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ON THE INVARIANT METHOD IN DIFFERENTIAL GEOMETRY OF SUBMANIFOLDS

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The invariant method for the investigation of submanifolds of homogeneous spaces was established by E. CARTAN, [1]. One part of the evaluations used in Cartan's method of moving frames is a "prolongation" procedure consisting in the exterior differentiation of the relations among the principal forms followed by the application of the structure equations and of Cartan's lemma. G. F. LAPTĚV pointed out that this prolongation procedure can be performed independently of the specialization of frames, [9]. In the course of such evaluations, one obtains the coordinate functions of certain geometric object fields on the submanifold, which are called the fundamental geometric object fields. Laptěv also developed a computational procedure for constructing the geometric objects of submanifolds based on the fundamental geometric objects. He and his disciples applied successfully this method to many concrete problems in differential geometry. Further, VASILJEV remarked that a modification of the above method can be used to study submanifolds of a space with fundamental Lie pseudogroup, [14]. The investigations of [9] and [14] are local and are written in an analogous form as the original papers by E. CARTAN, which is generally considered unsatisfactory nowadays. That is why we present an intrinsic and global explanation of these problems based on the theory of jets.

In § 1, we treat a submanifold V of an arbitrary differentiable manifold M, define the cap fields of V and justify a procedure for finding their coordinate functions. This algorithm is quite analogous to the prolongation procedure by Laptěv. In § 2, we show how to reduce the investigation of a submanifold of an arbitrary homogeneous space to the results of § 1. However, a simple direct algorithm can be obtained for certain special homogeneous spaces only (nevertheless, all the main homogeneous spaces treated in the "classical" differential geometry are of this special type). This is explained in § 3. To clear up the fundamental ideas, we use the frame field of order zero in § 1 – § 3. Since a convenient specialization of frames is practically inevitable, we add some remarks concerning this subject in § 4. In the last paragraph, we outline the application of the invariant method to the submanifolds of a space with fundamental Lie pseudogroup. The case of a flat pseudogroup of the first order is treated in all details. The appendix deals with Laptěv's method for the construction of geometric objects of submanifolds, [9]. We find that this method gives a local construction of equivariant mappings and we also deduce a theorem leading to global results. In this way we show that our general intrinsic definition of the geometric objects of submanifolds agrees with the computational procedures of the classical differential geometry.

The standard terminology and notation of the theory of jets are used throughout the paper, see [3]. In addition, j^s_r, $s < r$, denotes the canonical projection of r-jets into s-jets. Unless otherwise specified, our considerations are in the category C^∞.

1. Cap fields on a submanifold

We recall that a contact element of dimension m and of order r (shortly: a contact m^r-element) on a manifold M at a point $x \in M$ is the set XL^r_m, where X is an m^r-velocity on M at x, [3]. Such a contact element is called regular, if $m < n = \dim M$ and if X is a regular velocity. The fibred manifold of all regular contact m^r-elements on M will be denoted by $K^r_m(M)$. Obviously, $K^r_m(M)$ has a natural structure of an associated fibre bundle of the symbol $(M, K^r_{n,m}, L^r_n, H^r(M))$, where $K^r_{n,m}$ means the space of all regular contact m^r-elements on $\mathbf{R}^n$ at 0 and $H^r(M)$ is the r-th principal prolongation of M.

Let V be an m-dimensional submanifold of M and let r be a positive integer. Then V determines a regular contact m^r-element $k^r_x V$ at every $x \in V$. Using the expressive terminology by Bompiani, we shall say $k^r_x V$ to be the cap of order r (shortly: the r-cap) of V at x. Let $Q^r(V)$ denote the restriction of $H^r(M)$ over V, which is a principal fibre bundle $Q^r(V)(V, L^r_n)$. Further, let $K^r(V)$ be the restriction of $K^r_m(M)$ over V, so that $K^r(V)$ is an associated fibre bundle of the symbol $(V, K^r_{n,m}, L^r_n, Q^r(V))$. We have a canonical cross section $\sigma^r : V \to K^r(V)$, $x \mapsto k^r_x V$, which will be called the cap field of order r (or the r-th cap field) of V.

We shall now apply the concepts of § 1 of [8] to the above situation. Denote by $p : \mathbf{R}^n \to \mathbf{R}^m$ the natural projection and view $\mathbf{R}^n$ as a fibred manifold $(\mathbf{R}^n, p, \mathbf{R}^m)$. Let $\hat{K}^r_{n,m} \subset K^r_{n,m}$ be the subspace of all elements transversal with respect to this fibering. Then the elements of $\hat{K}^r_{n,m}$ can be identified with the elements of $J^r(\mathbf{R}^n, p, \mathbf{R}^m)$ (= the r-th prolongation of the fibred manifold $(\mathbf{R}^n, p, \mathbf{R}^m)$) with source $0 \in \mathbf{R}^m$ and target $0 \in \mathbf{R}^n$. This identification gives the coordinates $y^J_p, \ldots, y^J_{p_1 \ldots p_r}$ on $\hat{K}^r_{n,m}$. Any $X \in \hat{K}^r_{n,m}$ is identified with an r-jet of a mapping of the form

$$(1) \qquad b^J_p x^p + \ldots + \frac{1}{r!} b^J_{p_1 \ldots p_r} x^{p_1} \ldots x^{p_r}, \quad p, q, \ldots = 1, \ldots, m,$$

$$J, K, \ldots = m + 1, \ldots, n.$$

Then we set $y^J_p(X) = b^J_p, \ldots, y^J_{p_1 \ldots p_r}(X) = b^J_{p_1 \ldots p_r}$. Obviously, if $\bar{y}^J_p, \ldots, \bar{y}^J_{p_1 \ldots p_{r-1}}$ are the analogous coordinates on $\hat{K}^{r-1}_{n,m}$, then

(2) $$y^J_p = (j^{r-1}_r)^* \bar{y}^J_p, \ldots, y^J_{p_1 \ldots p_{r-1}} = (j^{r-1}_r)^* \bar{y}^J_{p_1 \ldots p_{r-1}} .$$

Introduce

$$\hat{Q}^r(V) = \{u \in Q^r(V);\ u^{-1}(k^r_x V) \in \hat{K}^r_{n,m},\ x = \beta u\} .$$

According to [8], we get the coordinate functions

(3) $$a^J_p, \ldots, a^J_{p_1 \ldots p_r} : \hat{Q}^r(V) \to \mathbf{R}$$

of the r-th cap field σ^r of V. Let $i_r : \hat{Q}^r(V) \to H^r(M)$ be the injection. On $H^r(M)$, there is a canonical $(\mathbf{R}^n \oplus \mathfrak{l}^{r-1}_n)$-valued form θ_r (where $\mathfrak{l}^{r-1}_n$ means the Lie algebra of L^{r-1}_n), see [5], [7]. Let $\tilde{\theta}_r = i^*_r \theta_r$ be its restriction to $\hat{Q}^r(V)$. We shall show that the components of $\tilde{\theta}_r$ play an essential role in the evaluation of the functions $a^J_p, \ldots$ $\ldots, a^J_{p_1 \ldots p_r}$. Nonetheless, we first deduce an auxiliary lemma.

Lemma 1. *Let $V \subset \mathbf{R}^n$, $0 \in V$ be an m-dimensional submanifold such that $k^r_0 V \in$ $\in \hat{K}^r_{n,m}$. Then*

(4) $$y^J_{p_1 \ldots p_{r-1} q}(k^r_0 V) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_0 y^J_{p_1 \ldots p_{r-1}}(k^{r-1}_0(\tau^{-1}_{\gamma_q(t)}(V))) ,$$

where $\tau_y : \mathbf{R}^n \to \mathbf{R}^n$ is the translation $x \mapsto x + y$ and $\gamma_q(t)$ is the curve on V projected into $\mathbf{R}^m$ in the curve $x^p = \delta^p_q t$.

Proof. Let $k^r_0 V = j^r_0[f^J(x^p)]$, where $f^J(x^p) = b^J_p x^p + \ldots + b^J_{p_1 \ldots p_r} x^{p_r} \ldots x^{p_r}$. Then $k^{r-1}_0(\tau^{-1}_{\gamma_q(t)}(V)) = j^{r-1}_0\{[b^J_p(x^p + \delta^p_q t) + \ldots + (1/r!)\, b^J_{p_1 \ldots p_r}(x^{p_1} + \delta^{p_1}_q t) \ldots (x^{p_r} +$ $+ \delta^{p_r}_q t)] - f^J(\delta^p_q t)\}$, so that $y^J_{p_1 \ldots p_{r-1}}(k^{r-1}_0(\tau^{-1}_{\gamma_q(t)}(V))) = b^J_{p_1 \ldots p_{r-1} q} t$, which implies our Lemma.

