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STRUCTURE AND IDEAL THEORY OF COMMUTATIVE SEMIGROUPS

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This is a continuation of the author's work on the ideal-theoretic properties and the structure of commutative semigroups in a series of papers, [2], [4], [6], [7] and [9]. For non-commutative semigroups the relation between maximal ideals and the existence of idempotents was established only for left cancellative semigroups [8]. But in the first section of this paper we could prove these results for commutative semigroups which are not necessarily cancellative semigroups. In [2], we have established several equivalent characterizations of cancellative chained monoids. In sections 2 and 3 we obtain some characterizations of arbitrary chained semigroups. We also develop the ideal theory of Noetherian semigroups. Tamura's concentric condition ($x^{\omega}S = \square$ for every x in a semigroup S) in [11] happens to be a very important concept and some classes of chained semigroups could be characterized completely by this property. Even Noetherian cancellative semigroups have this intrinsic property. This is the subject matter of the Sections 2 and 3. About the structure theory the important theorems in this paper are Theorems 1.5, 2.3, 2.4, 2.10 and 3.6. Throughout this paper all our semigroups are commutative. We refer the reader to [1] and in particular [7] for all the definitions and the theory of primary, prime, maximal ideals and radical of an ideal, etc. We treat the semigroup S itself as a prime ideal and call all prime ideals different from S as proper prime ideals. We recall from [7], that the radical of an ideal A of a semigroup S, denoted by $\sqrt{A}$, is the intersection of all prime ideals that contain A and if A is not contained in any proper prime ideal, then $\sqrt{A} = S$. Also $\sqrt{A} = \{x \in S : x^n \in A \text{ for some integer } n\}$. If a is an element in a semigroup S, then $aS^1 = a \cup aS$ and $\bigcap_{i=1}^{\infty} a^i S = a^{\omega} S$. If A is an ideal of S, then $A^{\omega} = \bigcap_{i=1}^{\infty} A^i$. A maximal ideal M of S is trivial if $S \setminus M$ contains only one element. S is called *Noetherian* if every ascending chain of ideals terminates at a finite stage or equivalently every ideal is of the form $\bigcup_{i=1}^{n} a_i S^1$. M^* denotes the intersection of all maximal ideals and Z denotes the set of all non-cancellable elements, which can be verified easily to be a prime ideal of S. A prime ideal P is called a *mini-*

mal prime divisor of an ideal A if $A \subset P$ and there exists no prime ideal Q such that $A < Q < P$ (here "$<$" denotes the proper inclusion). Monoids are those semigroups containing identity. If S is a cancellative monoid, then it admits a group K of quotients and if $x \in S$, x^{-1} denotes the inverse of x in K.

1. MAXIMAL IDEALS AND IDEMPOTENTS

1.1. Proposition. *If a semigroup S satisfies any one of the following conditions, then it contains idempotents:*

i) S is a globally idempotent semigroup with maximal ideals;

ii) S contains maximal ideals such that each one of which is trivial.

Proof. If M is a maximal ideal in a globally idempotent semigroup, then M is a prime ideal [10; 73]. So, if $a \in S \setminus M$, then $a^2 \in S \setminus M$ and hence $S = M \cup a \cup \cup aS = M \cup a^2 \cup a^2S$. Thus $a \in a^2S^1$, which implies the existence of an idempotent. (ii) is evident from (i) since maximal ideals which are not trivial are prime [3].

1.2. Corollary. *Any cancellative globally idempotent semigroup with maximal ideals contains an identity.*

Proof. By condition (i) of 1.1, S contains an idempotent, say e. Then for any $x \in S$, $ex = e^2x$ and $xe = xe^2$ and so $x = ex = xe$ since S is cancellative. Thus e is the identity of S.

