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Luděk Zajíček

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ON THE DIFFERENTIATION OF CONVEX FUNCTIONS IN FINITE AND INFINITE DIMENSIONAL SPACES

LUDĚK ZAJÍČEK, Praha

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1. INTRODUCTION

If f is a convex function defined in a Euclidean n-space E^n then the set $N(f)$ of all points at which f is not differentiable is small. There exist several works concerning the sets $N(f)$ ([1]) or, which is almost equivalent, the sets of all singular boundary points of convex bodies in E^n ([8], [5], [1], [3]). In the present article we give a characterization of the magnitude of sets $N(f)$ in E^n. By the same method we obtain also an infinite dimensional generalization of our result. We also characterize the magnitude of sets $S_k(f)$ defined in [1]. If we write in the sequel "Banach space", we mean "real Banach space".

We shall say that f is a convex function defined in a Banach space B if its domain D_f is an open convex subset of B and f is convex on D_f. If f is a convex function defined in E^n then for any $x \in D_f$ there exists [1] a maximal linear manifold L_x such that $x \in L_x$ and $f/D_f \cap L_x$ is differentiable at x. For $0 \leqq k \leqq n$, S_k is the set of all $x \in D_f$ for which $\dim L_x \leqq k$. It is proved in [1] that S_k is the union of countable many compact sets of finite k-dimensional Hausdorff measure. In [1] it is further proved that for $k \leqq 2$ the set S_k can be covered by countably many k-cells of finite k-measure. These results were obtained in [1] as consequences of theorems concerning "upper semi-continuous collections".

In the case $n = 2$, $k = 1$ the result of Anderson and Klee [1] was improved by BESICOVITCH [3]. He proved that in E^2 any set $S_1(f) = N(f)$ is countably rectifiable.

In the infinite dimensional case we are interested in continuous convex functions defined in a separable real Banach space B. Let f be such a function. It is well-known that the set $N(f)$ of all points at which f is not Gâteaux differentiable is of the first category. Further information concerning the sets $N(f)$ follows from results on differentiation of Lipschitz functions ([4], [7]). A more precise result was proved by ARONSZAJN in [2]. He proved that $N(f)$ belongs to the class U^0 which is defined as follows:

1. Let $o \neq a \in B$, $Z(a) = \{A \subset B,\ A$ is Borel, $A \cap \{x + at,\ t \in R\}$ is countable for any $x \in B\}$.
2. For any sequence $\{a_n\}$, $a_n \neq o$, $Z\{a_n\} = \{A \subset B,\ A = \bigcup_n A_n,\ A_n \in Z(a_n)\}$.
3. $U^0 = \bigcap Z\{a_n\}$, the intersection being taken over all sequences $\{a_n\}$ complete in B.

The result of Aronszajn implies that $\mu(N(f)) = 0$ for any Gaussian measure μ on B. Now we shall state our main results.

Definition 1. We shall say that $M \subset E^n$ is a *$(c - c)$-surface of dimension k* $(k = 1, \ldots, n - 1)$ if there exists a permutation π of the numbers $1, \ldots, n$ and $2n - 2k$ convex functions $f_{k+1}, g_{k+1}, \ldots, f_n, g_n$ defined on the whole space E^k such that M is the set of all $(x_1, \ldots, x_n) \in E^n$ such that $y_j = f_j(y_1, \ldots, y_k) - g_j(y_1, \ldots, y_k)$ for $j = k + 1, \ldots, n$ where $y_i = x_{\pi(i)}$ for $i = 1, \ldots, n$.

Notation. If M is a subset of a vector space, then Lin M is the linear hull of M.

Definition 2. Let B be an infinite dimensional Banach space. A set M is called a *$(c - c)$-hypersurface* if there exist a closed subspace $H \subset B$ and a vector $v \in B$ such that $B = H + \text{Lin}\{v\}$, and two Lipschitz convex functions f, g defined on the whole H such that

$$M = \{x + (f(x) - g(x))\, v,\ x \in H\}\,.$$

Theorem 1. *A set $M \subset E^n$ is a subset of the set $S_k(f)$ $(0 < k < a)$ for a convex function f defined in E^n iff M can be covered by countably many $(c - c)$-surfaces of dimension k.*

Theorem 2. *A subset M of a separable real Banach space B is a subset of the set $N(f)$ for a continuous convex function f defined in B iff M can be covered by countable many $(c - c)$-hypersurfaces.*

These theorems are quite analogous to each other and their proofs are almost identical. We shall also prove a generalization of Theorem 2 which is an analogue of Theorem 1 in the separable infinite dimensional case.

