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ACCESSIBILITY SPACES, k-SPACES AND INITIAL TOPOLOGIES

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1. Introduction. In 1937 E. ČECH introduced the following property of spaces: If $A \subset X$, $x \in A'$ then there is a closed $M \subset A \cup [x]$ such that $x \in M'$ (see e.g. [7], p. 444). WHYBURN [19] defined a topological space as being *accessible* (*approximately accessible*) by a family $\mathscr{M}$ of sets if for $x \in X$, $A \subset X$ such that $x \in A'$ there exists $M \in \mathscr{M}$ such that $x \in M'$ and $M \subset A \cup [x]$ ($x \in M'$ and $x \notin (M \sim A)'$). He defined an *accessibility space* as a T_1-space which is approximately accessible by closed sets and proved that a T_1-space X is an accessibility space iff every quotient map into X is a hereditary quotient map.

ARHANGEL'SKII [1] and RUDIN [15] proved that a T_2-space is accessible by compact sets iff it is a Fréchet space. We use this result to show that a T_2 accessibility space is a Fréchet space if it is a k-space. Other conditions are also investigated for an accessibility space to be a Fréchet space.

Some general contributions will be made to the theory of initial topologies studied by WILLARD [21], MCDONALD [13], MCDONALD and WILLARD [14], AULL [3] and WARRACK [18]. Then, it will be shown that if every $T_2(T_4)$ image of a T_4 space is a k-space (accessibility space) then the space is compact. Similar type theorems will be proved for k'-spaces, sequential spaces and Fréchet spaces.

2. Some contributions to the theory of initial topologies.

Definition 1[3]. A topological space X is *initially property* P *with respect to property* Q if every continuous image with property Q satisfies property P.

Theorem 1. *Let* $(X, \mathscr{T})$ *be* T_4 *and let every* T_2, (T_3), $[T_{3\frac{1}{2}}]$, $\{T_4\}$ *image space satisfy property* P *where* P *is hereditary with respect to closed sets. Then if there exists a* T_2, (T_3), $[T_{3\frac{1}{2}}]$, $\{T_4\}$ *space of infinite cardinal* $\mathscr{A}$ *that does not satisfy property* P, *then every subset of* X *of cardinality* $\mathscr{A}$ *has a limit point.*

We prove a more general lemma first.

Lemma 1. *Let $(X, \mathscr{T})$ be T_4 and let D be a closed set of X. Let f be a continuous function on X onto a space $(Z, \mathscr{U})$ such that $f(D)$ is closed and $X \sim D$ is homeomorphic to $f(X \sim D) = Z \sim f(D)$ and U is open in $f(X)$ if $f^{-1}(U)$ is open in $(X, \mathscr{T})$ and $U \cap f(D)$ is open in $f(D)$. If $f(D)$ is* (a) T_3 (b) $T_{3\frac{1}{2}}$ (c) T_4 (d) *completely T_2 then Z has the given property. Furthermore* (e) *if $f(D)$ is Urysohn or is T_2 and two separated sets in D are contained in disjoint open sets in X then Z is T_2.*

Proof. (a) It is immediate that $(Z, \mathscr{U})$ is T_1. The regularity of $(Z, \mathscr{U})$ follows much the same way as in the proof of lemma 4.2 of McDonald and Willard [14].

(b) From the construction of $(Z, \mathscr{U})$ a function h is continuous from Z to the real line if

(i) h is given by $g = h(f)$ where g is continuous on $(X, \mathscr{T})$ to the real line and
(ii) h restricted to $f(D)$ is continuous to the real line.

Suppose $z \in Z$, $A \subset Z$, A is closed and $z \notin A$. If $z \notin f(D)$ consider the function g such that $g(f^{-1}(z)) = 0$ and $g(f^{-1}(A \cup f(D)) = 1$. Then for h such that $g = h(f)$, $h(z) = 0$ and $h(A) = 1$. Suppose $z \in f(D)$; there exists a function h on $f(D)$ such that $h(z) = 0$ and $h(A \cap f(D)) = 1$. Also, there exists a function g on D such that $g = h(f)$ and this function can be extended to all of X. Then h can be extended to all of Z. The set $B = \{w : h(w) \leqq \frac{1}{2},\ w \in Z\} \cap A$ is closed and disjoint from z. Hence there exists a continuous function p such that $p(z) = 0$ and $p(B) = 1$. Set $q = \min(|p| + 2|h|, 1)$; q is continuous, $q(z) = 0$ and $q(A) = 1$.