Let

(5) $$\theta^i , \quad i, j, \ldots = 1, \ldots, n ,$$

be the components of the canonical form θ_1 of $H^1(M)$. Put $\tilde{\theta}^j = i^*_1 \theta^j$. By the definition of $\hat{Q}^1(V)$, $\tilde{\theta}^p$ are linearly independent and $\tilde{\theta}^J$ are some linear combinations of $\tilde{\theta}^p$.

Proposition 1. *It holds*

(6) $$\tilde{\theta}^J = a^J_p \tilde{\theta}^p ,$$

where $a^J_p : \hat{Q}^1(V) \to \mathbf{R}$ are the coordinate functions of the first cap field σ^1 of V.

Proof. Consider first an m-dimensional submanifold $W \subset \mathbf{R}^n$, $0 \in W$ such that $k^1_0 W \in K^1_{n,m}$ and denote by $\varphi : W \to \mathbf{R}^n$ the injection. Then

(7) $$(\varphi^*(\mathrm{d}x^J))_0 = y^J_p(k^1_0 W)\,(\varphi^*(\mathrm{d}x^p))_0 ,$$

since φ can be considered a local cross section of $(\mathbf{R}^n, p, \mathbf{R}^m)$ determined by $x^J = b^J_p x^p$ and it holds $(\varphi^*(\mathrm{d}x^p))_0 = (\mathrm{d}x^p)_0$, $(\varphi^*(\mathrm{d}x^J))_0 = b^J_p(\varphi^*(\mathrm{d}x^p))_0$, $b^J_p = y^J_p(k^1_0 W)$. Further, let $u \in Q^1(V)$, $u = j^1_0 \Psi$ and let $\lambda : V \to M$ be the injection. Then (7) implies

$$(8) \qquad ((\Psi^{-1}\lambda)^* (\mathrm{d}x^J))_0 = y^J_p(k^1_0(\Psi^{-1}(V)))\,((\Psi^{-1}\lambda)^* (\mathrm{d}x^p))_0 .$$

By the definition of θ_1, $((\Psi^{-1})^* (\mathrm{d}x^i))_0 = \theta^i_u$, so that $((\Psi^{-1}\lambda)^* (\mathrm{d}x^i))_0 = \tilde{\theta}^i_u$. Hence (8) can be rewritten as $\tilde{\theta}^J_u = a^J_p(u)\,\tilde{\theta}^p_u$, QED.

We shall need the equations of the fundamental distribution on $Q^1_x(V) \times K^1_{n,m}$, $x \in V$, see [8]. We recall that the structure equations of θ_1 are $\mathrm{d}\theta^i = \theta^j \wedge \theta'^i_j$, where (θ^i, θ'^i_j) is an admissible extension of θ_1, [7]. On $\hat{Q}^1(V)$, we have

$$(9) \qquad \mathrm{d}\tilde{\theta}^i = \tilde{\theta}^j \wedge \tilde{\theta}'^i_j , \quad \text{where} \quad \tilde{\theta}'^i_j = i^*_1 \theta'^i_j ,$$

and $\tilde{\theta}^p$, $\tilde{\theta}'^i_j$ satisfy the assumptions of Lemma 2 of [8]. The exterior differentiation of (6) yields

$$[\mathrm{d}a^J_p - a^J_q \tilde{\theta}'^q_p - a^J_q a^K_p \tilde{\theta}'^q_K + a^K_p \tilde{\theta}'^J_K + \tilde{\theta}'^J_p] \wedge \tilde{\theta}^p = 0 .$$

Hence the equations of the fundamental distribution on $Q^1_x(V) \times K^1_{n,m}$ are

$$(10) \qquad \mathrm{d}y^J_p - y^J_q \omega^q_p - y^J_q y^K_p \omega^q_K + y^K_p \omega^J_K + \omega^J_p := \mathrm{d}y^J_p + \Phi^J_p(y^J_p, \omega^i_j) = 0 ,$$

where ω^i_j is the canonical l^1_n-valued form of $Q^1_x(V)$ and the functions Φ^J_p are defined by equation (10) itself. – To simplify the following considerations, we change our notation: the coordinate functions of σ^1 will be denoted by $\bar{a}^J_p$.

Consider now the coordinate functions $a^J_p, a^J_{pq} : \hat{Q}^2(V) \to \mathbf{R}$ of the second cap field σ^2 of V. By (2), we find directly $a^J_p = (j^1_2)^* \bar{a}^J_p$. Further, let θ^i, θ^i_j be the components of the canonical form θ_2 of $H^2(M)$ and let $\tilde{\theta}^i$, $\tilde{\theta}^i_j$ be the induced forms on $\hat{\theta}^2(V)$. Then

$$(11) \qquad \mathrm{d}a^J_p + \Phi^J_p(a^J_p, \tilde{\theta}^i_j) = a^J_{pq}\tilde{\theta}^q .$$

Since this assertion is a special case of the following proposition, we need not prove it separately. Applying the exterior differentiation to (11), we find the equations of the fundamental distribution on $Q^2_x(V) \times K^2_{n,m}$, $x \in V$, and we can similarly proceed step by step on. Thus, let $\bar{a}^J_p, \ldots, \bar{a}^J_{p_1 \ldots p_{r-1}} : \hat{Q}^{r-1}(V) \to \mathbf{R}$ be the coordinate functions of the $(r-1)$-st cap field of V and let $\bar{\theta}_{r-1} = (\bar{\theta}^i, \bar{\theta}^i_j, \ldots, \bar{\theta}^i_{j_1 \ldots j_{r-1}})$ be the restriction of the canonical form of $H^{r-1}(M)$ to $\hat{\theta}^{r-1}(V)$. Assume by induction that it holds

$$(12) \qquad \begin{gathered} \bar{\theta}^J = \bar{a}^J_p \bar{\theta}^p , \\ \mathrm{d}\bar{a}^J_p + \Phi^J_p(\bar{a}^J_p, \bar{\theta}^i_j) = \bar{a}^J_{pq}\bar{\theta}^q , \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathrm{d}\bar{a}^J_{p_1 \ldots p_{r-2}} + \Phi^J_{p_1 \ldots p_{r-2}}(\bar{a}^J_p, \ldots, \bar{a}^J_{p_1 \ldots p_{r-2}}, \bar{\theta}^i_j, \ldots, \bar{\theta}^i_{j_1 \ldots j_{r-2}}) = a^J_{p_1 \ldots p_{r-2} q}\bar{\theta}^q , \end{gathered}$$

provided the equations of the fundamental distribution on $Q_x^{r-1}(V) \times K_{n,m}^{r-1}$, $x \in V$, are

$$(13) \qquad d\bar{y}_p^J + \Phi_p^J(\bar{y}_p^J, \bar{\omega}_j^i) = 0\,,$$

$$\dots\dots\dots\dots$$

$$d\bar{y}^J_{p_1\dots p_{r-2}} + \Phi^J_{p_1\dots p_{r-2}}(\bar{y}^J_p, \dots, \bar{y}^J_{p_1\dots p_{r-2}}, \bar{\omega}^i_j, \dots, \bar{\omega}^i_{j_1\dots j_{r-2}}) = 0\,,$$

$$d\bar{y}^J_{p_1\dots p_{r-1}} + \Phi^J_{p_1\dots p_{r-1}}(\bar{y}^J_p, \dots, \bar{y}^J_{p_1\dots p_{r-1}}, \bar{\omega}^i_j, \dots, \bar{\omega}^i_{j_1\dots j_{r-1}}) = 0\,,$$

where $(\bar{\omega}^i_j, \dots, \bar{\omega}^i_{j_1\dots j_{r-1}})$ is the canonical $\mathfrak{l}_n^{r-1}$-valued form of $Q_x^{r-1}(V)$.

Proposition 2. *Let θ_r be the canonical form of $H^r(M)$, let $\tilde{\theta}^i, \dots, \tilde{\theta}^i_{j_1\dots j_r}$ be the components of the induced form $i_r^*\theta_r$ on $\hat{Q}^r(V)$ and let $a^j_p, \dots, a^J_{p_1\dots p_r} : \hat{Q}^r(V) \to \mathbf{R}$ be the coordinate functions of the r-th cap field σ^r of V. Then*

$$(14) \qquad \tilde{\theta}^J = a^J_p\tilde{\theta}^p\,,$$

$$\dots\dots\dots\dots$$

$$da^J_{p_1\dots p_{r-2}} + \Phi^J_{p_1\dots p_{r-2}}(a^J_p, \dots, a^J_{p_1\dots p_{r-2}}, \tilde{\theta}^i_j, \dots, \tilde{\theta}^i_{j_1\dots j_{r-2}}) = a^J_{p_1\dots p_{r-2}q}\tilde{\theta}^q\,,$$

$$(15) \qquad da^J_{p_1\dots p_{r-1}} + \Phi^J_{p_1\dots p_{r-1}}(a^J_p, \dots, a^J_{p_1\dots p_{r-1}}, \tilde{\theta}^i_j, \dots, \tilde{\theta}^i_{j\dots j_{r-1}}) = a^J_{p_1\dots p_{r-1}q}\tilde{\theta}^q\,.$$