Theorem 1 immediately implies the results of [1] mentioned above since any difference of two convex functions in the finite dimensional space is locally Lipschitz and any Lipschitz image of a set of a finite k-dimensional Hausdorff measure is of a finite k-dimensional Hausdorff measure. The fact that S_k can be covered by countably many k-cells which is proved in [1] for $k \leqq 2$ clearly follows from Theorem 1 for any $0 < k < n$. It is almost obvious that Theorem 1 improves the results of [1] also in the cases $k = 1, 2$, as well as the result of [3] (see Example 1).

The proof of Theorem 1 immediately yields a result on singular boundary points of convex bodies in Euclidean spaces (Theorem 3).

Theorem 2 and the result of [2] mentioned above imply that any $(c - c)$-hypersurface belongs to U^0. We can obtain this fact also directly from the proof of Theorem 2 without using the results of [2]. Thus the Aronszajn's result mentioned above follows from Theorem 2. However, we do not know any example of a set from U^0 which cannot be covered by countably many $(c - c)$-hypersurfaces. Therefore we do not know whether Aronszajn's result characterizes the magnitude of sets $N(f)$.

Added in the proof. Let $f(x)$ be a continuously differentiable function defined on $(0, 1)$ for which the derivative $f'(x)$ is of unbounded variation on each subinterval $I \subset (0, 1)$. Then $\text{Graph} f \subset E^2$ is from U^0 but cannot be covered by countably many $(c - c)$-hypersurfaces. This example shows that our Theorem 2 improves Aronszajn's result.

2. LEMMAS

Let B be a Banach space, f a real function defined in B and $a \in B$, $v \in B$. We shall denote the derivative of f at the point a in the direction v by $D_v f(a)$. Thus

$$D_v f(a) = \lim_{h \to 0} (1/h) (f(a + hv) - f(a)) .$$

The assertions of the following proposition are well known.

Proposition 1. *Let B be a Banach space. Let f be a continuous convex function defined on an open convex set $D_f \subset B$. Let $a \in D_f$. Denote by L_a the set of all $v \in B$ such that there exists $D_v f(a)$. Then the following assertions hold:*

(i) *Let S be the set of all continuous affine functions s defined on B whose graphs support the graph of f at the point $(a, f(a))$ (in other words: $s(a) = f(a)$ and $s(x) \leqq f(x)$ for $x \in B$). Then*

$$L_a = \{v : s_1(a + v) = s_2(a + v) \quad \text{for any} \quad s_1 \in S,\, s_2 \in S\}$$

and $D_v f(a) = s(a + v) - s(a)$ for any $s \in S$ and $v \in L_a$.

(ii) *L_a is a closed linear subspace of B.*

(iii) *f is Gâteaux differentiable at a iff $L_a = B$. In this case and only in this case $S = \{s\}$ and the graph of s is the unique supporting hyperplane of the graph of f at the point $(a, f(a))$.*

(iv) *If B is finite dimensional then f is Gâteaux differentiable at a iff it is Fréchet differentiable at a.*

Proof. The assertion (i) follows easily from Theorem 43A from [9]. The assertions (ii), (iii) are easy consequences of (i). The assertion (iv) is well known (see e.g. Theorem 42D from [9]).

Lemma 1. *Let B be a Banach space and let $M \subset B \times R$ be a set such that for any point $m = (e, t) \in M$ there exists a continuous affine function g_m defined on B such* *that* $g_m(e) = t$ *and* $g_m(x) \leqq y$ *for any point* $(x, y) \in M$ (*in other words: the closed hyperplane* Graph g_m *"supports"* M *at* (e, t)). *Then there exists a sequence* $\{f_n\}_{n=1}^{\infty}$ *of convex Lipschitz functions defined on the whole* B *such that* $M \subset \bigcup_{n=1}^{\infty} \operatorname{Graph} f_n$.

Proof. For any integer n denote by A_n the set of all points $m \in M$ for which g_m is Lipschitz with the constant n. Then clearly $M \subset \bigcup_{n=1}^{\infty} A_n$, the function $f_n(x) = \sup_{m \in A_n} g_m(x)$ is a convex Lipschitz function on B and $A_n \subset \operatorname{Graph} f_n$. Therefore $M \subset \bigcup_{n=1}^{\infty} \operatorname{Graph} f_n$.