(c) Proofs can be constructed that are similar to either proofs of (a) or (b).

(d) Follows from (b) and the fact that if a space is completely Hausdorff its weak topology is Hausdorff.

(e) We consider only the case when $x, y \in f(D)$, $x \neq y$. Let G_x and G_y be open in $f(D)$ then $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are open in D and hence contained in disjoint $\mathscr{T}$-open sets T_x and T_y respectively. Then $U_x = f(T_x \sim (D \sim f^{-1}(G_x))$ and $U_y = f(T_y \sim (D \sim f^{-1}(G_y))$ are disjoint $\mathscr{U}$-open sets containing x and y respectively.

Proof of theorem. If a set of cardinal $\mathscr{A}$ in $(X, \mathscr{T})$ does not have a limit point then X contains a closed discrete set of cardinal $\mathscr{A}$. Then if there exists a set of cardinal $\mathscr{A}$ not having property P and satisfying separation axiom Q map D into this set and construct a new topology as given by lemma 1. The resulting topology will not satisfy property P but will satisfy separation axiom Q where Q is T_2, T_3, $T_{3\frac{1}{2}}$, T_4 or completely T_2.

It was shown in [3] that a T_3 space X is initially countably paracompact with respect to T_2 iff X is countably compact. More generally using the same proof we have:

Theorem 2. *If* P *is hereditary with respect to closed sets and there exists a denumerable T_2 space not satisfying property* P *then if a T_3 space $(X, \mathscr{T})$ is initially* P *with respect to T_2 then $(X, \mathscr{T})$ is countably compact.*

3. Applications to accessibility spaces and k-spaces. Accessibility spaces were defined in the introduction.

Definitions. A space X is a *k-space* if every k-closed space is closed. A set M is *k-closed* if $M \cap K$ is closed if K is compact. A space X is a k'-space if for $M \subset X$, $x \in M'$ there is a compact set K such that $x \in (M \cap K)'$. A space is S_0 if sequences converge to at most one point. A *sequential space* is a space such that sequentially closed sets are closed and a Fréchet space is one in which $x \in A'$ means there is a sequence in $A \sim [x]$ converging to x. An $S_5(S_6)$ space is a sequential (Fréchet) S_0 space. *Quasi-k-spaces* [15] (*Sequentially-k-spaces*) are defined analogously to k-spaces with countably compact (sequentially compact) replacing compact in the definition.

Relations between these properties and others may be found in [17] and [5].

Theorem 3. *If $(X, \mathcal{T})$ is $T_3(T_4)$ and initially property* P *with respect to $T_2(T_4)$ then X is countably compact when* P *is* (a) *k* (b) *k'* (c) *quasi-k* (d) *sequentially-k* (e) *sequential* (f) *Fréchet* (g) *accessibility.*

Proof. It will be sufficient to show the existence of a denumerable T_4-space that is not a quasi-k-space and one that is not an accessibility space.

The example of Arens [12, p. 77] of a denumerable T_4-space that is not sequential is not a quasi-k-space and hence not a k-space is an accessibility space. The following denumerable T_4-space (see e.g. [8]) is not an accessibility space. We use the result of Whyburn [19] that an accessibility space is accessible by closed sets if regular.

Example 1. Let $(X, \mathcal{T})$ consist of the points (m, n) with $m \geqq 0$ and $n > 0$ and the point $(0, 0)$. Let the points such that $m > 0$ be isolated. Let basic neighborhoods of $(0, n)$ consist of $(0, n)$ and all but a finite number of points (m, n) for fixed n. Let the basic neighborhoods of $(0, 0)$ be sets S containing $(0, 0)$ and all but a finite number of points of $\{(0, n)\}$ and such that if S contains $(0, k)$ then it contains all but a finite number of elements of the form (m, k). We omit the argument to show that $(X, \mathcal{T})$ is T_3 and hence T_4. We now show that $(X, \mathcal{T})$ is not accessible by closed sets. The point $z = (0, 0)$ is a limit point of $M = \{(m, n) : m > 0,\ n > 0\}$. If H is a set such that $[z] \cup H \subset M$ and $z \in H'$, H must contain an infinite number of points of the form (m, k) for some k so that $(0, k)$ is a limit point of H contrary to H being closed. So $(X, \mathcal{T})$ is not an accessibility space.