1.3. Proposition. *Let S be an Archimedean semigroup. Then an ideal is maximal iff it is trivial, and S has no maximal ideals if $S = S^2$.*

Proof. If M is a trivial ideal, then evidently M is a maximal ideal. Suppose that M is a non-trivial maximal ideal. Then $S \setminus M$ contains an idempotent e, by 1.1. We claim now that M is a prime ideal, which contradicts the Archimedean property that the only prime ideal is the whole semigroup itself [5; 148]. Suppose $xy \in M$ with $x, y \notin M$. Then $M \cup x \cup xS = M \cup y \cup yS = M \cup eS$. This implies $x = = ex = (y\varepsilon)\, x \in M$, which is a contradiction. The second part follows from the fact that maximal ideals are prime if $S = S^2$ (by [10]).

1.4. Proposition. *For the following semigroups S containing maximal ideals, we have $S \neq S^2$ and $S^2 = M^*$.*

i) S has no idempotents.

ii) S is an Archimedean semigroup.

Proof. Assume (i). Then by condition (i) of 1.1, $S \neq S^2$. By condition (ii) of 1.1, every maximal ideal is trivial. So, if M is a maximal ideal, then $S = M \cup a$, $a \notin M$ and $a \notin S^2$ and $S^2 = MS \cup aS$. Since $a \notin S^2$, $aS \subseteq M$ and hence $S^2 \subseteq M$. Thus

$S^2 \subseteq M^*$. Conversely, let $t \in M^*$. Then $t \notin S \setminus t$. Hence $t \in S^2$. To prove (ii), we observe by 1.3 that $S \neq S^2$. Now if $x \in S^2$ and $x \notin M^*$, then there exists a maximal ideal M not containing x and by 1.3, $M = S \setminus x$ with $x \notin S^2$, which is a contradiction. Thus $S^2 \subseteq M^*$. Clearly, $x \in M^*$ implies $x \notin S \setminus x$ and hence $x \in S^2$.

1.5. Theorem. *Let S be a semigroup with $S \neq Z$. Then the following are equivalent:*

i) Every proper prime ideal is maximal.

ii) S is a cancellative Archimedean semigroup not containing an identity, or S is an extension of an Archimedean semigroup by a group and S contains an identity, or S is a group.

Proof. (i) $\Rightarrow$ (ii). If S contains an identity and if S is not a group, then S contains a unique maximal ideal M, which is also the unique prime ideal by virtue of the hypothesis. Then $\sqrt{(a \cup aS)} = M$ for every $a \in M$. Then for $b \in M$, we have $b^n = a$ or $b^n = as$, $s \in S$ for some positive integer n. Thus $b^{n+1} \in aM$. Hence M is an Archimedean subsemigroup. Clearly $S \setminus M$ is a group. Assume now that S does not contain an identity. If $Z \neq \square$, then there exists a cancellable element b since $Z \neq S$. Since Z is a prime ideal and hence maximal, we have

$$Z \cup b \cup bS = S = Z \cup b^2 \cup b^2 S\,.$$

Therefore $b = b^2$ or $b = b^2 s$ for some $s \in S$, which implies that S contains a cancellable idempotent and hence S contains an identity, which is a contradiction. Thus $Z = \square$. Clearly S does not contain idempotents. We assert now that S has no proper prime ideals, which implies that S is Archimedean [5; 148]. Let P be a proper prime ideal. Then $P \cup xS^1 = S = P \cup x^2 S^1$, where $x \notin P$. This proves, as above, the existence of an idempotent.

(ii) $\Rightarrow$ (i). If S contains an identity and S is an extension of an Archimedean semigroup M by a group G, then clearly M is the maximal ideal of S. Then the Archimedean property of M forces that M is the unique prime ideal of S. The other cases are trivial.

A commutative semigroup S is called a *$\mathscr{U}$-semigroup* if A is an ideal of S with $\sqrt{A} = S$, then $A = S$. For any semigroup S, if $T = \{a \in S \mid \sqrt{(aS^1)} \neq S\}$, then it can be verified easily that T is a prime ideal. Moreover $S \setminus T$ is an Archimedean subsemigroup. For, if $a, b \in S \setminus T$, $\sqrt{(aS^1)} = \sqrt{(bS^1)} = S$. Hence $b^n = a$ or $b^n = as$ for some $s \in S$ and for some positive integer n. Hence $b^{n+1} \in a(S \setminus T)$. Since T is a prime ideal, similarly there exists a positive integer m such that $a^{m+1} \in b(S \setminus T)$. We note that in $\mathscr{U}$-semigroups any proper ideal is contained in a proper prime ideal. In the following we show the importance of this class of semigroups.