Proof. Obviously, it holds

$$(16) \qquad a^J_p = (j_r^{r-1})^*\,\bar{a}^J_p, \dots, a^J_{p_1\dots p_{r-1}} = (j_r^{r-1})^*\,\bar{a}^J_{p_1\dots p_{r-1}},$$

$$\tilde{\theta}^i = (j_r^{r-1})^*\,\bar{\theta}^i, \dots, \tilde{\theta}^i_{j_1\dots j_{r-1}} = (j_r^{r-1})^*\,\bar{\theta}^i_{j_1\dots j_{r-1}}\,,$$

so that (14) is a direct consequence of (12) and (16). It remains to deduce that, for every $u \in Q^r(V)$ and every $X \in T_u(Q^r(V))$, it is

$$(17) \qquad da^J_{p_1\dots p_{r-1}}(X) + \Phi^J_{p_1\dots p_{r-1}}(a^J_p(u), \dots, a^J_{p_1\dots p_{r-1}}(u), \xi^i_j, \dots, \xi^i_{j_1\dots j_{r-1}}) = a^J_{p_1\dots p_r}(u)\,\xi^{p_r}\,,$$

where $\xi^i = \tilde{\theta}^i(X), \dots, \xi^i_{j_1\dots j_{r-1}} = \tilde{\theta}^i_{j_1\dots j_{r-1}}(X)$. Put $\bar{u} = j_r^{r-1}u$, $\bar{X} = j_r^{r-1}X$, $u = j_0^r\Psi$. By definition, it is $\tilde{\theta}_r(X) = \theta_r(i_{r*}X) = \tilde{u}^{-1}(i_{r-1*}X)$. Consider a decomposition $\bar{X} = \bar{X}_1 + \bar{X}_2$ such that $\tilde{u}^{-1}(i_{r-1*}X_1) = \xi^i e_i$. Then $\bar{X}_2 \in T_{\bar{u}}(Q_x^{r-1}(V))$, $x = \beta u$ and Lemma 1 of [8] implies

$$(18) \qquad d\bar{a}^J_{p_1\dots p_{r-1}}(\bar{X}_2) = -\Phi^J_{p_1\dots p_{r-1}}(\bar{a}^J_p(\bar{u}), \dots, \bar{a}^J_{p_1\dots p_{r-1}}(\bar{u}), \xi^i_j, \dots, \xi^i_{j_1\dots j_{r-1}})\,.$$

Further, $i_{r-1*}\bar{X}_1$ can be written as $j_0^1[\Psi j_0^{r-1}\tau_{\gamma(t)}]$, where $\gamma(t)$ is the curve on $\Psi^{-1}(U)$ projected into $\mathbf{R}^m$ in the curve $x^p = \xi^p t$. By Lemma 1 and by the chain rule, we find

$$(19) \qquad d\bar{a}^J_{p_1\dots p_{r-1}}(\bar{X}_1) = j_0^1\bar{a}^J_{p_1\dots p_{r-1}}(\Psi j_0^{r-1}\tau_{\gamma(t)}) =$$

$$= \left(\frac{d}{dt}\right)_0 \bar{y}^J_{p_1\dots p_{r-1}}(k_0^{r-1}(\tau^{-1}_{\gamma(t)}(\Psi^{-1}(V)))) = y^J_{p_1\dots p_r}(u)\xi^{p_r}\,.$$

But (18) and (19) is equivalent to (17), QED.

To derive the equations of the fundamental distribution on $Q_x^r(V) \times K_{n,m}^r$, $x \in V$, we apply the exterior differentiation to (15). Using the structure equations of θ_r, [7], and taking into account that (13) is completely integrable, we deduce easily that we obtain certain relations of the form

$$(20) \qquad [\mathrm{d}a^J_{p_1 \ldots p_r} + \Phi^J_{p_1 \ldots p_r}(a^J_p, \ldots, a^J_{p_1 \ldots p_r}, \tilde{\theta}^i_j, \ldots, \tilde{\theta}^i_{j_1 \ldots j_{r-1}}, \tilde{\theta}'^i_{j_1 \ldots j_r})] \wedge \tilde{\theta}^{p_r} = 0 ,$$

where $\theta'^i_{j_1 \ldots j_r}$ determine an admissible extension of θ_r, $\tilde{\theta}'^i_{j_1 \ldots j_r} = i_r^* \theta'^i_{j_1 \ldots j_r}$ and $\Phi^J_{p_1 \ldots p_r}$ are certain linear combinations of $\tilde{\theta}$'s with coefficients in $a^J_p, \ldots, a^J_{p_1 \ldots p_r}$. By Lemma 2 of [8], we conclude that the equations of the fundamental distribution on $Q_x^r(V) \times$ $\times K_{n,m}^r$ are

$$(21) \qquad \mathrm{d}y^J_p + \Phi^J_p(a^J_p, \omega^i_j) = 0 ,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathrm{d}y^J_{p_1 \ldots p_{r-1}} + \Phi^J_{p_1 \ldots p_{r-1}}(y^J_p, \ldots, y^J_{p_1 \ldots p_{r-1}}, \omega^i_j, \ldots, \omega^i_{j_1 \ldots j_{r-1}}) = 0 ,$$

$$\mathrm{d}y^J_{p_1 \ldots p_r} + \Phi^J_{p_1 \ldots p_r}(y^J_p, \ldots, y^J_{p_1 \ldots p_r}, \omega^i_j, \ldots, \omega^i_{j_1 \ldots j_r}) = 0 ,$$

provided $(\omega^i_j, \ldots, \omega^i_{j_1 \ldots j_r})$ is the canonical $\mathfrak{l}_n^r$-valued form of $Q_x^r(V)$.

Remark 1. It should be underlined that the previous procedure gives the equations of the fundamental distribution on $Q_x^r(V) \times K_{n,m}^r$. We shall explain a practical advantage of this fact in the sequel.

2. Submanifolds of homogeneous spaces

Assume that a Lie group G acts transitively on the left on M. The transformation determined by $g \in G$ will be denoted by A_g, $A_g : M \to M$. Fix a point $c \in M$ and denote by H its stability group. Then G can be considered a principal fibre bundle $G(M, H)$ over M with the structure group H. Every r-frame $Y \in H_c^r(M)$ determines a principal fibre bundle homomorphism $\eta : G(M, H) \to H^r(M)$, $g \mapsto A_g Y$ (= the composition of the mapping A_g and the r-jet Y).

Proposition 3. *Let θ_r be the canonical form of $H^r(M)$ and let $Y \in H_c^r(M)$. Then $\eta^* \theta_r$ is an $(\mathbf{R}^n \oplus \mathfrak{l}_n^{r-1})$-valued left invariant form on G.*

Proof. Let $X = j_0^1 \gamma(t)$ be a tangent vector to G and let $A_{\gamma(0)} Y = j_0^r \varphi$. Then $\eta_* X = j_0^1(A_{\gamma(t)} Y)$ and $\theta_r(\eta_* X) = j_0^1[\varphi^{-1} A_{\gamma(t)} \bar{Y}]$, $\bar{Y} = j_r^{r-1} Y$, see [5], [7]. On the other hand, $L_{g*} X = j_0^1 g\, \gamma(t)$, $g \in G$, so that $\eta_*(L_{g*} X) = j_0^1[A_{g\gamma(t)} Y]$. Since $A_{g\gamma(0)} Y =$ $= j_0^r A_g \varphi$, we have $\theta_r(\eta_* L_{g*} X) = j_0^1[\varphi^{-1} A_g^{-1} A_{g\gamma(t)} \bar{Y}] = j_0^1[\varphi^{-1} A_{\gamma(t)} \bar{Y}] = \theta_r(\eta_* X)$, QED.

Lemma 2. *Let $Y, \hat{Y} \in H^r_c(M)$ satisfy $j^{r-1}_r Y = j^{r-1}_r \hat{Y} = \bar{Y}$ and let*

$$(22) \qquad \eta^*\theta_r = (\theta^i, \ldots, \theta^i_{j_1 \ldots j_{r-2}}, \theta^i_{j_1 \ldots j_{r-1}}),$$

$$\hat{\eta}^*\theta_r = (\theta^i, \ldots, \theta^i_{j_1 \ldots j_{r-2}}, \hat{\theta}^i_{j_1 \ldots j_{r-1}}).$$

Then

$$(23) \qquad \theta^i_{j_1 \ldots j_{r-1}} = \hat{\theta}^i_{j_1 \ldots j_{r-1}} + a^i_{j_1 \ldots j_{r-1}k}\theta^k,$$

where $a^i_{j_1 \ldots j_r}$ are the natural coordinates of the element $Z \in L^r_n$ determined by $Y = \hat{Y}Z$.