Corollary 3. *In the following conditions,* (a) $\leftrightarrow$ (b) *and* (c) $\leftrightarrow$ (d) *for a $T_3(T_4)$ space X.*

(a) *X is initially quasi-k with respect to $T_2(T_4)$.*
(b) *X is countably compact.*
(c) *X is initially sequentially k with respect to $T_2(T_4)$.*
(d) *X is sequentially compact.*

Proof. (a) $\leftrightarrow$ (b) and (d) $\rightarrow$ (c) is immediate and (c) $\rightarrow$ (d) follows from the fact that countably compact sequentially-k-spaces are sequentially compact. (See theorem 12).

4. Compactness of various initially P spaces.

McDonald and Willard [14] used advantageously the following lemma of KATĚTOV [11] to show that T_3 spaces that are initially T_3 with respect to T_2 are compact.

Lemma 4A. *Every nowhere dense closed set of a space X is compact iff the set of non-isolated points is compact.*

From this follows:

Lemma 4B. (McDonald and Willard [14]). *Every nowhere dense closed set of a countably compact space X is compact iff X is compact.*

In this section we use these lemmas to show that initially k-spaces and accessibility spaces are compact.

Theorem 4. *Let $(X, \mathscr{T})$ be a T_4 space. If X is countably compact and not compact then βX is not an accessibility space. If X is initially an accessibility space with respect to $T_{3\frac{1}{2}}$ then X is compact.*

Proof. Since X is countably compact by theorem 3 and not compact, by lemma 4B there is a nowhere dense noncompact closed subset of X say A. Let $z \in \beta X \sim X$ such that $z \in A'$. Consider $X \cup [z]$ as a subspace of βX. Since $X \sim A$ is dense in X, $z \in (X \sim A)'$, but since X is normal z is not limit point of any closed set contained in $(X \sim A)$. Therefore $X \cup [z]$ is not an accessibility space and hence βX is not. Let $y \in X \sim A$. There exists a closed neighborhood of y, $C_y \subset X \sim A$.

Form a quotient map f from $X \cup [z]$ onto X such that $f(x) = x$ for $x \in X$ and $f(z) = y$. The resulting quotient topology $(X, \mathscr{U})$ is weaker than $(X, \mathscr{T})$ and is completely regular and $(X, \mathscr{U})$ is not an accessibility space.

Theorem 5. *If $(X, \mathscr{T})$ is normal and initially k with respect to T_2 then X is compact.*

Proof. Again let A be the nowhere dense closed subset of the previous proof and let $X \cup [z]$ be a subset of βX where $z \in A'$. Form the smallest topology $(X \cup [z], \mathscr{V})$ containing the topology of $X \cup [z]$ as a subset of βX and the topology with $(X \sim A) \cup [z]$ as the only non-trivial open set. We first show that $(X \cup [z], \mathscr{V})$ is not a k-space by showing that X is k-closed but not closed in this topology. If K is a compact set such that $K \cap X$ is not closed, then $K \sim [z]$ is not compact. As a subset of βX, K must also be compact and closed. Since βX is a Wallman-compactification of X, $K \cap A \neq \emptyset$ and $K \cap A$ is closed in $(X \cup [z], \mathscr{V})$, there is a stronger topology on K as a subspace of $(X \cup [z], \mathscr{V})$ than as a subspace of βX contrary to K being compact in $(X \cup [z], \mathscr{V})$. So X is k-closed in $(X \cup [z], \mathscr{V})$ so this space is not a k-space. As in the previous theorem pick a point $y \in X \sim A$ and identify with z using a quotient map this time from $(X \cup [z], \mathscr{V})$. Again let C_y be a closed neighborhood of y disjoint from $A \cup [z]$. The set $\{X \sim (\text{int } C_y\text{: interior with respect to } (X, \mathscr{T}))\}$ is a k-closed set in the quotient topology that is not closed.

We note that in the case of accessibility spaces we considered $T_{3\frac{1}{2}}$ images but in the case of k-spaces we used T_2-image. The following theorem related to some theorems of WARRACK [18] shows that we can not use $T_{3\frac{1}{2}}$-images in the case of k-spaces.