1.6. Theorem. *A semigroup S is a $\mathscr{U}$-semigroup iff S has at least one proper prime ideal and if $\{P_\alpha\}$ is the set of all proper prime ideals, then $x \cup xS = S$ for every $x \in S \setminus \bigcup P_\alpha$, or S is a group.*

Proof. Let S be a $\mathscr{U}$-semigroup, which is not a group. If S has no proper prime ideals, then $\sqrt{(a \cup aS)} = S$ for every $a \in S$. This implies $a \cup aS = S$ and hence S is a group. So assume that S has proper prime ideals. Then for any $a \in S \setminus \bigcup P_\alpha$, $\sqrt{(a \cup aS)} = S$ since a does not belong to any proper prime ideal. Thus $a \cup aS = S$. Conversely, if A is an ideal different from S and if $x \in A \setminus \bigcup P_\alpha$, then $x \cup xS = S \subseteq A$. Hence $A \subseteq \bigcup P_\alpha$ and thus $\sqrt{A} \neq S$. Hence S is a $\mathscr{U}$-semigroup.

1.7. Theorem. *The following are equivalent on a semigroup S with maximal ideals*:

i) $S = S^2$.

ii) S is a $\mathscr{U}$-semigroup, or S has a unique maximal ideal which is prime.

Proof. (i) $\Rightarrow$ (ii). Let $T = \{a : \sqrt{(aS^1)} \neq S\}$. If $T = \square$, then for every $a \in S$, $\sqrt{(aS^1)} = S$ and so S has not proper prime ideals. But maximal ideals are prime by [10]. Hence this case is inadmissible. If $T \neq S$, then T is the unique maximal ideal. For, let M be any maximal ideal. Since $S = S^2$, M is a prime ideal and so $\sqrt{M} = M$. Now if $a \in M \setminus T$, then $S = \sqrt{(a \cup aS)} \subseteq \sqrt{M} = M$. Thus $M \subseteq T$ and so $M = T$. The only other possibility is $T = S$, in which case S is a $\mathscr{U}$-semigroup.

(ii) $\Rightarrow$ (i). Since $\sqrt{(S^2)} = S$, we must have $S = S^2$ if S is a $\mathscr{U}$-semigroup. The second assumption implies that $S = S^2$ by Schwarz's result [10].

2. CHAINED SEMIGROUPS

A commutative semigroup is said to be a chained semigroup if its ideals are linearly ordered by set inclusion. Chained semigroups need not have maximal ideals as can be seen in the semigroup $S = \{x_1, x_2, \ldots\}$, where $x_i x_j = x_j x_i = x_{\min(i,j)}$. But if they have maximal ideals, they must have a unique maximal ideal only.

2.1. Theorem. *For a chained semigroup S, the following are true*:

i) If P is a prime ideal, then $P = \bigcap_{n=1}^{\infty} x^n P$ *for every* $x \notin P$.

ii) For any ideal A, $\sqrt{A}$ *is a prime ideal.*

iii) $A = \{a \in S : a^\omega S = \square\}$ *is a prime ideal or empty.*

iv) If S has no idempotents, then for any $a \in S$, $a^\omega S = \square$, *or* $a^\omega S$ *is a prime ideal. Also if S is a cancellative semigroup with an identity, then for every non-unit a,* $a^\omega S = \square$, *or* $a^\omega S$ *is a prime ideal.*

Proof. (i) is evident because, if $x \notin P$, then for every positive integer n, $x^n \notin P$ and so $P \subseteq x^n S^1$. This implies $P = x^n P$, so that $P = \bigcap_{n=1}^{\infty} x^n P$. (ii) follows from the fact that in chained semigroups, every ideal can have at most one minimal prime divisor and so $\sqrt{A}$ is a prime ideal for every ideal A. (iii) follows from a direct verification. To prove (iv), let $a^\omega S \neq \square$ and $xy \in a^\omega S$ with $x, y \notin a^\omega S$. Then $x \notin a^m S$

and $y \notin a^n S$ for some positive integers m and n. Hence we may assume both x and $y \notin a^n S^1$ for some n. Then $a^n S \subseteq xS$ and $a^n S \in yS$ and $a^{4n} = a^{2n}$. $a^{2n} \in (xy) S^1 \subseteq$ $\subseteq a^{8n} S$, which implies the existence of an idempotent. The proof of the last part is similar to the above.

2.2. Theorem. *For a chained semigroup S with $S \neq S^2$, the following are true:*

i) $S = x \cup xS = S^2 \cup x$, $x \notin S^2$ and $S^2 = xS$ is the unique maximal ideal.

ii) If $a \notin x^\omega S$, then $a = x^n$ for $n > 1$. If $a \in x^\omega S$, then $a = x^r$ for some positive integer r, or $a = x^n s_n$ and $s_n \in x^\omega S$ for every positive integer n. If $S \neq Z$, then every element in $x^\omega S$ is of the form x^r.

iii) If $S \neq Z$, then x is a cancellable element and $x^\omega S$ is a prime ideal, or empty.

iv) $S \setminus x^\omega S = \{x, x^2, \ldots\}$ or $S \setminus x^\omega S = \{x, x^2, \ldots, x^r\}$.

Proof. (i) $S \setminus S^2$ contains only one element, say x, since ideals are comparable and $S \setminus x$ is a maximal ideal. By chained condition, $S^2 \subseteq x \cup xS$ and hence $S^2 = xS$. Therefore $S = S^2 \cup x = x \cup xS$.

(ii) Let $a \notin x^\omega S$. Since $x \notin S^2$, $a \neq x$. Since $a \in xS$, we must have $a \in x^{n-1} S \setminus x^n S$ for some positive integer n. Therefore $a = x^{n-1} s$, $s \in xS^1$ and so $a = x^n$. If $a \in x^\omega S$, then $a = x^n s_n$ for every positive integer n. If some $s_i \notin x^\omega S$, from the above, $s_i = x^r$ for some positive integer r and hence a is of the required form.

(iii) If x is a non-cancellable element, then $S = x \cup xS \subseteq Z$ and hence $S = Z$. Now let $a, b \notin x^\omega S$ with $ab \in x^\omega S$. By (ii), $a = x^n$ and $b = x^m$ for some positive integers n and m. Therefore $x^{n+m} \in x^\omega S$ and so $x^{n+m} = x^{n+m+2} s$ for some $s \in S$. Since x is cancellable, we have $x = x^2 s \in S^2$, which is a contradiction.

(iv) Suppose that x^r is the least power contained in $x^\omega S$. Then $x^n \in x^\omega S$ for all $n > r$ and so $S \setminus x^\omega S$ contains $x, x^2, \ldots, x^r$. Thus $S \setminus x^\omega S = \{x, x^2, \ldots, x^r\}$ by (ii). If no power of x is in $x^\omega S$, as above, $S \setminus x^\omega S = \{x, x^2, \ldots\}$.

Now we provide some structure theorems for chained semigroups. Firstly we shall prove a result which is of independent interest.

2.3. Theorem. *The following are equivalent on a semigroup $S = x \cup xS$, $x \in S$:*

i) $S = \{x, x^2, \ldots\}$.

ii) S is a Noetherian cancellative semigroup with $x \notin xS$.

iii) S is a Noetherian cancellative semigroup with no idempotents.

iv) $a^\omega S = \square$ for every $a \in S$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial.

(iii) $\Rightarrow$ (iv). Let $b \in a^\omega S$. Then $b = a^i s_i$ for some $s_i \in S$, which implies $s_i = a s_{i+1}$ for every i, by cancellation property. By Noetherian condition the chain $s_1 S^1 \subseteq$ $\subseteq s_2 S^1 \subseteq \ldots$ terminates. So $s_n S^1 = s_{n+1} S^1$. Hence $s_n = s_{n+1} t = s_n st$ and so st is an idempotent, which is a contradiction.

(iv) $\Rightarrow$ (i). Let $a \neq x^n$ for any n. Then $a = x s_1 = x^2 s_2 = \ldots$. Thus $a \in x^\omega S = \square$, which is a contradiction.

Combining the result (i) of 2.2 and 2.3, we have:

2.4. Theorem. *Let S be a chained semigroup with $S \neq S^2$. Then the following are equivalent:*

i) $S = \{x, x^2, \ldots\}$, where $x \in S \setminus S^2$.
ii) S is a cancellative semigroup without idempotents.
iii) $a^\omega S = \square$ for every $a \in S$.

2.5. Theorem. *Let S be a chained semigroup. Then S is an Archimedean semigroup with no idempotents iff $a^\omega S = \square$ for every $a \in S$.*

Proof. Suppose $a^\omega S \neq \square$ for some $a \in S$. Then by 2.1, $a^\omega S$ is a prime ideal. If S is an Archimedean semigroup without idempotents, then $a^\omega S = S$. Hence $a \in a^2 S$, which implies that S has idempotents, which is a contradiction. Conversely, let $a^\omega S = \square$ for every $a \in S$. This implies that S has no idempotents. Suppose P is a prime ideal different from S. There exists an $x \notin P$. Since ideals are linearly ordered, $P \subseteq xS^1$ and hence $P = xP$, so that $P \subseteq x^\omega S = \square$. Thus S has no proper prime ideals, which implies that S is Archimedean.

2.6. Theorem. *A cancellative Archimedean chained semigroup is a group if $a^\omega S \neq \square$ for some $a \in S$.*

Proof. If S contains an identity, then S is a group since S is Archimedean. So it suffices to show that S has an identity. If S has an idempotent, by the cancellative condition, this idempotent is the identity. So assume that S has no idempotents. Then by 2.1, $a^\omega S$ is a non-empty prime ideal and hence $a^\omega S = S$ by Archimedean property. Thus S contains idempotents as in the proof of 2.5, which is a contradiction.

Note. In 2.6, the condition that $a^\omega S \neq \square$ for some $a \in S$ is essential as can be seen in the example $S = \{x, x^2, \ldots\}$.

2.7. Theorem. *Let S be a chained semigroup containing cancellable elements. Then S is a cancellative semigroup if $a^\omega S = \square$ for every $a \in S$. The converse is true if S is a Noetherian semigroup without idempotents.*

Proof. Clearly, if $Z \neq \square$, then $Z \subseteq aS^1$ for every cancellable element a. Since Z is a prime ideal, we must have $Z = aZ$ and hence $Z \subseteq a^\omega S = \square$. Suppose that S is a Noetherian cancellative semigroup without idempotents. Let $a^\omega S \neq \square$ for some a. Then by 2.1, $a^\omega S$ is a prime ideal. No power of a belongs to $a^\omega S$ since otherwise $a^n \in a^\omega S$ implies $a^n = a^{n+1} b$ for some $b \in S$ and hence $a = ab$, which implies b is an idempotent. Now $x \in a^\omega S$ implies $x = a^i s_i$, $s_i \in S$. Since S is cancellative, we have $s_i = a s_{i+1}$. Also every $s_i \in a^\omega S$ since $a^i \notin a^\omega S$ and $a^\omega S$ is a prime ideal. Because of the non-existence of idempotents $s_i \neq s_{i+1}$. By Noetherian condition the chain $s_1 S^1 \subseteq s_2 S^1 \subseteq \ldots$ terminates. Hence $s_i S^1 = s_{i+1} S^1$, which implies $s_{i+1} = s_{i+1} t$, $t \in S$ and thus an idempotent exists, which is a contradiction.

We observe now that the converse of 2.7 without Noetherian condition depends on a new concept, which we develop here. Moreover we characterize the chained semigroups with identity enjoying this property.

If S is a commutative monoid (semigroup with identity) with a unique maximal ideal M and if S contains some cancellable non-units, then $K = \{a/b : a, b \in S, b \in S \setminus Z\}$ is a commutative monoid containing an isomorphic copy of S, called the *quotient monoid* S. If a/b, $c/d \in K$, then $a/b = c/d$ iff $adu = bcv$ for some units $u, v \in S$. The *almost integral closure* of S in K is the set, $\{x \in K \mid rx^n \in S$ for some cancellable element $r \in S$ and for every $n\}$. An element $x \in K$ is said to be *almost integral* over S if there exists a finite of elements $x_1, x_2, \ldots, x_n$ of K such that $\bigcup_{i=1}^{n} x_i S$ contains all powers of x.

2.8. Proposition. *The almost integral closure of S in K is the set of all almost integral elements of K over S.*

Proof. If x is an almost integral element of K over S, then $x^t \in \bigcup_{i=1}^{n} r_i/s_i S$ for all positive integers t, where $r_i, s_i \in S$, such that s_i's are cancellable. Set $s = s_1 s_2 \ldots s_n$. Then $sx^t \in S$, where s is cancellable. Thus x belongs to the almost integral closure of S. Conversely, if x is in the almost integral closure of S, then $rx^n \in S$ for every positive integer n and for some cancellable element r. Hence s has an inverse in K, say r^{-1} and so $x^n \in r^{-1}S$ for every n. Therefore x is almost integral over S.

S is said to be *almost integrally closed* if the almost integral closure of S in K is S itself.

2.9. Theorem. *If S is an almost integrally closed monoid, then for every non-unit x in S, every element in $x^{\omega}S$ is a non-cancellable element. Furthermore if S is cancellative, then $x^{\omega}S = \Box$.*

Proof. Let $r \in x^{\omega}S$. Then $r = x^n s_n$, $s_n \in S$. If x is cancellable, then $r(x^{-1})^n \in S$ for every n which implies that r is not cancellable since otherwise $x^{-1} \in S$ by the property of almost integral closedness of S and this contradicts that x is a non-unit. Hence we must have $xy = xz$ for some $y, z \in S$ and $y \neq z$. Then, since $r = xs$ for some $s \in S$, we have $xys = xzs$ and so $ry = rz$. Thus r is non-cancellable.

2.10. Theorem. *Let S be a cancellative chained monoid with a unique maximal ideal M. Then the following are equivalent:*

i) S is almost integrally closed.

ii) M is the only prime ideal.

iii) S is an extension of an Archimedean semigroup by a group.

Proof. (i) $\Rightarrow$ (ii). Let Q be a prime ideal $\neq M$. If $x \in M \setminus Q$, then $x^n \notin Q$ and hence $Q \subseteq x^{\omega}S$. But by 2.9 $x^{\omega}S = \Box$. Thus M is the only prime ideal.

(ii) $\Rightarrow$ (iii). Let M be the only prime ideal of S. Then for $x, y \in M$, we have $\sqrt{(xS)} = \sqrt{(yS)} = M$ and hence for some n $y^n \in xS$, i.e., $y^{n+1} \in xyS \subseteq M$. Thus M is an Archimedean semigroup and clearly $S \setminus M$ is a group.

(ii) $\Rightarrow$ (i). Let $x \in K \setminus S$ and $rx^n \in S$ for all positive integers n and for some $r \in S$. By a property of cancellative chained monoids, which are called *Prüfer monoids* in [2], we have the inverse of x, namely, $x^{-1} \in S$. If $(x^{-1})^{\omega} S \neq \Box$, then it is a prime ideal by 2.1 and hence by hypothesis $(x^{-1})^{\omega} S = M$, which implies $x^{-1} = x^{-2}s$ for some $s \in S$ and $x = s \in S$, a contradiction. Thus we have $(x^{-1})^{\omega} S = \Box$. Then for any $r \in S$, there exists a positive integer n such that $rS \nsubseteq (x^{-1})^n S$. Hence $rx^n \notin S$ for some n, which is a contradiction. Thus S is almost integrally closed.

(iii) $\Rightarrow$ (ii). Let S/A be a group, with A being an Archimedean semigroup. Clearly A is a maximal ideal. Since A is an Archimedean semigroup, it can be verified easily that A is a prime ideal.

3. NOETHERIAN SEMIGROUPS

As in commutative ring theory, one can show easily that in Noetherian semigroups every ideal is an intersection of finite number of primary ideals. We study in this section characterizations of Noetherian semigroups satisfying some prescribed properties.

3.1. Lemma. *Let A be a finitely generated ideal of a semigroup S. If $A = AB$ for some ideal B and if $a \in A$, then $a = ab$ for some $b \in B$.*

Proof. Let $\{x_1, x_2, \ldots, x_n\}$ be a minimal set of generators of A, i.e., $x_i \notin x_jS$ for $i \neq j$. Then $x_i \cup x_iS = (x_i \cup x_iS) B = \bigcup x_iB$. Let $a \in A$. If $a = x_i$, $x_i \in x_iB$ since $x_i \notin x_jS$ for $i \neq j$. Hence $a = ab$ for some $b \in B$. Similarly if $a \in x_iS$, we have the desired result.

3.2. Lemma. *Let S be a semigroup containing no idempotents except perhaps the identity 1. If P is a finitely generated prime ideal contained properly in xS for some $x \in S$ and $xS \neq S$, then P does not contain any concellable element. Also if A is a finitely generated ideal containing a cancellable element, then $A \neq AB$ for any proper ideal B.*

Proof. Clearly $P < xS$ implies $P = x^nP$. If a is a cancellable element in P, then as in 3.1, $a = ab$, $b \in xS$ and so $ab = ab^2$ and $b = b^2$, which leads to a contradiction. Similar observation proves the second part.

3.3. Proposition. *Let A be a finitely generated ideal and let $B = A^{\omega}$ such that $AB = \bigcap Q_\lambda$, where Q_λ are primary ideals. Then $AB = B$.*

Proof. It suffices to show that $B \subseteq AB$, i.e., $B \subseteq Q_\lambda$ for every λ. Let $\sqrt{Q_\lambda} = P_\lambda$. Clearly P_λ is a prime ideal. If $A \nsubseteq P$, then $B \subseteq Q_\lambda$ since $AB \subseteq Q_\lambda$. Let $A \subseteq P_\lambda$.

Since $A = \bigcup_{i=1}^{n} x_i S$, $x_i \in P_\lambda = \sqrt{Q_\lambda}$ for $i = 1, 2, \ldots, n$. Then $x_i^{r_i} \in Q_\lambda$ for $i = 1, 2, \ldots, n$. If $m = \max(r_1, r_2, \ldots, r_n)$; then $A^n \subseteq Q_\lambda$. But $B \subseteq A^n$, so that $B \subseteq Q_\lambda$.

3.4. Theorem. *Let S be a Noetherian semigroup without idempotents except perhaps identity. Then for any ideal A, $A^\omega \subseteq Z$, and $A^\omega = \square$ if S is cancellative.*

Proof. If $A^\omega = \square$, then trivially $A^\omega \subseteq Z$. If $A^\omega \neq \square$, then by 3.3, $AA^\omega = A^\omega$ and hence by 3.2, A^ω does not contain any cancellable element. The other part is evident.

3.5. Theorem. *Let S be a Noetherian monoid with a unique maximal ideal $M = mS$ for some $m \in S$. Then the following are true.*

i) If $x \in M$, then $x = m^r u$, u unit, or $x \in M^\omega$ with $x = mxs$.
ii) $P \subseteq M^\omega$ for every prime ideal P.
iii) M^ω is a prime ideal if S has no idempotents except 1.
iv) If $Z \neq M$, $Z = M^\omega$.
v) If S is cancellative, then S is a direct product $N \times G$, where N is the additive semigroup of all non-negative integers and G is an abelian group.

Proof. (i) Let $x \in M$. Then $x = mt_1$, $t_1 \in S$. If t_1 is not a unit, then $t_1 = mt_2$ and so $x = m^2 t_2$. Proceeding in this manner we have either $x = m^r t_r$, t_r being a unit, or $x = m^n t_n$ for $n = 1, 2, \ldots$. Hence in the second case $x \in M^\omega$. Since the chain $t_1 S^1 \subseteq t_2 S^1 \subseteq \ldots$ terminates, we have $t_n S^1 = t_{n+1} S^1$ for some n. This implies $m^{n-1} t_n = m^n t_n s = xs$ for some s and hence $m^n t_n = mxs$ and so $x = mxs$.

(ii) is trivial. (iii) and (iv) can be deduced easily from (i). To prove (v), observe $x = mxs$ is inadmissible for any x. Thus every non-unit is of the form $m^r u$, u being a unit. Hence every element of S can be identified with (r, u), where $r \in N$ and $u \in S \setminus M = G$, which is a group.

3.6. Theorem. *The following are equivalent on a semigroup S.*

i) S is a Noetherian cancellative chained semigroup.
ii) S is a group, or $S = \{x, x^2, \ldots\}$, or $S \approx N \times G$, where N is the additive semigroup of non-negative integers and G is an abelian group.

Proof. (i) $\Rightarrow$ (ii). Let S contain an identity and S be not a group. Then S has a unique maximal ideal M. By condition (i), we must have $M = x \cup xS$. By virtue of 3.5, for proving $S \approx N \times G$ it suffices to show $M^\omega = \square$. We note first that S has no idempotents except 1. Suppose $y \in M^\omega$. Then $y = xs_1$. Since $xS \subseteq s_1 S$ implies that $x^2 \in x^\omega S$, we have $x^2 = x^4 s$ for some $s \in S$, and so $x^2 s = 1$. Hence $s_1 S < xS$, i.e., $s_1 = xs_2$. Because of the non-existence of idempotents, we have $s_1 S < s_2 S$ with $s_2 S < xS$. Proceeding in this manner we obtain a non-terminating strictly ascending chain of ideals $s_1 S < s_2 S < \ldots$, which contradicts the Noetherian condition. Now suppose that S does not have an identity. Since S is a Noetherian chained semigroup, $S = x \cup xS$ for some $x \in S$. Then by 2.3, the result follows. (ii) $\Rightarrow$ (i) is trivial.

References

[1] *Clifford, A. H.,* and *Preston, G. B.:* The Algebraic Theory of Semigroups, Vol. 1, Amer. Math. Soc. (1961).

[2] *Dorofeeva, M. P., Mannepalli, V. L.,* and *Satyanarayana, M.:* Prüfer and Dedekind monoids, Semigroup Forum, *9* (1975), 294—309.

[3] *Grillet, P. A.:* Intersections of maximal ideals in semigroups, Amer. Math. Monthly, *76* (1969), 503—509.

[4] *Mannepalli, V. L.,* and *Satyanarayana, M.:* Monoids of Dedekind type, Semigroup Forum, *9* (1974), 19—27.

[5] *Petrich, M.:* On the structure of a class of commutative semigroups, Czech. Math. Jour., *14* (*89*) (1964), 147—153.

[6] *Satyanarayana, M.:* A class of commutative semigroups in which the idempotents are linearly ordered, Czech. Math. Jour., *21* (*96*) (1971), 633—637.

[7] *Satyanarayana, M.:* Commutative primary semigroups, Czech. Math. Jour., *22* (*97*) (1972), 509—516.

[8] *Satyanarayana, M.:* On left cancellative semigroups, Semigroup Forum, *6* (1973), 317—329.

[9] *Satyanarayana, M.:* On commutative semigroups which are unions of a finite number of principal ideals, to appear in Czech. Math. Jour.

[10] *Schwarz, Štefan:* Prime ideals and maximal ideals in semigroups, Czech. Math. Jour., *19* (*94*) (1969), 72—79.

[11] *Tamura, T.:* The study of closets and free contents related to semilattice decompositions of semigroups, Semigroups, edited by K. W. Folley, Academic Press (1969), 221—260.

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