Notation. If X is a Banach space, then X' is its dual space.

The main idea of the present article is contained in the following lemma.

Lemma 2. *Let* B *a Banach space (finite or infinite dimensional). Let* K *be a proper subspace of* B *of a finite dimension* u, *let* $\{e_1, \ldots, e_u\}$ *be a basis of* K. *Let* $x_i \in B'$, $i = 1, \ldots, u$ *be continuous linear functionals for which* $x_i(e_j) = \delta_{ij}$. *Let* $H = \{v \in B : x_i(v) = 0, i = 1, \ldots, u\}$. *Then* $B = K + H$ *and for any* $v \in B$ *we have* $v = x_i(v)\, e_i + \pi(v)$ *where* π *is the projection on* H *"in the direction of* K*". Let* f *be a continuous convex function defined on a convex subset* D_f *of* B. *Let* $A \subset D_f$ *be a set such that for any point* $a \in A$ *and any* $v \in K$ *the derivative* $D_v f(a)$ *does not exist. Then there exist convex Lipschitz functions* $Z_i^j(h)$, $T_i^j(h)$, $i = 1, \ldots, u$, $j = 1, 2, \ldots$, *defined on the whole subspace* H *such that any point* $a \in A$ *fulfils the equations*

$$x_1(a) = Z_1^j(\pi(a)) - T_1^j(\pi(a)),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_u(a) = Z_u^j(\pi(a)) - T_u^j(\pi(a))$$

for some integer j.

Proof. Let $a \in A$ and let M_a be the set of all functionals $g \in K'$ such that $f(a) + g(x) \leqq f(a + x)$ for any $x \in K$. The set M_a is evidently convex and closed. We shall prove that it is a u-dimensional convex set. Suppose on the contrary that this is not true. Then there exists $g_0 \in M_a$ and $g_1 \in K', \ldots, g_{u-1} \in K'$ such that for any $g \in M_a$ there exist real numbers $c_1, \ldots, c_{u-1}$ such that $g = g_0 + \sum_{c=1}^{u-1} c_i g_i$. Therefore there exists a nonzero $v \in K$ such that $g(v) = g_0(v)$ for any $g \in M_a$. Therefore by Proposition 1, (i) $D_v f(a)$ exists and this is a contradiction. Since M_a is u-dimensional there exist points $m_i \in M_a$, $i = 0, 1, \ldots, u$, such that the vectors $m_j - m_0$, $j = 1, \ldots, u$, are linearly independent and m_i have rational coordinates with respect to the basis x_i/K, $i = 1, \ldots, u$. Since there are only countably many possibilities for $\{m_i, i = 0, 1, \ldots, u\}$ we can suppose without any loss of generality that the vectors m_i,

$i = 0, 1, \ldots, u$, lie in M_a for any $a \in A$. Let p_i, $i = 0, \ldots, u$, be continuous linear functionals on B (they exist by Theorem 43A from [9]) such that p_i extends m_i and $p_i(x) \leqq f(a + x) - f(a)$ for any $x \in B$ and $i = 0, 1, \ldots, u$. Let $G = \{(x, f(x)) \in B \times R, x \in A\}$. Let $V_i \subset B \times R$ be the graph of p_i, $i = 0, \ldots, u$. Clearly $B \times R = H \times R + V_i$ for $i = 0, \ldots, u$. Let π_i be the projection of $B \times R$ on $H \times R$ "in the direction of V_i". Thus if $z \in B \times R$ and $z = (h + k, y)$ where $h \in H$, $k \in K$, we have $\pi_i(z) = (h, y - m_i(k))$. Let $b = (a, f(a)) \in G$, $a = h_0 + k_0$. Since the closed hyperplane $T_i = b + V_i$, $i = 0, \ldots, u$ supports the graph of f at the point b, $\pi_i(T_i) = T_i \cap H \times R$ is a closed hyperplane in $H \times R$ which "supports" the set $\pi_i(G)$ at the point $\pi_i(b)$ in the sense of Lemma 1. In fact, $\pi_i(T_i)$ is thc graph of the continuous affine function $s(h) = f(a) + p_i(h - a)$, $\pi_i(b) = (h_0, f(a) - m_i(k_0))$ and $s(h_0) = f(a) - m_i(k_0)$. Further, any point $c \in \pi_i(G)$ is of the form $(h_1, f(h_1 + k_1) - m_i(k_1))$, $h_1 \in H$, $k_1 \in K$, and we have $s(h_1) = f(a) + p_i(h_1 - a) \leqq f(h_1 + k_1) - m_i(k_1)$. By Lemma 1 there exist convex Lipschitz functions $C_i^j(h)$, $j = 1, 2, \ldots$, defined on H such that $\pi_i(G)$ is covered by the union of the graphs of the functions $C_i^j(h)$. Consequently, for any point $(a, y) \in G$, $a = h + k$, the equations