Theorem 6. *Let $(X, \mathscr{T})$ be a $T_{3\frac{1}{2}}$-space, then X is initially locally compact with respect to $T_{3\frac{1}{2}}$ iff $\beta X \sim X$ is finite.*

Proof. If $\beta X \sim X$ is finite then for any $T_{3\frac{1}{2}}$ image Y of X, $\beta Y \sim Y$ is finite. If $\beta X \sim X$ is infinite and locally compact $\beta X \sim X$ is compact and there exists $z \in \beta X \sim \sim X$ such that $X \cup [z]$ is not locally compact. Again using a quotient map as in the previous two theorems, we obtain a $T_{3\frac{1}{2}}$ image of X that is not locally compact.

We return now to k-spaces. From Theorem 5 and the fact that compact $\rightarrow$ locally compact $\rightarrow k' \rightarrow k$ we obtain:

Theorem 7. *The following are equivalent for a normal space $(X, \mathscr{T})$,*

(a) *X is initially k with respect to T_2.*

(b) *X is initially k' with respect to T_2.*

(c) *X is initially locally compact with respect to T_2.*

(d) *X is compact.*

We note that Warrack [18] first proved the equivalence of (c) and (d). With sequential spaces we can easily show that even with T_4 images we obtain compactness.

Theorem 8. *The following are equivalent for a T_4 space X,*

(a) *$(X, \mathscr{T})$ is compact and sequential,*

(b) *$(X, \mathscr{T})$ is initially sequential with respect to T_4.*

Prove: (a) $\rightarrow$ (b) follows from Franklin's [9] result that quotients of sequential spaces are sequential. (b) $\rightarrow$ (a). If $(X, \mathscr{T})$ is not compact there is a weaker T_4 topology $(X, \mathscr{U})$ (see [6]). If $(X, \mathscr{U})$ is sequential there is a sequentially closed set in $(X, \mathscr{T})$ that is not sequentially closed in $(X, \mathscr{U})$ so that there is a convergent sequence of distinct points in $(X, \mathscr{U})$ which is a closed discrete set in $(X, \mathscr{T})$. But this contradicts theorem 3 which says that $(X, \mathscr{T})$ is countably compact.

Theorem 9. *The following are equivalent for a T_4-space $(X, \mathscr{T})$.*

(a) *$(X, \mathscr{T})$ is compact and Fréchet.*

(b) *$(X, \mathscr{T})$ is initially an accessibility space with respect to T_2.*

(c) *$(X, \mathscr{T})$ is initially an accessibility space with respect to $T_{3\frac{1}{2}}$.*

(d) *$(X, \mathscr{T})$ is initially Fréchet with respect to T_2.*

(e) *$(X, \mathscr{T})$ is initially Fréchet with respect to T_4.*

Proof. $(b) \Rightarrow (c)$ and $(b) \Rightarrow (d) \Rightarrow (e)$ are immediate. $(c) \Rightarrow (a)$ follows from theorem 4 and the result that a T_2, accessibility k-space is a Fréchet space (see theorem 11). $(e) \Rightarrow (a)$ follows from theorem 8. Finally $(a) \Rightarrow (b)$ follows from Franklin's [7] result that every closed image of a Fréchet space is a Fréchet space.

5. Relations between the various properties.

Arhangel'skii [1] and Rudin [15]) proved that a T_2-space is accessible by compact sets iff it is a Fréchet space. We use this result to prove that a T_2 accessibility space is a k-space iff it is a Fréchet space.

Theorem 10. *If a T_2 space is approximately accessible by compact sets it is accessible by compact sets.*

Proof. Let $x \in M'$. There exists a compact set K such that $x \in K'$ and $x \notin (K \sim M)'$. Since $(K \sim M)' \subset K$, $(K - M)'$ is compact. There exists a closed neighborhood C_x of x such that $C_x \cap \overline{(K \sim (M \cup [x])} = \emptyset$. Then $K \cap C_x \subset M \cup [x]$ is compact and $x \in (K \cap C_x)'$.

Corollary 10. *A T_2 space is approximately accessible by compact sets iff it is a Fréchet space.*

Theorem 11. *Let X be a T_2 accessibility space. Then X is a Fréchet space iff X a k-space.*

This theorem will be a consequence of corollary 10 and the following lemma:

Lemma 11. *Let X be approximately accessible by closed sets. Then if X is k (quasi-k) [sequentially-k] and every compact (countable compact) [sequentially compact] set is closed then X is approximately accessible by compact (countably compact) [sequentially compact] sets.*

Proof. We prove the compact case only. Suppose $x \in M'$; there exist a closed set C such that $x \in C'$ and $x \notin (C - M)'$. Suppose there does not exist a compact set $K \subset C$ such that $x \in K'$. Then for any compact set K, $K \cap (C \sim [x])$ is closed so that $C \sim [x]$ is k-closed and hence closed by the k-space property contrary to $x \in C'$.