Proof. If $X \in T(G)$, $X = j^1_0\, \gamma(t)$ and $A_{\gamma(0)}X = j^r_0\varphi$, $A_{\gamma(0)}\hat{Y} = j^r_0\hat{\varphi}$, then $Z = j^r_0(\hat{\varphi}^{-1}\varphi)$. Hence $\theta_r(\eta_* X) = j^1_0[\varphi^{-1}A_{\gamma(t)}\bar{Y}]$ and $\theta_r(\hat{\eta}_* X) = j^1_0[\hat{\varphi}^{-1}\varphi\varphi^{-1}A_{\gamma(t)}\bar{Y}]$, so that (23) is a direct consequence of Lemma 2 of [7], QED.

From Proposition 3 and Lemma 2 we derive the following algorithm for the evaluation of the induced forms on G. First of all, for every $Y_1 \in H^1_c(M)$ one can choose a basis ω^i, ω^λ of $\mathfrak{g}^*$ such that

$$(24) \qquad \omega^i = \eta_1^*\theta^i$$

and that $\omega^i = 0$ are the differential equations of H. Let

$$(25) \qquad \mathrm{d}\omega^i = \tfrac{1}{2}c^i_{jk}\omega^j \wedge \omega^k + c^i_{j\lambda}\omega^j \wedge \omega^\lambda, \quad \lambda, \mu, \ldots = n+1, \ldots, \dim G,$$

$$\mathrm{d}\omega^\lambda = \tfrac{1}{2}c^\lambda_{\alpha\beta}\omega^\alpha \wedge \omega^\beta, \qquad \alpha, \beta, \ldots = 1, \ldots, \dim G,$$

be the structure equations of G. Take an element $\hat{Y}_2 \in H^2_c(M)$ such that $j^1_2\hat{Y}_2 = Y_1$. Then $\hat{\eta}_2^*\theta^i = \omega^i$ and the structure equations of the canonical form θ_2 of $H^2(M)$ imply

$$(26) \qquad \mathrm{d}\omega^i = \omega^j \wedge \hat{\eta}_2^*\theta^i_j.$$

We shall compare (25_1) and (26). Using Proposition 3 and Cartan's lemma, we obtain

$$(27) \qquad \hat{\eta}_2^*\theta^i_j = c^i_{j\lambda}\omega^\lambda + \tfrac{1}{2}c^i_{jk}\omega^k + a^i_{jk}\omega^k,$$

where a^i_{jk} are any constants satisfying $a^i_{jk} = a^i_{kj}$. By Lemma 2, all possible values of a^i_{jk} are in a one-to-one correspondence with all frames of $H^2_c(M)$ over Y_1. Hence one can choose e.g. an element $Y_2 \in H^2_c(M)$ such that $a^i_{jk} = 0$. Then

$$(28) \qquad \eta_2^*\theta^i_j = c^i_{j\lambda}\omega^\lambda + \tfrac{1}{2}c^i_{jk}\omega^k.$$

Take an element $\hat{Y}_3 \in H^3_c(M)$ satisfying $j^2_3\hat{Y}_3 = Y_2$. Then

$$(29) \qquad \hat{\eta}_3^*\theta^i = \omega^i, \quad \hat{\eta}_3^*\theta^i_j = c^i_{j\lambda}\omega^\lambda + \tfrac{1}{2}c^i_{jk}\omega^k$$

and the structure equations of the canonical form θ_3 of $H_3(M)$ imply

$$\mathrm{d}(c^i_{j\lambda}\omega^\lambda + \tfrac{1}{2}c^i_{jk}\omega^k) = (c^k_{j\lambda}\omega^\lambda + \tfrac{1}{2}c^k_{jl}\omega^l) \wedge (c^i_{k\mu}\omega^\mu + \tfrac{1}{2}c^i_{km}\omega^m) + \omega^k \wedge \hat{\eta}^*_3\theta^i_{jk}\,. \tag{30}$$

Comparing (25) and (30), one finds

$$\hat{\eta}^*_3\theta^i_{jk} = c^i_{(j|\mu|}c^\mu_{k)\lambda}\omega^\lambda + (c^i_{(j|\lambda|}c^\lambda_{k)l} + \tfrac{1}{2}c^i_{(j|m|}c^m_{k)l})\,\omega^l + a^i_{jkl}\omega^l\,, \tag{31}$$

where a^i_{jkl} are arbitrary constants symmetric in all subscripts. According to Lemma 2, all possible values of a^i_{jkl} are in a one-to-one correspondence with all elements of $H^3_c(M)$ over Y_2. Hence one can choose e.g. an element Y_3 such that $a^i_{jkl} = 0$; and so on.

Consider now an m-dimensional submanifold V of M. Let $Q(V)$ be the restriction of the principal fibre bundle $G(M, H)$ over V, which is a principal fibre bundle $Q(V)(V, H)$. The fibred manifold $K^r(V)$ can be naturally considered an associated fibre bundle of the symbol $(V, K^r_{m,c}(M), H, Q(V))$, where $K^r_{m,c}(M)$ means the fibre of $K^r_m(M)$ over c. Then the coordinate functions of the r-th cap field σ^r of V are some functions on a subspace of $Q(V)$ introduced as follows. Let ω^α be the above basis of $\mathfrak{g}^*$ and let e be the unit of G. Then $(\omega^i)_e$ is identified with a basis of $T^*_c(M)$. Denote by $\hat{Q}(V) \subset Q(V)$ the subspace of all $g \in Q(V)$ such that the tangent space of the submanifold $A^{-1}_g(V)$ at c is complementary to the subspace $(\omega^p)_e = 0$ of $T_c(M)$. Let $i : \hat{Q}(V) \to G$ be the injection and let $\tilde{\omega}^\alpha = i_*\omega^\alpha$. Consider further an element $Y_r \in H^r_c(M)$ such that $\eta^*_r\theta^i = \omega^i$ and define $\tilde{\eta}_r : \hat{Q}(V) \to \hat{Q}^r(V)$, $g \mapsto A_gY_r$. Obviously, it is $\eta_r i = i_r\tilde{\eta}_r$. If $a^J_p, \ldots, a^J_{p_1\ldots p_r} : \hat{Q}^r(V) \to \mathbf{R}$ are the coordinate functions of σ^r in the sense of § 1, then the functions

$$\tilde{a}^J_p = \tilde{\eta}^*_r a^J_p, \ldots, \tilde{a}^J_{p_1\ldots p_r} = \tilde{\eta}^*_r a^J_{p_1\ldots p_r} : \hat{Q}(V) \to \mathbf{R} \tag{32}$$

will be called the coordinate functions of σ^r with respect to Y_r. Assume that we have found the induced forms

$$\eta^*_r\theta^i = \omega^i,\ \eta^*_r\theta^i_j, \ldots, \eta^*_r\theta^i_{j_1\ldots j_{r-1}}\,. \tag{33}$$

Since $\eta_r i = i_r\tilde{\eta}_r$, we obtain the forms $\tilde{\eta}^*_r\tilde{\theta}^i_j, \ldots, \tilde{\eta}^*_r\tilde{\theta}^i_{j_1\ldots j_r}$ on $\hat{Q}(V)$ when replacing ω^α by $\tilde{\omega}^\alpha$ in (33). Then Proposition 2 implies immediately the following assertion, which gives an algorithm for finding the functions (32).

Proposition 4. *The coordinate functions* (32) *of the r-th cap field of a submanifold V with respect to* $Y_r \in H^r_c(M)$ *satisfy*

$$\tilde{\omega}^J = \tilde{a}^J_p\tilde{\omega}^p\,, \tag{34}$$

$$\mathrm{d}\tilde{a}^J_p + \Phi^J_p(\tilde{a}^J_p, \tilde{\eta}^*_r\tilde{\theta}^i_j) = \tilde{a}^J_{pq}\tilde{\theta}^q\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathrm{d}\tilde{a}^J_{p_1\ldots p_{r-1}} + \Phi^J_{p_1\ldots p_{r-1}}(\tilde{a}^J_p, \ldots, \tilde{a}^J_{p_1\ldots p_{r-1}}, \tilde{\eta}^*_r\tilde{\theta}^i_j, \ldots, \tilde{\eta}^*_r\tilde{\theta}^i_{j_1\ldots j_{r-1}}) = \tilde{a}^J_{p_1\ldots p_r}\tilde{\omega}^{p_r}\,.$$

Obviously, if we restrict all quantities of (34) to a fibre $Q_x(V)$, $x \in V$, then we obtain the equations of the fundamental distribution on $Q_x(V) \times K^r_{n,m}$.

