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SCHUR COMPLEMENTS OF DIAGONALLY DOMINANT MATRICES

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1. DEFINITIONS

We shall deal principally with square complex matrices. For positive integer n, let $\langle n \rangle = \{1, 2, \ldots, n\}$. A matrix $A \in \mathbb{C}^{n,n}$, the set of $n \times n$ complex matrices, is *(row) diagonally dominant if*

(1) $$|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad i \in \langle n \rangle,$$

a *positive diagonal matrix* if

(2) $$A = \operatorname{diag}(d_1, \ldots, d_n), \quad \text{with} \quad d_i > 0, \quad i \in \langle n \rangle,$$

an *H-matrix* (cf. [6], [9]) if

(3) $$AD \text{ is diagonally dominant for some positive diagonal } D,$$

a *Z-matrix* (cf. [5]) if

(4) $$a_{ij} \leqq 0, \quad i, j \in \langle n \rangle, \quad i \neq j,$$

an *M-matrix* if

(5) $$A \text{ is both an } H\text{-matrix and a Z-matrix, and } a_{ii} > 0, \; i \in \langle n \rangle.$$

We shall denote by $\mathscr{DD}^{(n)}$, $\mathscr{D}^{(n)}$, $\mathscr{H}^{(n)}$, $\mathscr{Z}^{(n)}$, and $\mathscr{M}^{(n)}$, respectively, the sets of matrices of order n satisfying (1), (2), (3), (4), and (5). We shall denote by $\mathscr{PD}^{(n)}$ the set of all positive definite hermitian matrices of order n. If the order of the matrix is not in question, we will sometimes suppress the superscript (n).

For $A \in \mathbb{C}^{n,n}$, we define the *inertia* of A to be

$$\operatorname{In} A = (\pi(A), \nu(A), \delta(A)),$$

*) The research of this author took place while he was a visiting faculty member at Oregon State University, Winter and Spring, 1977.

where $\pi(A)$, $\nu(A)$, and $\delta(A)$ are, respectively, the number of characteristic roots of A with positive, negative, and zero real part.

Given α, $\phi \subseteq \alpha \subseteq \langle n \rangle$, we let $|\alpha|$ denote the cardinality of α. Given $A \in \mathbb{C}^{n,n}$ and α, β, $\phi \subset \alpha, \beta \subseteq \langle n \rangle$, we let $A[\alpha; \beta]$ denote the submatrix of A with rows indexed by α and columns indexed by β; if $\alpha = \beta$, we write $A[\alpha]$ for $A[\alpha; \beta]$.

An equivalent (cf. [5]), and more standard definition of $A \in \mathcal{M}^{(n)}$ is that A be a Z-matrix and satisfy

$$\det A[\alpha] > 0\,, \quad \phi \subset \alpha \subseteq \langle n \rangle\,; \tag{6}$$

it is sufficient to show (instead of [6]) that

$$\det A[1, \ldots, k] > 0\,, \quad k \in \langle n \rangle\,. \tag{7}$$

2. SCHUR COMPLEMENTS

Given $A \in \mathbb{C}^{n,n}$, partitioned into blocks as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

with $A_{11} \in \mathbb{C}^{k,k}$ and nonsingular. Then the *Schur complement of* A_{11} *in* A is the matrix

$$A/A_{11} = A_{22} - A_{21} A_{11}^{-1} A_{12} \in \mathbb{C}^{n-k,n-k}\,. \tag{8}$$

It is known [4] that if $A_{11} = A[\alpha]$ for some $\alpha = \langle k \rangle$, $1 \leqq k < n$, then $A/A_{11} = B = (b_{ij})_{i,j=k+1}^{n}$, where

$$b_{ij} = \det A[1, \ldots, k, i;\ 1, \ldots, k, j] / \det A[1, \ldots, k]\,, \quad i, j \in \langle n \rangle \setminus \langle k \rangle\,. \tag{9}$$

