitz continuity with respect to u in B_ε with the constant $L(\mu) = L_1\mu$ we can make the operators $G_\varepsilon(\mu)(u)$, $\varepsilon \in \langle 0, \varepsilon_0 \rangle$, contractive and mapping any fixed ball of B_ε with center in $u = 0$ into itself choosing μ sufficiently small (independently of $\varepsilon \in \langle 0, \varepsilon_0 \rangle$). Indeed, if u is in B_ε and $\|u\|_\varepsilon \leqq r$, $(r > 0)$ then

$$\|G_3(\mu)(u)\|_\varepsilon \leqq \|G_\varepsilon(\mu)(u) - G_\varepsilon(\mu)(0)\|_\varepsilon + \|G_\varepsilon(\mu)(0)\|_\varepsilon \leqq$$

$$\leqq L_1\mu r + 2\mu \left\| \int_{-\infty}^0 K_\varepsilon(\tau) F(\varepsilon)(\cdot + \tau, 0) \, d\tau \right\|_\varepsilon.$$

Since

$$\left\| \int_{-\infty}^0 K_\varepsilon(\tau) F(\varepsilon)(\cdot + \tau, 0) \, d\tau \right\|_\varepsilon \leqq c_6$$

in virtue of (A_0) it suffices to choose $0 < \mu_0 < r(L_1 r + 2c_6)^{-1}$. Note that the constants L_1 and c_k, $k = 2, \ldots, 6$ may be estimated by the only constant c_7 which is of the form $c_7 = c_7(m, \varepsilon_0) = c_8(m^{-1} + \varepsilon_0)$.

To prove the theorem on convergence of $u_\varepsilon(\mu)$ to $u_0(\mu)$ analogous to the theorem 1.1 we shall make the following additional assumptions:

(A_0') The functions $\|F_1(\varepsilon)(u)(\cdot)\|_{(0,1/2)}$, $\|F_2(\varepsilon)(u)(\cdot)\|_{(0,1/2)}$ are bounded for $\varepsilon \in \langle 0, \varepsilon_0 \rangle$ and $u \in U_1(R)$, $\|u\|_{U_1} \leqq r$.

(A_1') The operator $F_1(\varepsilon)(u)(t)$ fulfils the condition $\|F_1(\varepsilon)(u_1)(\cdot) - F_1(\varepsilon)(u_2)(\cdot)\|_{(0,1/2)} \leqq L\|u_1 - u_2\|_{(0,1/2)}$, $\varepsilon \in (0, \varepsilon_0\rangle$, $u_1, u_2 \in U_1(R)$.

(A_2') The operator $F_2(\varepsilon)(u)(t)$ fulfils the condition $\|F_2(\varepsilon)(u_1)(\cdot) - F_2(\varepsilon)(u_2)(\cdot)\|_{(0,0)} \leqq L\varepsilon^{1/2}\|u_1 - u_2\|_{(0,1/2)}$, $\varepsilon \in (0, \varepsilon_0\rangle$, $u_1, u_2 \in U_1(R)$.

Theorem 2.2. *Let A satisfy the assumptions of Proposition 1.1 and let F satisfy* (A_0), (A_0'), (A_1), (A_1'), (A_2), (A_2') *and* $F_3(\varepsilon)(u)(t) \equiv 0$. *Then for any* $\nu \in \langle 0, 1)$ *there exists a constant $k > 0$ and a $\mu_1 \in (0, \mu_0)$ such that*

$$\|u_\varepsilon(\mu) - u_0(\mu)\|_{(0,1/2)} \leqq k\mu\left(\frac{\varepsilon^\nu}{1-\nu} + \psi(\varepsilon)\right),$$

where $\mu \in \langle 0, \mu_1\rangle$, $\varepsilon \in \langle 0, \varepsilon_0\rangle$ *and*

$$\psi(\varepsilon) = \sup\left\{\sum_{i=1}^{2}\|F_i(\varepsilon)(u) - F_i(0)(u)\|_{(0,0)};\ u \in U_1(R),\ \|u\|_{U_1} \leqq r\right\}.$$