3. Special homogeneous spaces

The considerations of § 2 can be summarized to a simple direct algorithm for those homogeneous spaces, whose structure equations in a suitable basis ω^α of $\mathfrak{g}^*$ have a special form without any products $\omega^i \wedge \omega^j$ in (25), i.e.

$$\text{(35)} \qquad \begin{aligned} d\omega^i &= c^i_{j\lambda}\omega^j \wedge \omega^\lambda\,, \\ d\omega^\lambda &= c^\lambda_{j\mu}\omega^j \wedge \omega^\mu + \tfrac{1}{2}c^\lambda_{\mu\nu}\omega^\mu \wedge \omega^\nu\,. \end{aligned}$$

This is equivalent to the fact that $\omega^\lambda = 0$ is an Abelian subgroup $K \subset G$. In other words, the localization of G to M is a flat pseudogroup. (One verifies directly that the structure equations of n-dimensional Euclidean, affine and projective spaces as well as of the spaces of their linear submanifolds are of the type (35)). In this case, one deduces easily from the structure equations of the canonical form θ_r of $H^r(M)$ that there is an element $Y_r \in H^r_c(M)$ such that

$$\text{(36)} \qquad \eta_r^*\theta^i = \omega^i\,, \quad \eta_r^*\theta^i_j = c^i_{j\lambda}\omega^\lambda\,,$$
$$\eta_r^*\theta^i_{jk} = c^i_{j\lambda}c^\lambda_{k\mu}\omega^\mu, \ldots, \eta_r^*\theta^i_{j_1\ldots j_{r-1}} = c^i_{j_1\lambda_1}c^{\lambda_1}_{j_2\lambda_2}\ldots c^{\lambda_{r-2}}_{j_{r-1}\lambda_{r-1}}\omega^{\lambda_{r-1}}\,;$$

this Y_r corresponds to the values $a^i_{j_1\ldots j_s} = 0$ for all $s = 2, \ldots, r$. We shall show that Y_r is the r-jet of a simple coordinate system on M. Consider the canonical coordinates on the group K determined by the basis e_i of its Lie algebra in a neighbourhood U of e. Assume that U is sufficiently small so that this coordinate system on U is projected by the bundle projection of $G(M, H)$ into a coordinate system $\varkappa$ on M in a neighbourhood of c.

Proposition 5. *The element Y_r corresponding to* (36) *is the r-jet of $\varkappa$ at* 0, *i.e.*

$$\text{(37)} \qquad Y_r = j^r_0\varkappa\,.$$

Proof. Equations (35) are equivalent to

$$\text{(38)} \qquad [e_i, e_j] = 0\,, \quad [e_j, e_\lambda] = -c^i_{j\lambda}e_i - c^\mu_{j\lambda}e_\mu\,, \quad [e_\lambda, e_\mu] = -c^\nu_{\lambda\mu}e_\nu\,.$$

If we denote by X_i, X_λ the vector fields on M determined by e_i, e_λ, then

$$\text{(39)} \qquad [X_i, X_j] = 0\,, \quad [X_j, X_\lambda] = c^i_{j\lambda}X_i + c^\mu_{j\lambda}X_\mu\,, \quad [X_\lambda, X_\mu] = c^\nu_{\lambda\mu}X_\nu\,.$$

Let x^i be the coordinates of $\varkappa$ on M. Since K is an Abelian group, it is

$$X_i = \frac{\partial}{\partial x^i}. \tag{40}$$

Consider the power series expansions

$$X_\lambda = \left(a^i_{j\lambda} x^j + \ldots + \frac{1}{r!} a^i_{j_1 \ldots j_r \lambda} x^{j_1} \ldots x^{j_r} + \ldots\right) \frac{\partial}{\partial x^i}. \tag{41}$$

The coefficients $a^i_{j\lambda}, \ldots, a^i_{j_1 \ldots j_r \lambda}$ can be determined as follows. On the one hand, we find by (40) and (41)

$$([X_{j_1}, [\ldots, [X_{j_{r-1}}, [X_{j_r}, X_\mu]] \ldots]])_0 = a^i_{j_1 \ldots j_r \mu} (X_i)_0 . \tag{42}$$

On the other hand, (39) implies

$$([X_{j_1}, [\ldots, [X_{j_{r-1}}, [X_{j_r}, X_\mu]] \ldots]])_0 = c^i_{j_1 \lambda_1} c^{\lambda_1}_{j_2 \lambda_2} \ldots c^{\lambda_{r-1}}_{j_r \mu} (X_i)_0 . \tag{43}$$

Comparing (42) and (43), we obtain

$$a^i_{j\mu} = c^i_{j\mu}, \quad a^i_{j_1 \ldots j_s \mu} = c^i_{j_1 \lambda_1} c^{\lambda_1}_{j_2 \lambda_2} \ldots c^{\lambda_{s-1}}_{j_s \mu} \quad \text{for} \quad s \geqq 2 . \tag{44}$$

Further, let $J^r_c\, T(M)$ be the space of all r-jets at $c \in M$ of the cross sections of the tangent bundle of M. According to a theorem by Libermann, [10], Proposition 1, there is a natural identification of $J^r_c\, T(M)$ and $T_{Y_r}(H^r(M))$. Denote by $E_i, E^j_i, \ldots \ldots, E^{j_1 \ldots j_r}_i$ the vectors of $T_{Y_r}(H^r(M))$ corresponding to the r-jets of the vector fields

$$\frac{\partial}{\partial x_i}, \; x^j \frac{\partial}{\partial x^i}, \; \ldots, \; \frac{1}{r!} x^{j_1} \ldots x^{j_r} \frac{\partial}{\partial x^i}.$$

Then (41) and (44) imply

$$\begin{aligned} \eta_{r*} e_i &= E_i , \\ \eta_{r*} e_\mu &= c^i_{j\mu} E^j_i + \ldots + c^i_{j_1 \lambda_1} c^{\lambda_1}_{j_2 \lambda_2} \ldots c^{\lambda_{r-1}}_{j_r \mu} E^{j_1 \ldots j_r}_i . \end{aligned} \tag{45}$$

On the other hand, let θ_r be the canonical form of $H^r(M)$. If $v \in T_{Y_r}(H^r(M))$, $v = v^i E_i + v^i_j E^j_i + \ldots + v^i_{j_1 \ldots j_r} E^{j_1 \ldots j_r}_i$, then

$$\theta_r(v) = (v^i, v^i_j, \ldots, v^i_{j_1 \ldots j_{r-1}}) , \tag{46}$$

see [7]. Comparing (45), (46) and (36), we prove Proposition 5.

Hence we have justified the following direct algorithm for finding the coordinate functions $\tilde{a}^J_p, \ldots, \tilde{a}^J_{p_1 \ldots p_r} : \hat{Q}(V) \to \mathbf{R}$ of the r-th cap field σ^r of V with respect to $j^r_0\varkappa$ (shortly: with respect to $\varkappa$). First of all, we have

$$\tilde{\omega}^J = \tilde{a}^J_p \tilde{\omega}^p . \tag{47}$$

The exterior differentiation of (47) yields

$$[\mathrm{d}\tilde{a}^J_p + \Psi^J_p(\tilde{a}^J_p, \tilde{\omega}^\lambda)] \wedge \tilde{\omega}^p = 0 ,$$

where $\Psi^J_p(\tilde{a}^J_p, \tilde{\omega}^\lambda) = \Phi^J_p(\tilde{a}^J_p, c^i_{j\lambda}\tilde{\omega}^\lambda)$. Let π^λ be the components of the canonical $\mathfrak{h}$-valued form on $Q_x(V)$, $x \in V$. Then

$$\mathrm{d}y^J_p + \Psi^J_p(y^J_p, \pi^\lambda) = 0 \tag{48}$$

are the equations of the fundamental distribution on $Q_x(V) \times K^1_{n,m}$. Further, it holds

$$\mathrm{d}\tilde{a}^J_p + \Psi^J_p(\tilde{a}^J_p, \tilde{\omega}^\lambda) = \tilde{a}^J_{pq}\tilde{\omega}^q . \tag{49}$$

Assume by induction that in the $(s - 1)$-st step of this procedure we have deduced the relation

$$\mathrm{d}\tilde{a}^J_{p_1 \ldots p_{s-1}} + \Psi^J_{p_1 \ldots p_{s-1}}(\tilde{a}^J_p, \ldots, \tilde{a}^J_{p_1 \ldots p_{s-1}}, \tilde{\omega}^\lambda) = \tilde{a}^J_{p_1 \ldots p_s}\tilde{\omega}^{p_s} . \tag{50}$$

The exterior differentiation of (50) yields

$$[\mathrm{d}\tilde{a}^J_{p_1 \ldots p_s} + \Psi^J_{p_1 \ldots p_s}(\tilde{a}^J_p, \ldots, \tilde{a}^J_{p_1 \ldots p_s}, \tilde{\omega}^\lambda)] \wedge \tilde{\omega}^{p_s} = 0 , \tag{51}$$

where $\Psi^J_{p_1 \ldots p_s}$ are certain linear combinations of $\tilde{\omega}^\lambda$ with coefficients in $\tilde{a}^J_p, \ldots, \tilde{a}^J_{p_1 \ldots p_s}$. Then

$$\begin{gathered} \mathrm{d}y^J_p + \Psi^J_p(y^J_p, \pi^\lambda) = 0 , \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \mathrm{d}y^J_{p_1 \ldots p_{s-1}} + \Psi^J_{p_1 \ldots p_{s-1}}(y^J_p, \ldots, y^J_{p_1 \ldots p_{s-1}}, \pi^\lambda) = 0 , \\ \mathrm{d}y^J_{p_1 \ldots p_s} + \Psi^J_{p_1 \ldots p_s}(y^J_p, \ldots, y^J_{p_1 \ldots p_s}, \pi^\lambda) = 0 \end{gathered} \tag{52}$$

are the equations of the fundamental distribution on $Q_x(V) \times K^s_{n,m}$ and it holds, moreover,

$$\mathrm{d}\tilde{a}^J_{p_1 \ldots p_s} + \Psi^J_{p_1 \ldots p_s}(\tilde{a}^J_p, \ldots, \tilde{a}^J_{p_1 \ldots p_s}, \tilde{\omega}^\lambda) = \tilde{a}^J_{p_1 \ldots p_s q}\tilde{\omega}^q . \tag{53}$$

Remark 2. We should like to emphasize that the values $a^J_p(u), \ldots, a^J_{p_1 \ldots p_r}(u)$, $u \in \hat{Q}(V)$, are the coefficients of the power series expansions of the equations (with respect to $\varkappa$) of V in the frame u. Though this fact is of great practical importance, it has not been known.