$$\begin{aligned} & y - m_0(k) = C_0^{j_0}(h) , \\ & \cdots\cdots\cdots\cdots \\ & y - m_u(k) = C_u^{j_u}(h) \end{aligned} \tag{1}$$

hold for a multiindex $(j_0, \ldots, j_u)$. The equations

$$\begin{aligned} & (m_1 - m_0)(k) = (C_0^{j_0} - C_1^{j_1})(h) , \\ & \cdots\cdots\cdots\cdots \\ & (m_u - m_0)(k) = (C_0^{j_0} - C_u^{j_u})(h) \end{aligned} \tag{2}$$

follow immediately from (1). The linear functionals $m_1 - m_0, \ldots, m_u - m_0$ are linearly independent and the set of all multiindices $(j_0, \ldots, j_u)$ is countable. Thus if we solve the equations (2) with respect to the unknowns $x_1(a) = x_1(k), \ldots, x_u(a) = x_u(k)$ we obtain the assertion of Lemma 2 since the set of all functions on H of the form $C - C'$ where C, C' are Lipschitz convex functions forms a linear space.

3. THE INFINITE DIMENSIONAL SEPARABLE CASE

Definition. 3 Let B be an infinite dimensional Banach space. Let $M \subset B$. We shall say that M is an $(\infty - u)$*-dimensional* $(c - c)$*-surface* if there exist $K, H, x_1, \ldots, x_u, \pi$ as in Lemma 2 and Lipschitz convex functions $Z_1, T_1, \ldots, Z_u, T_u$ defined on H such that M is the set of all points $y \in B$ for which

$$\begin{aligned} & x_1(y) = Z_1(\pi(y)) - T_1(\pi(y)) , \\ & \cdots\cdots\cdots\cdots \\ & x_u(y) = Z_u(\pi(y)) - T_u(\pi(y)) . \end{aligned}$$

Evidently the notion of the $(\infty - 1)$-dimensional $(c - c)$-surface coincides with the notion of the $(c - c)$-hypersurface defined in the first part of the present article.

Proposition 2. *Let B be an infinite dimensional separable Banach space. Let f be a continuous convex function defined on an open convex subset D_f of B. Let u be an integer. Let $A \subset D_f$ be the set of all points $a \in A$ for which there exists a u-dimensional subspace $K_a \subset B$ such that for any $o \neq v \in K_a$, $D_v f(a)$ does not exist. Then A can be covered by a countable union of $(\infty - u)$-dimensional $(c - c)$-surfaces.*

Proof. Let C be a countable dense subset of B. Let L_a be the set of all $v \in B$ for which there exists $D_v f(a)$. The set L_a is a closed linear subspace of B by Proposition 1, (ii). Since $L_a \cap K_a = \{o\}$, there exists a u-dimensional subspace K_a^* such that $L_a \cap K_a^* = \{o\}$ and K_a^* has a basis $c_1, \ldots, c_u$ where $c_1 \in C, \ldots, c_u \in C$. For any u-tuple $(c_1, \ldots, c_u)$ of linearly independent elements of C, denote by $A(c_1, \ldots, c_u)$ the set of all $a \in A$ for which $L_a \cap \operatorname{Lin}(c_1, \ldots, c_u) = \{o\}$. Clearly $A = \bigcup A(c_1, \ldots, c_u)$. By Lemma 2 any set $A(c_1, \ldots, c_u)$ can be covered by a countable union of $(\infty - u)$-dimensional $(c - c)$-surfaces. This implies the assertion of Proposition 2 immediately.