Proof. Expressing $A^{1/2}(u_\varepsilon(\mu)(t) - u_0(\mu)(t))$ in the form

$$A^{1/2}(u_\varepsilon(\mu)(t) - u_0(\mu)(t)) = -\mu\sum_{i=1}^{2}\left[2\int_{-\infty}^{0} K_\varepsilon(\tau)\, A^{1/2}(F_i(\varepsilon)(u_\varepsilon(\mu))(t+\tau) - F_i(\varepsilon)(u_0(\mu))(t+\tau))\,\mathrm{d}\tau + \int_{-\infty}^{0}(2K_\varepsilon(\tau) + T(\tau))\, A^{1/2}\, F_i(\varepsilon)(u_0(\mu))(t+\tau)\,\mathrm{d}\tau - \int_{-\infty}^{0} A^{1/2}\, T(\tau)\,(F_i(\varepsilon)(u_0(\mu)) - F_i(0)(u_0(\mu)))\,\mathrm{d}\tau\right]$$

we find

$$\|u_\varepsilon(\mu) - u_0(\mu)\|_{(0,1/2)} \leqq \mu\Big[2L\int_{-\infty}^{0}\|K_\varepsilon(\tau)\|\,\mathrm{d}\tau\,\|u_\varepsilon(\mu) - u_0(\mu)\|_{(0,1/2)} + 2L\int_{-\infty}^{0}\|A^{1/2}K_\varepsilon(\tau)\|\,\mathrm{d}\tau\,\varepsilon^{1/2}\|u_\varepsilon(\mu) - u_0(\mu)\|_{(0,1/2)} + \left\|\int_{-\infty}^{0}(2K_\varepsilon(\tau) + T(\tau))\,A^{1/2}(F_1(\varepsilon)(u_0(\mu))(t+\tau) + F_2(\varepsilon)(u_0(\mu))(t+\tau))\,\mathrm{d}\tau\right\| + \int_{-\infty}^{0}\|A^{1/2}\,T(\tau)\|\,\mathrm{d}\tau\sum_{i=1}^{2}\|F_i(\varepsilon)(u_0(\mu)) - F_i(0)(u_0(\mu))\|_{(0,0)}\Big].$$

We have proven in paragraph 1 the existence of a constant c_8 such that

$$\sup_{t\in R} \left\| \int_{-\infty}^{0} (2K_\varepsilon(\tau) + T(\tau))\, A^{1/2}(F_1(\varepsilon)(u_0(\mu))(t+\tau) + F_2(\varepsilon)(u_0(\mu))(t+\tau))\,\mathrm{d}\tau \right\| \leqq$$

$$\leqq \frac{c_8 \varepsilon^\nu}{1-\nu} (\|F_1(\varepsilon)(u_0(\mu))\|_{(0,1/2)} + \|F_2(\varepsilon)(u_0(\mu))\|_{(0,1/2)})$$

for $\nu \in \langle 0, 1)$ arbitrary. Further the terms $\|F_i(\varepsilon)(u_0(\mu))\|_{(0,1/2)}$ $(i = 1, 2)$ are by (A_0') bounded by some constant c_9. Using (2.4) and (2.5) we find

$$\|u_\varepsilon(\mu) - u_0(\mu)\|_{(0,1/2)} \leqq \mu \left[2L(c_2 + c_3) \|u_\varepsilon(\mu) - u_0(\mu)\|_{(0,1/2)} + \frac{2c_8 c_9}{1-\nu} \varepsilon^\nu + c_5\, \psi(\varepsilon) \right].$$

Choosing $0 < \mu_1 < \min([2L(c_2 + c_3)]^{-1}, \mu_0)$ we have

$$\|u_\varepsilon(\mu) - u_0(\mu)\|_{(0,1/2)} \leqq \mu \left[1 - \frac{\mu_1}{2L(c_2 + c_3)} \right]^{-1} \left(\frac{2c_8 c_9}{1-\nu} \varepsilon^\nu + c_5\, \psi(\varepsilon) \right),$$

$$0 \leqq \mu \leqq \mu_1, \quad \varepsilon \in \langle 0, \varepsilon_0 \rangle.$$

Corollary. *If the operators* $F_i(\varepsilon)(u)(t)$ $(i = 1, 2)$ *are equi-continuous in* $u \in U_1(R)$, $\|u\|_{U_1(R)} \leqq r$ *in the point* $\varepsilon = 0$, *i.e. if* $\lim_{\varepsilon\to 0+} \|F_i(\varepsilon)(u) - F_i(0)(u)\|_{(0,0)} = 0$ *uniformly for* $u \in U_1(R)$, $\|u\|_{U_1} \leqq r$, *then clearly* $\lim_{\varepsilon\to 0+} \|u_\varepsilon(\mu) - u_0(\mu)\|_{(0,1/2)} = 0$.

Bibliography

[1] *I. Straškraba, O. Vejvoda,* Periodic solutions to abstract differential equations, Czech. Math. J., *23 (98)*, 1973, pp. 635—669.

[2] *M. Kopáčková-Suchá,* On the weakly nonlinear wave equation involving a small parameter at the highest derivative, Czech. Math. J., *19 (94)*, 1969, Praha.

[3] *H. Bourbaki,* Integrating, measures, integrating of measures, 1967, Moscow, (Russian).

Authors' address: 115 67 Praha 1, Žitná 25, ČSSR (Matematický ústav ČSAV).