Remark 3. We shall show in the appendix that equations (52) can be used for an analytic construction of the equivariant mappings of the H-space $K^r_{m,c}(M)$. Starting from an analysis of some computational procedures by Laptěv, [9], we define a geometric object of order r for m-dimensional submanifolds of M (shortly: a geometric m^r-object on M) as an equivariant mapping μ of the H-space $K^r_{m,c}(M)$ into another H-space W. Denote by E the associated fibre bundle $(M, W, H, G(M, H))$. Further, let $E(V)$ be the restriction of E over an m-dimensional submanifold $V \subset M$. The mapping μ is extended to a base-preserving morphism $\mu_2 : K^r_m(M) \to E$, [6]. If σ^r is the r-th cap field of V, then the composition $\mu_2\sigma^r : V \to E(V)$ will be said to be the value of μ_2 on V. Let z^A be some local coordinates on W and let $z^A = z^A(y^J_p, \ldots, y^J_{p_1\ldots p_r})$ be the coordinate expression of μ. Then $z^A(\tilde{a}^J_p, \ldots, \tilde{a}^J_{p_1\ldots p_r})$ are the coordinate functions of the value $\mu_2\sigma^r$ of μ on V (where $\tilde{a}^J_p, \ldots, \tilde{a}^J_{p_1\ldots p_r}$ are the coordinate functions of σ^r).

4. Remarks on the specialization of frames

A simple case of specialization of frames is based on the following well known fact. In general, let $Q(B, H, \pi)$ be a principal fibre bundle, let H act transitively on the left on a manifold F, let ϱ be a cross section of the associated fibre bundle (B, F, H, Q) and let $p \in F$ be a point. Then

$$Q_1 = \{u \in Q;\ u^{-1}(\varrho(\pi(u))) = p\} \tag{54}$$

is a reduction of Q to the stability group H_1 of p. We shall say that Q_1 is the reduction determined by the pair (ϱ, p). Let y^A, $A = 1, \ldots, \dim F$, be some local coordinates on F, let π^λ be a basis of $\mathfrak{h}$ and let $\mathrm{d}y^A + \eta^A_\lambda(y^A)\,\pi^\lambda = 0$ be the equations of the fundamental distribution on $H \times F$. Further, let a^A be the coordinate functions of ϱ, [8], and let y^A_0 be the coordinates of p. Then $Q_1 \subset Q$ is characterized by

$$a^A = y^A_0\,. \tag{55}$$

One finds also directly that the differential equations of the stability group H_1 of p are

$$\eta^A_\lambda(y^A_0)\,\pi^\lambda = 0\,. \tag{56}$$

If one studies a submanifold V, then it is natural to take a cap field of V of a convenable order for the above field ϱ. For instance, assume that H acts transitively on $K^1_{m,c}(M)$ and denote by $p_1 \in K^1_{m,c}(M)$ the element $y^J_p = 0$. Then the reduction $Q_1(V) \subset Q(V)$ determined by the pair (σ^1, p_1) is usually called the frame field of the first order of V (while $Q(V)$ itself is sometimes said to be the frame field of order zero of V). The reduction $Q_1(V)$ is characterized by $a_p^J = 0$. In other words, if $\tilde{\bar{\omega}}^\alpha$ denotes the restriction of ω^α to $Q^1(V)$, then $\tilde{\bar{\omega}}^J = 0$. Such a specialization of frames is practically inevitable for a concrete evaluation and was used by Laptěv and his disciples in all their investigations. Naturally, it is convenient to specialize the frames further if possible.

However, if one constructs a reduction $\bar{Q}(V)$ of $Q(V)$ to a subgroup $\bar{H}$ of H in the above manner and if one continues in the prolongation procedure on $\bar{Q}(V)$, then one obtains the equations of the fundamental distribution on an $\bar{H}$-invariant subspace S of $K^r_{m,c}(M)$ only. Hence the method explained in the appendix of the present paper enables us to construct the $\bar{H}$-equivariant mappings of S only. Nevertheless, every $\bar{H}$-equivariant mapping of S can be naturally extended to an H-equivariant mapping of $K^r_{m,c}(M)$ as follows. In general, consider a homogeneous space F with a fundamental group H and denote by $\bar{H}$ the stability group of a point $p \in F$. Let F_1 be another H-space and let $\lambda : F_1 \to F$ be an equivariant surjection. Set $F_0 = \lambda^{-1}(p)$, which is an $\bar{H}$-space. Consider another $\bar{H}$-space $\bar{F}_0$ and an $\bar{H}$-equivariant mapping $\varphi : F_0 \to \bar{F}_0$. As H, has a natural structure of a principal fibre bundle $H(F, \bar{H})$, we can construct an associated fibre bundle $\bar{F}_1$ of the symbol $(F, \bar{F}_0, H, H(F, \bar{H}))$. Since every element of $\bar{F}_1$ is an equivalence class $\{(h, y)\}$, $h \in H$, $y \in F_0$, with respect to the equivalence relation $(h, y) \sim (h\bar{h}, \bar{h}^{-1}y)$, $\bar{h} \in \bar{H}$, we introduce a left action of H on $\bar{F}_1$ by $h'\{(h, y)\} = \{(h'h, y)\}$, $h' \in H$. This definition is correct, since $h'\{(h\bar{h}, \bar{h}^{-1}y)\} = \{(h'h\bar{h}, \bar{h}^{-1}y)\} = \{(h'h, y)\}$. Then we define an H-equivariant mapping $\tilde{\varphi} : F_1 \to \bar{F}_1$ by $\tilde{\varphi}(hy) = h\,\varphi(y)$, $y \in F_0$. Even this is a correct definition, since $hy = h'y'$, $y, y' \in F_0$ implies $h^{-1}h' \in \bar{H}$, so that $\tilde{\varphi}(h'y') = h'\,\varphi(y') = hh^{-1}h'\,\varphi(y') = h\,\varphi(y)$. The H-equivariant mapping $\tilde{\varphi} : F_1 \to \bar{F}_1$ is the above mentioned natural extension of an $\bar{H}$-equivariant mapping $\varphi : F_0 \to \bar{F}_0$.

5. Submanifolds of a space with fundamental Lie pseudogroup

Let Γ be a transitive Lie pseudogroup on a manifold M, [2]. Let $\Pi^r(\Gamma)$ be the groupoid of all r-jets of the transformations of Γ and let G^r_x denote the isotropy group of $\Pi^r(\Gamma)$ over $x \in M$. Obviously, $\Pi^r_x(\Gamma) = \{X \in \Pi^r(\Gamma);\ \alpha X = x\}$ is a principal fibre bundle over M with the structure group G^r_x. Fix a local chart $\varkappa$ on M with the centre x. Then

$$H^r(\Gamma, \varkappa) = \{X(j^r_0\varkappa);\ X \in \Pi^r_x(\Gamma)\}$$

is a reduction of $H^r(M)$ to a subgroup $G^r \subset L^r_n$ isomorphic to $\Pi^r_x(\Gamma)$. Let φ_{r+1} be the restriction of the canonical form θ_{r+1} of $H^{r+1}(M)$ to $H^{r+1}(\Gamma, \varkappa)$. According to [4], $\varphi_{r+1}(T_u(H^{r+1}(\Gamma, \varkappa)))$ is a fixed subspace $V_r \subset \mathbf{R}^n \oplus \mathfrak{l}^r_n$ for all $u \in H^{r+1}(\Gamma, \varkappa)$. Since $H^r_y(\Gamma, \varkappa)$, $y \in M$, is a fibre of a principal fibre bundle with the structure group G^r, there is a canonical $\mathfrak{g}^r$-valued form Ψ_r on $H_y^r(\Gamma, \varkappa)$. Further, let $(\varphi_{r+1})_y$ denote the restriction of φ_{r+1} to $H_y^{r+1}(\Gamma, \varkappa)$.