We note that theorem 11 was proved by E. Shirley with T_2 replaced by T_3.

We now turn to results connected with sequentially and countably compact spaces. We use the following result from [4].

Lemma 12. *If every subsequence of a sequence $\{x_n\}$ has a side point then $\{x_n\}$ has an uncountable number of side points.*

Theorem 12. *Let X be S_0 (sequences converge to at most one point) and let X be* *either sequentially-k or accessible by countably compact sets; then the sequentially compact and the countably compact subsets are identical.*

Proof. Suppose X is accessible by countably compact sets. Let M be countably compact and $\{x_n\}$ a sequence of distinct points in M with accumulation point y. There exists a countably compact subset $K \subset [y] \cup \bigcup [x_n]$ such that $y \in K'$ and since K is denumerable K is sequentially compact by Lemma 12 so that $\{x_n\}$ has a convergent subsequence.

Let X be sequentially-k and again let M be countably compact and let $\{x_n\}$ be a sequence of distinct points of M without convergent subsequences. Then the only sequentially compact subsets that intersect $N = \bigcup \{x_n\}$ intersect N in a finite number of points so that N is closed and has at most a denumerable number of side points by lemma 12 contrary to M being countably compact.

Corollary 12. *Let X be accessible by countably compact sets, then X is Fréchet iff every sequentially compact set is closed.*

Proof. From the theorem every countably compact set is closed. If $x \in M'$ there exists a countably compact set K such that $x \in K'$ and $K \cup [x] \subset M$. If no sequence of distinct points of K converges to x then $K \sim [x]$ is countably compact and hence closed so $x \notin K'$.

Theorem 13. *The following are equivalent for a space X.*

(a) *X is S_0 and sequential (S_5),*

(b) *X is sequentially-k and S_3 (sequentially compact subsets are closed).*

Proof. Let M be sequentially closed. If S is sequentially compact, $S \cap M$ is sequentially compact and hence closed so that M must be closed.

In theorem 12 and corollary 12 we noted that sequences tend to be more important if countably compact subsets are closed. The next theorem also illustrates this tendency.

Definition. A space X is said *to satisfy the weak sequential property* if either one of the equivalent conditions is satisfied.

(a) If $x \in F'$ where F is closed then exists a sequence of points in $F \sim [x]$ converging to x.

(b) If $x \in A'$, there is a sequence of points in $\bar{A} \sim [x]$ converging to X.

Theorem 14. *Let X be quasi-k and S_4 (every countably compact subset is closed) or sequential; then X is weak sequential. If an accessibility space is weak sequential it is a Fréchet space.*

Proof. If X is sequential and $x \in F'$ where F is closed, $F \sim [x]$ will be sequentially closed unless there is a sequence of points in $F \sim [x]$ converging to X. If X is S_4

and quasi-k then if K is countably compact $K \cap F$ is closed and countably compact; if no sequence in $F \sim [x]$ converges to F then $K \cap (F \sim [x])$ will be countably compact and hence closed so that $F \sim [x]$ is closed contradicting $x \in F'$.

If X is an accessibility space and weak sequential then if $x \in M'$ there exists a closed set F such that $x \in F'$ and $x \notin (F \sim M)'$ so that there is a sequence of points in $(F \cap M) \sim [x]$ converging to x.

Theorem 15. *An accessibility space X that is E_1 [2] (every point is the intersection of a countable number of closed neighborhoods) is Fréchet iff it is a quasi-k space.*

Proof. Analogous to the proof of theorem 10 we can prove that an E_1-space is accessible by countably compact sets if it is approximately accessible by countably compact sets, and by lemma 11 X is approximately accessible by countably compact sets. Noting that in E_1 spaces countably compact sets are closed, an application of theorem 14 completes the proof.

We summarize many of the results of this section as follows.

(a) T_2 + accessibility + $k \Rightarrow$ Fréchet.

(b) E_1 + accessibility + quasi-$k \Rightarrow$ Fréchet.

(c) Accessible by countably compact sets + $S_3 \Leftrightarrow S_6$ (Fréchet + S_0).

(d) S_3 + sequentially-$k \leftrightarrow$ (sequential + S_0).

(e) Accessibility + weak sequential $\leftrightarrow$ Fréchet.

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