Sylvester's formula (cf. [7, Vol. I, p. 33]) tells us that

$$\begin{gathered} \det B[i_1, \ldots, i_t;\ j_1, \ldots, j_t] = \\ = \det A[1, \ldots, k, i_1, \ldots, t_t;\ 1, \ldots, k, j_1, \ldots, j_t] / \det A[1, \ldots, k]\,, \\ k + 1 \leqq i_1 < \ldots < i_t \leqq n\,, \quad k + 1 \leqq j_1 < \ldots < j_t \leqq n\,. \end{gathered} \tag{10}$$

For $\alpha = \langle k \rangle$, let $\hat{\alpha} = \langle n \rangle - \langle k \rangle$; it is known (cf. [2]) that

$$(A/A[\alpha])^{-1} = A^{-1}[\hat{\alpha}]\,. \tag{11}$$

Schur complements of other nonsingular principal submatrices in A can be defined using permutation similarities of A.

It is known [8] that if $A \in \mathbb{C}^{n,n}$ is hermitian, $\phi \subset \alpha \subset \langle n \rangle$, and $A[\alpha]$ is nonsingular, then

$$\operatorname{In} A = \operatorname{In} A[\alpha] + \operatorname{In} A/A[\alpha],$$

and thus (see also [1])

$$A \in \mathscr{PD} \quad \text{iff} \quad A[\alpha] \in \mathscr{PD} \quad \text{and} \quad A/A[\alpha] \in \mathscr{PD}.$$

If $A \in Z$ and $\phi \subset \alpha \subset \langle n \rangle$, then

$$A \in \mathscr{M} \quad \text{iff} \quad A[\alpha] \in \mathscr{M} \quad \text{and} \quad A/A[\alpha] \in \mathscr{M}.$$

If $A \in \mathscr{M}$, clearly $A[\alpha] \in \mathscr{M}$; that $A/A[\alpha] \in \mathscr{M}$ is due to CRABTREE [3]. The converse follows by applying (10) to prove (7).

We shall study analogous results for other classes of matrices.

3. PRINCIPAL SUBMATRICES AND SCHUR COMPLEMENTS OF DIAGONALLY DOMINANT MATRICES

Our first result is

Theorem 1. *Given* $A \in \mathscr{DD}^{(n)}$ *and* $\alpha, \phi \subset \alpha \subset \langle n \rangle$. *Then* $A[\alpha] \in \mathscr{DD}$ *and* $A/A[\alpha] \in \mathscr{DD}$.

Proof. That $A[\alpha] \in \mathscr{DD}$, and is nonsingular, is obvious. To show that $A/A[\alpha] \in \mathscr{DD}$ for all α, $\phi \subset \alpha \subset \langle n \rangle$, it is sufficient to consider $\alpha = \{1, \ldots, k\}$, $1 \leqq k < n$.

Our proof will be by induction on n. We first, however, prove the result for $k = 1$ and arbitrary n. In this case $A[\alpha] = a_{11}$. Let $M = (m_{ij}) \in \mathbb{C}^{n,n}$ be defined by

$$m_{ij} = \begin{cases} |a_{ii}|, & i = j \in \langle n \rangle \\ -|a_{ij}|, & i, j \in \langle n \rangle, \quad i \neq j. \end{cases}$$

Clearly $M \in \mathscr{DD}$, and

$$\sum_{j=1}^{n} m_{ij} > 0, \quad i \in \langle n \rangle.$$

Let $B = A/a_{11} = (b_{ij})_{i,j=2}^{n}$, where $b_{ij} = a_{ij} - a_{i1}a_{11}^{-1}a_{1j}$, $i, j \in \langle n \rangle \setminus \langle 1 \rangle$. For $i \in \langle n \rangle \setminus \langle 1 \rangle$,

$$|b_{ii}| - \sum_{\substack{j=2 \\ j \neq i}}^{n} |b_{ij}| \geqq (|a_{ii}| - |a_{11}|^{-1}(-|a_{i1}|)(-|a_{1i}|)) +$$

$$+ \sum_{\substack{j=2 \\ j \neq i}}^{n} ((-|a_{ij}| - |a_{11}|^{-1}(-|a_{i1}|)(-|a_{1j}|)) =$$

$$= \sum_{j=2}^{n} m_{ij} - m_{11}^{-1} m_{i1} \sum_{j=2}^{n} m_{1j} = \sum_{j=1}^{n} m_{ij} - m_{11}^{-1} m_{i1} \sum_{j=1}^{n} m_{1j} > 0,$$

i.e., $B \in \mathscr{DD}$.

For $n = 2$, the result follows from the case $k = 1$. Assume the result for matrices of order less than n. Fix k such that $1 < k < n$; and let $\alpha = \langle k \rangle$. As $A \in \mathscr{D}\mathscr{D}$, we know that A, $A[\alpha]$, and (a_{11}) are nonsingular. By the quotient formula of CRABTREE and HAYNSWORTH [4],

$$A/A[\alpha] = (A/(a_{11}(/(A[\alpha]/(a_{11})) ;$$

but $A/(a_{11}) \in \mathscr{D}\mathscr{D}$ by the case $k = 1$, and then $(A/(a_{11})/(A[\alpha]/(a_{11})) \in \mathscr{D}\mathscr{D}$ by induction. ■

Corollary 1. *Given* $A \in \mathscr{H}^{(n)}$ *and* α, $\phi \subset \alpha \subset \langle n \rangle$. *Then* $A[\alpha] \in \mathscr{H}$ *and* $A/A[\alpha] \in \mathscr{H}$.

Proof. That $A[\alpha] \in \mathscr{H}$ is obvious. As $A \in \mathscr{H}$, there exists $D \in \mathscr{D}$ for which $AD \in \mathscr{D}\mathscr{D}$. Now $D/D[\alpha] \in \mathscr{D}$, and calculation shows that

$$(A/A[\alpha])\,(D/D[\alpha]) = AD/(AD)\,[\alpha] \in \mathscr{D}\mathscr{D} ,$$

hence $A/A[\alpha] \in \mathscr{H}$. ■

The converses of Theorem 1 and Corollary 1 are false; take

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}; \quad A[1] = 1 , \quad A/A[1] = -3 .$$

For a set $\mathscr{S}$ of nonsingular matrices of $\mathbb{C}^{n,n}$, let

$$\mathscr{S}^{-1} = \{A^{-1} \in \mathbb{C}^{n,n} \mid A \in \mathscr{S}\} .$$

Corollary 2. *Given* $A \in \mathscr{S}^{-1}$ *for* $\mathscr{S} \in \{\mathscr{D}\mathscr{D}^{(n)}, \mathscr{H}^{(n)}, \mathscr{M}^{(n)}\}$ *and* α, $\phi \subset \alpha \subset \langle n \rangle$. *Then* $A[\alpha] \in \mathscr{S}^{-1}$ *and* $A/A[\alpha] \in \mathscr{S}^{-1}$.

Proof. Applying formula (11), we have for α and $\hat{\alpha} = \langle n \rangle \setminus \alpha$,

$$A/A[\alpha] = (A^{-1}[\hat{\alpha}])^{-1} , \quad A[\alpha] = (A^{-1}/A^{-1}[\hat{\alpha}])^{-1} .$$

The result then follows immediately from Theorem 1, Corollary 1, and the Crabtree result. ■

4. INERTIAL RESULTS FOR H-MATRICES WITH REAL DIAGONAL MATRICES

Suppose first that $A \in \mathscr{D}\mathscr{D}^{(n)}$, with real diagonal entries. Clearly $a_{ii} \neq 0$, $i \in \langle n \rangle$. For $i \in \langle n \rangle$, let

$$C_i = \{z \in \mathbb{C} \mid |z - a_{ii}| \leqq \sum_{j \neq i} |a_{ij}|\}$$

be the Gerschgorin circle with center at the diagonal entry a_{ii} and radius $\sum_{j \neq i} |a_{ij}|$. It is easy to see that if $a_{ii} > 0$, C_i lies in the open right half-plane of $\mathbb{C}$, and, if $a_{ii} > 0$, C_i lies in the open left half-plane of $\mathbb{C}$. Further, each characteristic root of A lies in either