Lemma 3. *The values of $(\varphi_{r+1})_y$ lie in $\mathfrak{g}^r$ and the following diagram commutes.*

$$
\begin{tikzcd}
T(H_y^{r+1}(\Gamma, \varkappa)) \arrow[d, "j_{r+1*}^r"'] \arrow[dr, "(\varphi_{r+1})_y"] & \\
T(H_y^r(\Gamma, \varkappa)) \arrow[r, "\Psi_r"] & \mathfrak{g}^r
\end{tikzcd}
\tag{57}
$$

Proof. Let $X \in T_u(H_y^{r+1}(\Gamma, \varkappa))$, $u = j_0^{r+1}\mu$, $X = j_0^1\,\gamma(t)$, where γ is a curve on $H_y^{r+1}(\Gamma, \varkappa)$. Put $\bar{u} = j_{r+1}^r u$, $\bar{\gamma}(t) = j_{r+1}^r\,\gamma(t)$, $\bar{X} = j_{r+1*}^r X$. By the definition of θ_{r+1}, it is $\varphi_{r+1}(X) = \theta_{r+1}(X) = j_0^1[\mu^{-1}\,\bar{\gamma}(t)]$. Since y is the target of every $\gamma(t)$, we have $\mu^{-1}\,\bar{\gamma}(t) = \bar{u}^{-1}\,\bar{\gamma}(t)$. On the other hand, $\Psi_r(\bar{X}) = j_0^1[\bar{u}^{-1}\,\bar{\gamma}(t)]$ by the definition of Ψ_r, QED.

We find a more specific situation for flat pseudogroups. We recall that Γ is said to be flat if, for every $y \in M$, there exists a local chart $\varkappa$ on M in a neighbourhood of y such that the $\varkappa$-images of all translations in $\mathbf{R}^n$ belong to Γ. A coordinate system having this property will be also said to be flat.

Lemma 4. *If Γ is a flat pseudogroup and $\varkappa$ is a flat coordinate system, then $V_r = \mathbf{R}^n \oplus \mathfrak{g}^r$.*

Proof. Let $X \in T_u(H^{r+1}(\Gamma, \varkappa))$, $X = j_0^1\,\gamma(t)$. Hence $\gamma(t) = j_0^{r+1}\gamma_t\varkappa$, $\gamma_t \in \Gamma$, and $u = j_0^{r+1}\mu\varkappa$, $\mu \in \Gamma$. By the definition of θ_{r+1}, we have $\varphi_{r+1}(X) = j_0^1[\varkappa^{-1}\mu^{-1}(j_{r+1}^r\,\gamma(t))] = j_0^1[j_0^r(\varkappa^{-1}\mu^{-1}\gamma_t\varkappa)]$, where $j_0^r(\varkappa^{-1}\mu^{-1}\gamma_t\varkappa)$ is a curve on $H^r(\mathbf{R}^n)$. The principal fibre bundle $H^r(\mathbf{R}^n)$ is identified with $\mathbf{R}^n \times L_n^r$ by means of the projections $\beta : H^r(\mathbf{R}^n) \to \mathbf{R}^n$ and $\pi : H^r(\mathbf{R}^n) \to L_n^r$, $X \mapsto \tau_{\beta X}^{-1}X$. Set $\beta(j_0^r(\varkappa^{-1}\mu\gamma_t\varkappa)) = \varrho(t)$. Then

$$
\pi(j_0^r(\varkappa^{-1}\mu\gamma_t\varkappa)) = j_0^r(\tau_{\varrho(t)}^{-1}\varkappa^{-1}\mu^{-1}\gamma_t\varkappa)\,. \tag{58}
$$

Since $\varkappa$ is flat, $\varkappa\tau_{\varrho(t)}\varkappa^{-1} \in \Gamma$ and (58) can be written as $j_0^r[\varkappa^{-1}(\varkappa\tau_{\varrho(t)}^{-1}\varkappa^{-1}\mu^{-1}\gamma_t)\,\varkappa]$, which is a curve on G^r. We have thus proved $V_r \subset \mathbf{R}^n \oplus \mathfrak{g}^r$. The converse inclusion can be deduced quite similarly, QED.

Consider now an m-dimensional submanifold V of M. Let $Q^{r+1}(V, \Gamma, \varkappa)$ be the restriction of $H^{r+1}(\Gamma, \varkappa)$ over V. Then $K^{r+1}(V)$ is also an associated fibre bundle of the symbol $(V, K_{n,m}^{r+1}, G^{r+1}, Q^{r+1}(V, \Gamma, \varkappa))$. Put $\hat{Q}^{r+1}(V, \Gamma, \varkappa) = H^{r+1}(\Gamma, \varkappa) \cap \cap\, \hat{Q}^{r+1}(V)$ and denote by $\lambda : \hat{Q}^{r+1}(V, \Gamma, \varkappa) \to \hat{Q}^{r+1}(V)$ the injection. If $a_p^J, \ldots, a_{p_1 \ldots p_{r+1}}^J$ are the coordinate functions of the $(r + 1)$-st cap field σ^{r+1} of $Q^{r+1}(V)$, then

$$
\tilde{a}_p^J = \lambda^* a_p^J, \ldots, \tilde{a}_{p_1 \ldots p_{r+1}}^J = \lambda^* a_{p_1 \ldots p_{r+1}}^J \tag{59}
$$

are the coordinate functions of σ^{r+1} on $\hat{Q}^{r+1}(V, \Gamma, \varkappa)$. Then (14) and (15) imply

(60)
$$\tilde{\varphi}^J = \tilde{a}^J_p \tilde{\varphi}^p ,$$
$$\mathrm{d}\tilde{a}^J_p + \Phi^J_p(\tilde{a}^J_p, \tilde{\varphi}^i_j) = \tilde{a}^J_{pq}\tilde{\varphi}^q ,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mathrm{d}\tilde{a}^J_{p_1 \ldots p_r} + \Phi^J_{p_1 \ldots p_r}(\tilde{a}^J_p, \ldots, \tilde{a}^J_{p_1 \ldots p_r}, \tilde{\varphi}^i_j, \ldots, \tilde{\varphi}^i_{j_1 \ldots j_r}) = \tilde{a}^J_{p_1 \ldots p_r q}\tilde{\varphi}^q .$$

If $\Psi^i_j, \ldots, \Psi^i_{j_1 \ldots j_r}$ are the components of Ψ_r, then the equations of the fundamental distribution on $H^r_y(\Gamma, \varkappa) \times K^r_{n,m}$ are

(61)
$$\mathrm{d}y^J_p + \Phi^J_p(y^J_p, \Psi^i_j) = 0 ,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\mathrm{d}y^J_{p_1 \ldots p_r} + \Phi^J_{p_1 \ldots p_r}(y^J_p, \ldots, y^J_{p_1 \ldots p_r}, \Psi^i_j, \ldots, \Psi^i_{j_1 \ldots j_r}) = 0 .$$

Moreover, if Γ and $\varkappa$ are flat, then even the values of $(\tilde{\varphi}^i_j, \ldots, \tilde{\varphi}^i_{j_1 \ldots j_r})$ lie in $\mathfrak{g}^r$ by Lemma 4.

Thus, if we want to use the invariant method to the investigation of submanifolds of a space with a fundamental Lie pseudogroup Γ, we must first find the Lie algebra $\mathfrak{g}^r$ of G^r. For this purpose, the following relation between $\mathfrak{g}^r_x$ and the sheaf $\mathscr{L}$ of germs of the infinitesimal transformations of Γ can be sometimes used. Let $\mathscr{L}_x$ be the stalk of $\mathscr{L}$ over $x \in M$ and let $\mathscr{L}^i_x$ be the kernel of the jet projection $j^r_x : \mathscr{L}_x \to \to J^r_x\, T(M)$, [10], [12]. The space $\mathscr{L}^0_x/\mathscr{L}^r_x$ has the following Lie algebra structure. The elements of $\mathscr{L}^0_x/\mathscr{L}^r_x$ are of the form $j^r_x\xi$, where $\xi \in \mathscr{L}^0_x$, so that $\xi(x) = 0 \in T_x(M)$. If $\xi, \eta \in \mathscr{L}^0_x$, then $j^r_x([\xi, \eta])$ is quite determined by $j^r_x\xi$ and $j^r_x\eta$. This defines the bracket operation in $\mathscr{L}^0_x/\mathscr{L}^r_x$. We further introduce a mapping $\mathscr{L}^0_x/\mathscr{L}^r_x \to \mathfrak{g}^r_x$ as follows. Let $\xi \in \mathscr{L}^0_x$ be the germ of a Γ-field X defined in a neighbourhood $U \subset M$ of x. The field X is prolonged to a field X^r on $\beta^{-1}(U) \subset \Pi^r_x(\Gamma)$, [4]. Since $X(x) = 0$, the restriction of X^r to the fibre G^r_x of $\Pi^r_x(\Gamma)$ is tangent to G^r_x and one finds easily that this is a left invariant vector field on G^r_x. This field will be denoted by $i(j^r_x X)$.

Lemma 5. *The mapping* $i : \mathscr{L}^0_x/\mathscr{L}^r_x \to \mathfrak{g}^r_x$ *is a Lie algebra isomorphism.*

Proof. This follows directly from the fact that the mapping $X \mapsto X^r$ is bracket-preserving, see e.g. [13].

In particular, Lemma 5 can be used to determine $\mathfrak{g}^r$ in the case of a flat pseudogroup Γ of the first order. Let $\mathfrak{g} \subset \mathfrak{l}^1_n$ be the Lie algebra of G^1. Then the standard stalk of the sheaf of germs of the infinitesimal transformations of Γ is of the form

$$\mathbf{R}^n + \mathfrak{g} + p(\mathfrak{g}) + \ldots + p^r(\mathfrak{g}) + \ldots ,$$

where $p^r(\mathfrak{g})$ means the r-th prolongation of $\mathfrak{g}$, [12]. Hence Lemma 5 implies

(62)
$$\mathfrak{g}^r = \mathfrak{g} + p(\mathfrak{g}) + \ldots + p^{r-1}(\mathfrak{g}) .$$

For example, our results show how to use the invariant method for the investigation of real submanifolds of a complex n-dimensional manifold, since the pseudogroup of all holomorphic transformations on the underlying real $2n$-dimensional manifold is a flat pseudogroup of the first order. Naturally, even here it is useful to apply a convenient specialization of frames as explained in § 4.

Remark 4. Analogously to Remark 3, we define a geometric m^r-object on M as an equivariant mapping of a G^r-space $K^r_{n,m}$, [6]. Then the construction of the induced geometric object fields on m-dimensional submanifolds of M is quite similar to that of Remark 3.

Appendix. An analytic construction of equivariant mappings

Let $\mathscr{T}(F)$ be the Lie algebra of all vector fields on a manifold F and let H be a connected Lie group. We define a right infinitesimal action of H on F as a homomorphism of the Lie algebra $\mathfrak{h}$ of H into $\mathscr{T}(F)$, while a left infinitesimal action is introduced as an antihomomorphism of $\mathfrak{h}$ into $\mathscr{T}(F)$. Every left or right action of H on F determines a left or right infinitesimal action of H on F respectively; the converse problem is treated in [11]. In the sequel, we shall investigate the left infinitesimal actions only. Let Ψ or $\bar{\psi}$ be an infinitesimal action of H on a manifold F or $\bar{F}$ respectively and let $\varphi : F \to \bar{F}$ be a mapping. We shall say that Ψ and $\bar{\psi}$ are φ-related, if the vector fields $\Psi(X)$ and $\bar{\psi}(X)$ are φ-related for every $X \in \mathfrak{h}$.

Let F and $\bar{F}$ be two manifolds such that there are some global coordinates y^i on F and z^p on $\bar{F}$ and let $\varphi : F \to \bar{F}$ be a mapping with a coordinate expression

$$(63) \qquad z^p = z^p(y^i)\,, \quad i, j = 1, \ldots, \dim F\,, \quad p, q = 1, \ldots, \dim \bar{F}\,.$$

Let $Y = \eta^i(y^j)\,(\partial/\partial y^i)$ or $Z = \zeta^p(z^q)\,(\partial/\partial z^p)$ be a vector field on F or $\bar{F}$ respectively. Then Y and Z are φ-related if and only if

$$(64) \qquad \frac{\partial z^p(y^j)}{\partial y^i}\,\eta^i(y^j) = \zeta^p(z^q(y^j))$$

for every $y \in F$. Conversely, if Y is a given vector field on F, then there exists a vector field φ-related with Y if and only if the expressions $(\partial z^p(y^j)/\partial y^i)\,\eta^i(y^j)$ can be written in the form $\zeta^p(z^q(y^j))$, where ζ^p are functions on $\bar{F}$. If φ is surjective, then the latter field is uniquely determined.

Consider now an infinitesimal action Ψ of H on F and a mapping $\varphi : F \to \bar{F}$ of the form (63). Let e_λ be a basis of $\mathfrak{h}$ and let

$$(65) \qquad \Psi(e_\lambda) = \eta^i_\lambda(y^j)\,\frac{\partial}{\partial y^i}\,.$$

We shall first investigate whether there exist vector fields φ-related with (65). According to (64), we may proceed as follows. Write formally the relations

$$(66) \qquad \mathrm{d}y^i + \eta^i_\lambda(y^j)\,\pi^\lambda = 0\,.$$

Differentiating (63) and replacing $\mathrm{d}y^i$ according to (66), we obtain

$$(67) \qquad \mathrm{d}z^p + \frac{\partial z^p(y^j)}{\partial y^i}\,\eta^i_\lambda(y^j)\,\pi^\lambda = 0\,.$$

If (67) can be written in the form

$$(68) \qquad \mathrm{d}z^p + \zeta^p_\lambda(z^q(y^i))\,\pi^\lambda = 0\,,$$

then the vector fields $\zeta^p_\lambda(z^q)\,(\partial/\partial z^p)$ on $\bar{F}$ are φ-related with (65). Moreover, if φ is surjective, then one finds easily that $\bar{\psi} : \mathfrak{h} \to \mathscr{T}(F)$, $v^\lambda e_\lambda \mapsto v^\lambda\,\eta^p_\lambda(z^q)\,(\partial/\partial z^p)$ is an infinitesimal action of H on F. Comparing with [9], p. 301, we see that we have explained the foundations of a procedure due to Laptěv. Starting from the above facts, one can develop a practical procedure for finding the pairs $(\varphi, \bar{\psi})$ to a given infinitesimal action Ψ in the same way as in [9].

The above local construction can be sometimes globalized by virtue of the following simple proposition. We recall that an infinitesimal action of H on F is said to be proper, if it is determined by an action (i.e. global action) of H on F, [11].

Proposition 6. *Let $\varphi : F \to \bar{F}$ be a surjective mapping and let Ψ or $\bar{\psi}$ be an infinitesimal action of H on F or $\bar{F}$ respectively. Assume that Ψ and $\bar{\psi}$ are φ-related. If Ψ is proper, then $\bar{\psi}$ is also proper and φ is an equivariant mapping of the corresponding H-spaces.*

Proof. We shall use freely the terminology and the results of [11]. Let Ψ or $\bar{\Psi}$ be the infinitesimal graph of Ψ or $\bar{\psi}$ respectively and let $\Phi = \mathrm{id} \times \varphi : H \times F \to \to H \times \bar{F}$. Since Ψ and $\bar{\psi}$ are φ-related, the differential of Φ maps $\Psi_{(h,p)}$ bijectively onto $\bar{\Psi}_{(h,\varphi(p))}$ for every $h \in H$, $p \in F$. Hence the leaves of Ψ are transformed into the leaves of $\bar{\Psi}$ and the restriction of Φ to a leaf of Ψ is a local diffeomorphism into the corresponding leaf of $\bar{\Psi}$. Let $\pi : H \times F \to H$ and $\bar{\pi} : H \times \bar{F} \to H$ be the product projections. By Corollary 3 to Theorem XII of Chapter III of [11], the restriction of π to the leaf of Ψ containing (e, p) is bijective for every $p \in F$. Conversely, by the same Corollary, if the restriction of $\bar{\pi}$ to the leaf $\bar{\lambda}$ or $\bar{\Psi}$ containing $(e, \bar{p})$ is bijective for every $\bar{p} \in \bar{F}$, then $\bar{\psi}$ is proper. Consider a point $p \in F$ such that $\varphi(p) = \bar{p}$ and denote by λ the leaf of Ψ passing through (e, p). Since we have $\pi(a) = \pi(\Phi(a))$ for every $a \in H \times F$ and the restriction of π to λ is bijective, it suffices to deduce $\Phi(\lambda) = \bar{\lambda}$. Since the restriction of Φ to λ is a local diffeomorphism into $\bar{\lambda}$, $\Phi(\lambda)$ is an open subset of $\bar{\lambda}$. Further, let μ be another leaf of Ψ such that $\Phi(\lambda) \cap \Phi(\mu) \neq \emptyset$.

Then the set of all points $a \in \lambda$ satisfying $\Phi(a) \in \Phi(\mu)$ is both open and closed in λ. But λ is connected, so that $\Phi(\lambda) \subset \Phi(\mu)$. In the same way we find $\Phi(\mu) \subset \Phi(\lambda)$. Hence the complement of $\Phi(\lambda)$ in $\bar{\lambda}$ is a union of open sets. Since $\bar{\lambda}$ is also connected, we deduce $\Phi(\lambda) = \bar{\lambda}$. Finally, the Corollary to Theorem VIII of Chapter III of [11] implies directly that φ is an equivariant mapping of the corresponding H-spaces, QED.

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