Proposition 3. *Let B be an infinite dimensional Banach space. Let u be an integer. Let $A = \bigcup_{n=1}^{\infty} A_n$ where any A_n is an $(\infty - u)$-dimensional $(c - c)$-surface. Then there exists a continuous convex function f such that for any $a \in A$ there exists a u-dimensional subspace K_a such that for any $o \neq v \in K_a$, $D_v f(a)$ does not exist.*

Proof. Let u be an integer. Let $K, H, x_1, \ldots, x_u, Z_1, T_1, \ldots, Z_u, T_u$ be as in Definition 3 and

$$A_n = \{y : x_i(y) = Z_i(\pi(y)) - T_i(\pi(y)),\ i = 1, \ldots, u\}.$$

Put

$$g_0(y) = T_1(\pi(y)) + \ldots + T_u(\pi(y)) + x_1(y) + \ldots + x_u(y)$$

and

$$g_i(y) = Z_i(\pi(y)) + g_0(y) - T_i(\pi(y)) - x_i(y)$$

for $i = 1, \ldots, u$, $y \in B$.

The functions $g_0, \ldots, g_u$ are clearly Lipschitz and convex. Put $f_n = \max(g_0, \ldots, g_u)$. Then f_n is Lipschitz and convex and for any point $a \in A_n$ and any $o \neq v \in K$, $D_v f_n(a)$ does not exist. In fact, if $a \in A_n$, then we have $g_0(a) = g_1(a) = \ldots = g_u(a) = f_n(a)$ and therefore for $y = a + k$, $k \in K$, we have

$$\begin{aligned} f_n(y) &= f_n(a) + \max((x_1(k) + \ldots + x_u(k)) - x_1(k), \ldots \\ &\quad \ldots, (x_1(k) + \ldots + x_u(k)) - x_u(k), x_1(k) + \ldots + x_u(k)) = \\ &= f_n(a) + h_n(k). \end{aligned}$$

If $v \in K$, then $D_v f_n(a)$ exists iff $D_v h_n(o)$ exists. Further, $D_v h_n(o)$ exists for $v = v_1 e_1 + \ldots + v_u e_u$ iff there exists $D_w h(0, \ldots, 0)$, where $w = (v_1, \ldots, v_u)$ and $h(x_1, \ldots, x_u) = \max((x_1 + \ldots + x_u) - x_1, \ldots, (x_1 + \ldots + x_u) - x_u, x_1 + \ldots + x_u) = x_1 + \ldots + x_u - \min(0, x_1, \ldots, x_u)$.

This follows from the fact that the function h_n corresponds to the function h in the isomorphism between E^u and K defined by the identification $(c_1, \ldots, c_u) = c_1 e_1 + \ldots + c_u e_u$. If we put $g(x_1, \ldots, x_u) = \min(0, x_1, \ldots, x_u)$ then it is easy to see that $D_w g(0, \ldots, 0)$ exists for no $o \neq w \in E^u$. Therefore $D_w h(0, \ldots, 0)$ exists for no $o \neq w \in E^u$. Now it is clearly sufficient to put $f = \sum_{h=1}^{\infty} c_n f_n$, where $c_n > 0$ are sufficiently small numbers. It is possible to put $c_n = n^{-2} (\sup_{\|x\| \leqq n} |f_n(x)|)^{-1}$. Theorem 2 which is stated in the first part is a consequence of Proposition 2 and Proposition 3 in the case $u = 1$.

Note 1. If we write in the definition of the $(c - c)$-hypersurface (Definition 2) "continuous convex functions f, g" instead of "convex Lipschitz functions f, g", Theorem 2 also holds. It follows easily from Lemma 1.

4. THE FINITE DIMENSIONAL CASE

Proof of Theorem 1. We must prove the following assertions:

(A) Let f be a convex function defined in E^n. Then $S_k(f)$ can be covered by countably many $(c - c)$-surfaces of dimension k.

(B) Let $M \subset E^n$ be a countable union of $(c - c)$-surfaces of dimension k. Then there exists a convex function f defined on E^n such that $M \subset S_k(f)$.

Let f be a convex function defined in E^n. Let $x \in S_k(f)$. Since $\dim L_x \leqq k$ there exists a permutation π of the numbers $1, \ldots, n$ such that $L_x \cap \mathrm{Lin}(e_{\pi(k+1)}, \ldots, e_{\pi(n)}) = \{o\}$ where e_j is the j-th unit coordinate vector. If we use Lemma 2 for $K_\pi = \mathrm{Lin}(e_{\pi(k+1)}, \ldots, e_{\pi(n)})$, $H_\pi = \mathrm{Lin}(e_{\pi(1)}, \ldots, e_{\pi(k)})$ and for all possible permutations π we obtain the assertion (A).