$$C_+ = \bigcup_{\{i \in <n> \mid a_{ij} > 0\}} C_i$$

or

$$C_- = \bigcup_{\{i \in <n> \mid a_{ij} < 0\}} C_i$$

As $C_+ \cap C_- = \emptyset$, if $|\{i \in \langle n \rangle \mid a_{ij} > 0\}| = k$, then (cf. [10, p. 147]) k characteristic roots of A lie in C^+, and $n - k$ lie in C^-.

Theorem 2. *Suppose $A \in \mathscr{H}^{(n)}$, with real diagonal entries. Then*

$$(12) \qquad \pi(A) = |\{i \in \langle n \rangle \mid a_{ii} > 0\}|, \quad \nu(A) = |\{i \in \langle n \rangle \mid a_{ii} < 0\}|, \quad \delta(A) = 0;$$

A is positive stable (i.e., $\pi(A) = n$) iff $a_{ii} > 0$, $i \in \langle n \rangle$.

Also, if A has all real principal minors, and α is given, $\phi \subset \alpha \subset \langle u \rangle$, then

$$(13) \qquad \text{In } A = \text{In } A[\alpha] + \text{In } A/A[\alpha],$$

and A is positive stable iff $A[\alpha]$ and $A/A[\alpha]$ are positive stable.

Note. The second statement of this result extends Theorem VII of Taussky's famous paper, *A* recurring theorem on determinants [11].

Proof. Given $A \in \mathscr{H}^{(n)}$ with real diagonal entries. Then there exists a $D \in \mathscr{D}$ for which $AD \in \mathscr{D}\mathscr{D}$. It follows that $D^{-1}AD \in \mathscr{D}\mathscr{D}$, with real diagonal entries. By our discussion above, it is clear that (12) holds for $D^{-1}AD$ and thus also for A.

Suppose now that A (and hence also AD) has all real principal minors. Suppose $\alpha = \langle k \rangle$, $1 \leqq k < n$. Then $A[\alpha] \in \mathscr{D}\mathscr{D}$ is nonsingular, and by a simple continuity argument $a_{11} \cdot \ldots \cdot a_{kk}$ and $\det A[\alpha]$ have the same sign. By Corollary 1, $B = A/A[\alpha] \in \mathscr{D}\mathscr{D}$. Also, for $i \in \langle n \rangle \setminus \alpha$,

$$b_{ii} = \det A[1, \ldots, k, i] / \det A[1, \ldots, k] \in \mathbb{R},$$

with the same sign as $a_{11} \cdot \ldots \cdot a_{kk} \cdot a_{ii} / (a_{11} \cdot \ldots \cdot a_{kk}) = a_{ii}$. The desired conclusions now follow. ■

Corollary 3. *Suppose $A \in (\mathscr{H}^{(n)})^{-1}$, with real principal minors. Then all the conclusions of Theorem 2 hold.*

Proof. Let $B = A^{-1} \in \mathscr{H}$. Clearly, by Theorem 2, $\delta(A) = \delta(B) = 0$, and $\pi(A) = \pi(B) = |\{i \in \langle n \rangle \mid b_{ii} > 0\}|$,

$$\nu(A) = \nu(B) = |\{i \in \langle n \rangle \mid b_{ii} < 0\}|.$$

Also, for $i \in \langle n \rangle$, the sign of $a_{ii} = \det B[1, \ldots, \hat{\imath}, \ldots, n]/\det B$ is the sign of $b_{11} \ldots \hat{b}_{ii} \ldots b_{nn}/b_{11} \ldots b_{ii} \ldots b_{nn} = 1/b_{ii}$, i.e., is the sign of b_{ii}. That (13) holds for $A \in \mathscr{H}^{-1}$ follows, using (11), from the fact that (13) holds for $B = A^{-1} \in \mathscr{H}$.

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