The proof of the assertion (B) is essentially the same as the proof of Proposition 3.

Note 2. If we write in the definition of the $(c - c)$-surface of dimension k (Definition 1) "convex continuous functions $f_{k+1}, g_{k+1}, \ldots, f_n, g_n$" instead of "convex Lipschitz functions $f_{k+1}, g_{k+1}, \ldots, f_n, g_n$", Theorem 1 also holds. It follows immediately from Lemma 1.

We shall now prove that for any $0 < k < n$ there exists a countably k rectifiable set (for definition see [6]), which cannot be covered by a countable union of $(c - c)$-surfaces of dimension k. Thus Theorem 1 improves the results of [1] and [3].

Example 1. Let $0 < k < n$. Let g be a Lipschitz function defined on $\langle 0, 1\rangle$ which is not differentiable at any point of a perfect set $P \subset \langle 0, 1\rangle$. It is well known that such a function g exists. For example, the function f from [10], p. 136, Remarque 3 is such a function. Define $f : \langle 0, 1\rangle^k \to E^{n-k}$ by the equation $f(x_1, \ldots, x_k) = (g(x_1), 0, \ldots, 0)$. Clearly f is Lipschitz. Put $M = \operatorname{Graph} f \subset \langle 0, 1\rangle^k \times E^{n-k} \subset E^n$. The set M is clearly countably k rectifiable. We shall prove that M cannot be covered by a countable union of $(c - c)$-surfaces of dimension k. Suppose that $M \subset \bigcup_{s=1}^{\infty} A_s$, where A_s is a $(c - c)$-surface of dimension k for any integer s. Since M is a complete subspace of E^n and any set $M \cap A_s$ is closed in M there exists an open ball B in E^n and an index s_0 such that $M \cap B \subset A_{s_0} \cap B$. Let D be the set of all $x \in M \cap B$ for which $\operatorname{Tan}(M, x)$ ([6], p. 233) is a linear space (in other words: there exists a linear tangent manifold of M at x). From the definition of M it follows easily that D is not of σ-finite $(k - 1)$-dimensional Hausdorff measure. But Theorem 1 easily implies that D is of σ-finite $(k - 1)$-dimensional measure and this is a contradiction.

Note 3. Let $M \subset E^n$, $0 < k < n$ and let there exist $2n - 2k$ convex functions $f_{k+1}, g_{k+1}, \ldots, f_n, g_n$ defined on the whole space E^k and such a system of orthonormal coordinates $y_1, \ldots, y_n$ that M is the set of all points of E^n for which

$$y_j = f_j(y_1, \ldots, y_k) - g_j(y_1, \ldots, y_k) \quad \text{for} \quad j = k + 1, \ldots, n\,.$$

From the proof of Proposition 3 it is easily seen that M is the set $S_k(f)$ for a convex function f in E^n. Therefore M can be covered by countably many $(c - c)$-surfaces of dimension k.

5. SINGULAR BOUNDARY POINTS OF CONVEX BODIES IN E^n

Theorem 3. *Suppose C is a convex body in E^{n+1} and for each boundary point x of C, let H_x be the intersection of all hyperplanes which support C at x. For $0 < k < n$ let $B_k = \{x : x \in \operatorname{Bd}(C)$ and $\dim H_x \leqq k\}$. Then B_k can be covered by countably many $(c - c)$-surfaces of dimension k.*

Proof. Near to a boundary point x, the surface of the body can be represented by means of a convex function defined on a hyperplane supporting the body at x. Note 3 implies that it is sufficient to prove that for any convex function f in E^n the set $\{[x, y] : x \in S_k(f),\ y = f(x)\}$ can be covered by countably many $(c - c)$-surfaces of dimension k in E^{n+1}. We shall show that the last proposition follows from the proof of Lemma 2. For this it is sufficient to prove from (1) and (2) that $y = C(h) - C^*(h)$ where C, C^* are Lipschitz convex functions defined on H. But this immediately follows from the equation $y = m_0(k) + C_0^{j_0}(h)$ since $m_0(k)$ is a fixed linear combination of $x_1(a), \ldots, x_u(a)$.

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Author's address: 186 00 Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK).