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ON THE MEASURABLE SOLUTIONS OF CERTAIN FUNCTIONAL EQUATIONS

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1. INTRODUCTION

Let $\Delta_n = \{(p_1, p_2 \dots p_n) : p_i \geqq 0 \text{ for } i = 1, \dots, n, \sum_{i=1}^{n} p_i = 1\}$ for $n \geqq 1$ denote the set of all n-ary probability distributions.

Let $f : [0, 1] \to R$ (Reals) be measurable (or continuous at one point or bounded in a small interval) and satisfies the functional equation

$$(1.1) \quad \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i y_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i^\alpha f(y_j) + \sum_{i=1}^{m} \sum_{j=1}^{n} y_j^\beta f(x_i), \quad \alpha \neq \beta; \quad \alpha, \beta > 0$$
$$m = 2, 3; \quad n = 2, 3,$$

where $X \in \Delta_m$, $Y \in \Delta_n$.

Let $F : [0, 1] \times [0, 1] \to R$ (Reals) be measurable in each variable and satisfies the functional equation

$$(1.2) \qquad \sum_{i=1}^{m} \sum_{j=1}^{n} F(x_i y_j, u_i v_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i^\alpha u_i^\beta F(y_j, v_j) + \sum_{i=1}^{m} \sum_{j=1}^{n} y_j^r v_j^\delta F(x_i, u_i)$$
$$\alpha, \beta, r, \delta > 0, \ (\alpha - r)(\delta - \beta) < 0,$$

where $X, U \in \Delta_m$, $Y, V \in \Delta_n$, $m = 2, 3$; $n = 2, 3$ and $F(1, 0) = 0$.

The object of this paper is to find the measurable (or continuous at one point, or bounded in a small interval) solutions of (1.1) and (1.2). So far functional equation (1.1) was solved under continuity while (1.2) under continuity for

$$\sum_{i=1}^{m} x_i = 1 = \sum_{j=1}^{n} y_i, \quad \sum_{i=1}^{m} u_i \leqq 1 \quad \text{and} \quad \sum_{j=1}^{n} v_j \leqq 1,$$

by SHARMA and TANEJA [5]. Their treatment is not clear at several places because the domains of the parameters are not properly defined.

2. THE SOLUTION OF (1.1)

In order to derive the measurable solutions of (1.1), we prove some lemmas in what follows.

Lemma 2.1.

$$f(0) = f(1) = 0.$$

Proof. Let $(x_1, x_2) = (0, 1)$, $(y_1, y_2, y_3) = (0, 1, 0)$ in (1.1). Then

$$2f(0) = f(1)\,. \tag{2.1}$$

Substituting $(x_1, x_2) = (0, 1)$, $(y_1, y_2, y_3) = (y, 1 - y, 0)$ for $y \in (0, 1)$ in (1.1) yields,

$$3f(0) = [y^\beta + (1 - y)^\beta]\,.\,[3f(0)]\,, \quad y \in (0, 1)\,, \quad \beta \neq 1\,.$$

Hence, $f(0) = 0$ and therefore from (2.1), we have

$$f(1) = f(0) = 0\,. \tag{2.2}$$

Lemma 2.2. *For $X \in \Delta_n$, $n = 2, 3$*

$$\sum_{i=1}^{n} f(x_i) = C \sum_{i=1}^{n} [x_i^\alpha - x_i^\beta]\,, \quad \alpha, \beta > 0\,, \quad \alpha \neq \beta\,,$$

where C is an arbitrary constant.

Proof. For $n = 2$ or 3, since

$$\sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i y_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} f(y_j x_i)$$

from (1.1), we have

$$\left(\sum_{i=1}^{n} x_i^\alpha\right)\left(\sum_{j=1}^{n} f(y_j)\right) + \left(\sum_{j=1}^{n} y_j^\beta\right)\left(\sum_{i=1}^{n} f(x_i)\right) =$$

$$= \left(\sum_{i=1}^{n} y_i^\alpha\right)\left(\sum_{j=1}^{n} f(x_j)\right) + \left(\sum_{j=1}^{n} x_j^\beta\right)\left(\sum_{i=1}^{n} f(y_i)\right).$$

Thus

$$\frac{\sum_{j=1}^{n} f(y_j)}{\sum_{j=1}^{n} (y_j^\alpha - y_j^\beta)} = \frac{\sum_{i=1}^{n} f(x_i)}{\sum_{i=1}^{n} (x_i^\alpha - x_i^\beta)} = C\,, \quad \text{for} \quad x_i, y_j \in (0, 1)\,.$$

Note that since $\alpha \neq \beta$, the denominator will not vanish. Thus,

$$\sum_{i=1}^{n} f(x_i) = C \sum_{i=1}^{n} [x_i^\alpha - x_i^\beta]\,, \quad x_i \in (0, 1)\,. \tag{2.3}$$

Which together with the fact that $f(1) = f(0) = 0$ makes (2.3) true for all $x_i \in [0, 1]$ with $\sum_{i=1}^{n} x_i = 1$.

Remark. Lemma 2.2 is proved above without any regularity condition. When $\alpha = 1$ and $f(\frac{1}{2}) = \frac{1}{2}$, Lemma 2.2 gives

$$\sum_{i=1}^{n} f(x_i) = \frac{1 - \sum_{i=1}^{n} x_i^\beta}{1 - 2^{1-\beta}}$$

which is the non-additive entropy of order β. See another functional equation in [4].

The same result as a theorem on p. 213 was proved by Sharma and Taneja [5] assuming continuity.

Lemma 2.3. *For fixed* $x \in [0, 1]$, *if*

$$(2.4)\qquad A_x(t) = f(xt) + f((1-x)\,t) - [x^\alpha + (1-x)^\alpha]\,f(t) - \\ - t^\beta[f(x) + f(1-x)]\,, \quad t \in [0, 1]$$

then

$$A_x(u + v) = A_x(u) + A_x(v)\,, \quad u, v \in [0, 1]\,.$$

Proof. Let $(x_1, x_2) = (x, 1-x)$, $(y_1, y_2, y_3) = (u, v, 1-u-v)$ in (1, 1) for $x, u, v, u + v \in [0, 1]$, then

$$(2.5)\qquad f(xu) + f((1-x)\,u) + f(xv) + f((1-x)\,v) + f(x(1-u-v)) + \\ + f((1-x)(1-u-v)) = [x^\alpha + (1-x)^\alpha]\,. \\ .\,[f(u) + f(v) + f(1-u-v)] + [u^\beta + v^\beta + (1-u-v)^\beta]\,.\,[f(x) + f(1-x)]\,.$$

Let $(x_1, x_2) = (x, 1-x)$, $(y_1, y_2, y_3) = (u + v, 1-u-v, 0)$ in (1.1), for x, u, v as above, then

$$(2.6)\qquad f(x(u+v)) + f((1-x)(u+v)) + f(x(1-u-v)) + \\ + f((1-x)(1-u-v)) = [x^\alpha + (1-x)^\alpha]\,.\,[f(u+v) + f(1-u-v)] + \\ + [(u+v)^\beta + (1-u-v)^\beta]\,.\,[f(x) + f(1-x)]\,.$$

Substracting (2.5) from (2.6) and using (2.4), we have, for fixed $x \in [0, 1]$.

$$(2.7)\qquad A_x(u+v) = A_x(u) + A_x(v)\,, \quad \text{for} \quad u, v, u + v \in [0, 1]\,.$$

This implies that $A_x(t)$ is additive in t.

Now, we can prove the following theorem.

Theorem 2.1. *If* $f : [0, 1] \to R$ *(Reals) satisfies the functional equation* (1.1) *and* f *has any of the following properties*:

(a) f *is continuous at a point,*

(b) f *is bounded in a small interval,*

(c) f *is measurable,*

then, $f(x) = C[x^\alpha - x^\beta]$, $\alpha, \beta > 0$, $\alpha \neq \beta$, *where* C *is an arbitrary constant.*

Proof. As $A_x(t)$ defined in (2.4) is measurable (or continuous at a point, or bounded in a small interval), we conclude by [2] that

$$(2.8)\qquad A_x(t) = A_x(1)\,.\,t$$

from the fact that $f(1) = 0$, we see that $A_x(1) = 0$, hence from (2.4) and Lemma 2.2, we get

$$(2.9)\qquad f(xu) + f((1-x)\,u) = [x^\alpha + (1-x)^\alpha]\,f(u) + u^\beta[f(x) + f(1-x)] = \\ = [x^\alpha + (1-x)^\alpha]\,f(u) + u^\beta[x^\alpha - x^\beta + (1-x)^\alpha - (1-x)^\beta]\,.\,C\,.$$

From Lemma 2.2, we know that

$$f(1-u)+f(xu)+f((1-x)u = \tag{2.10}$$
$$= C[(1-u)^\alpha-(1-u)^\beta+(xu)^\alpha-(xu)^\beta+((1-x)u)^\alpha-((1-x)u)^\beta].$$

Substracting (2.9) from (2.10), and then substituting $x=\frac{1}{2}$, it is easy to see that

$$f(1-u)=[2.\left(\frac{u}{2}\right)^\alpha-2.\left(\frac{u}{2}\right)^\beta+(1-u)^\alpha-(1-u)^\beta].C- \tag{2.11}$$
$$-2^{1-\alpha}f(u)-u^\beta[2^{1-\alpha}-2^{1-\beta}].C.$$

But from (2.3),

$$f(u)+f(1-u)=[u^\alpha-u^\beta+(1-u)^\alpha-(1-u)^\beta].C \tag{2.12}$$

combining (2.11) and (2.12), we get

$$f(u)=C[u^\alpha-u^\beta],\quad \alpha\neq 1,\quad u\in[0,1]. \tag{2.13}$$

For $\alpha=1$, the Lemma 2.2 gives

$$\sum_{i=1}^n f(x_i)=C[1-\sum_{i=1}^n x_i^\beta]. \tag{2.14}$$

Hence (1.1) and (2.14) yields

$$\sum_{i=1}^m\sum_{j=1}^m f(x_iy_j)=\sum_{j=1}^m f(y_j)+C(\sum_{j=1}^m y_j^\beta)(1-\sum_{i=1}^m x_i^\beta)= \tag{2.15}$$
$$=\sum_{j=1}^m[f(y_j)+Cy_j^\beta]-C\sum_{i=1}^m\sum_{j=1}^m x_i^\beta y_j^\beta.$$

Let

$$h(t)=f(t)+ct^\beta \tag{2.16}$$

then using $f(0)=0=f(1)$, we get $h(0)=0$, $h(1)=C$. Hence (2.15) becomes

$$\sum_{i=1}^m\sum_{j=1}^m h(x_iy_j)=\sum_{j=1}^m h(y_j),\quad m=2,3.$$

But since

$$\sum_{i=1}^m\sum_{j=1}^m h(x_iy_j)=\sum_{j=1}^m\sum_{i=1}^m h(y_jx_i),$$

it is clear that

$$\sum_{j=1}^m h(y_j)=\sum_{i=1}^m h(x_i)=C,\quad X,Y\in\Delta_m,\quad \text{where}\quad m=2,3.$$

Since $h(0)=0$, we get

$$\sum_{i=1}^m h(x_i)=C\quad\text{for}\quad m=2,3. \tag{2.17}$$

For fixed $x \in [0, 1]$, if we take

$$A_x(t) = h(xt) + h((1 - x)\, t)\,, \quad t \in [0, 1] \tag{2.18}$$

and use the method employed in Lemma 2.3, we obtain,

$$A_x(u + v) = A_x(u) + A_x(v)\,, \quad \text{for} \quad u, v, u + v \in [0, 1]\,.$$

Again by [2]

$$A_x(u) = A_x(1)\,.\,u\,.$$

Since $A_x(1) = C$, we have

$$h(xu) + h((1 - x)\, u) = Cu\,, \quad u \in [0, 1]\,, \quad x \in [0, 1]\,.$$

For $x = 1$, $h(u) = Cu$. Thus

$$f(u) = C[u - u^{\beta}]\,, \quad u \in [0, 1]\,. \tag{2.19}$$

Thus (2.13) and (2.19) prove theorem 2.1.

3. THE SOLUTION OF (1.2)

Let $(x_1, x_2) = (0, 1) = (u_1, u_2)$, $(y_1, y_2, y_3) = (0, 1, 0) = (v_1, v_2, v_3)$ in (1.2) then we have

$$2\, F(0, 0) = F(1, 1)\,. \tag{3.1}$$

Let $(x_1, x_2) = (0, 1) = (u_1, u_2)$, $(y_1, y_2, y_3) = (y, 1 - y, 0)$ and $(v_1, v_2, v_3) = (v, 1 - v, 0)$ for $y, v \in [0, 1]$ in (1.2), then

$$3\, F(0, 0) = [F(0. 0) + F(1, 1)]\,.\,[y^r v^{\delta} + (1 - y)^r\, (1 - v)^{\delta}]\,. \tag{3.2}$$

Combining (3.1) and (3.2), we get

$$F(0, 0) = F(1, 1) = 0\,. \tag{3.3}$$

The following lemmas can be proved by the method employed for proving Lemma 2.2 and 2.3.

Lemma 3.1. *For* $x, u \in \Delta_m$, $m = 2, 3$

$$\sum_{i=1}^{m} F(x_i, u_i) = C \sum_{i=1}^{m} (x_i^{\alpha} u_i^{\beta} - x_i^{r} u_i^{\delta})\,, \quad \alpha, \beta, \gamma, \delta > 0 \quad (\alpha - \gamma)\,(\delta - \beta) < 0\,. \tag{3.4}$$

Lemma 3.2. *For fixed* $x, u \in [0, 1]$ *if*

$$A_{xu}(p, q) = F(xp, uq) + F((1 - x)\, p, (1 - u)\, q) - F(p, q)\,. \tag{3.5}$$
$$.\, [x^{\alpha} u^{\beta} + (1 - x)^{\alpha}\, (1 - u)^{\beta}] - p^r q^{\delta} [F(x, u) + F(1 - x, 1 - u)]\,,$$
$$p, q \in [0, 1]$$

then

(3.6)

$$A_{xu}(y + \omega, v + t) = A_{xu}(y, v) + A_{xu}(\omega, t), \; y, \omega\, v, t, y + \omega, v + t \in [0, 1] .$$

Lemma 3.3. *For* $x, u \in [0, 1]$, *if* $A_{xu}(\cdot, \cdot)$ *is measurable (or continuous at one point, or bounded in a small interval) in each of its variables and satisfies* (3.6), *then*

$$A_{xu}(p, q) = A_{xu}(1, 0) . p + A_{xu}(0, 1) . q . \tag{3.7}$$

Proof. Putting $v = 0 = t$ in (3.6), we have

$$A_{xu}(y + \omega, 0) = A_{xu}(y, 0) + A_{xu}(\omega, 0) , \quad y, \omega, y + \omega \in [0, 1] .$$

As before,

$$A_{xu}(y, 0) = A_{xu}(1, 0) . y .$$

Similarly,

$$A_{xu}(0, t) = A_{xu}(0, 1) . t .$$

Hence,

$$A_{xu}(p, q) = A_{xu}(p, 0) + A_{xu}(0, q) = A_{xu}(1, 0) . p + A_{xu}(0, 1) . q .$$

Lemma 3.4. *For all* $p \in [0, 1]$, *we have*

(3.8) (a) $F(p, 0) = pF(1, 0)$, $F(0, p) = pF(0, 1)$,

(b) $F(1, 0) = F(0, 1) = F(p, 0) = F(0, p) = 0$.

Proof. The equation (3.7) for $q = 0$ yields

$$A_{xu}(p, 0) = A_{xu}(1, 0) . p$$

which on using (3.5) gives

$$F(xp, 0) + F(1 - x)\, p, 0) - F(p, 0) . [x^\alpha u^\beta + (1 - x)^\alpha (1 - u)^\beta] = \tag{3.9}$$
$$= p\{F(x, 0) + F(1 - x, 0) - F(1, 0) . [x^\alpha u^\beta + (1 - x)^\alpha (1 - u)^\beta]\} .$$

In (3.9), let $x = 1$. Then

$$F(p, 0) = pF(1, 0) , \quad p \in [0, 1] . \tag{3.10}$$

Similarly, we can get

$$F(0, p) = pF(0, 1) , \quad p \in [0, 1] . \tag{3.11}$$

Let $(x_1, x_2) = (1, 0)$, $(u_1, u_2) = (0, 1)$, $(y_1, y_2, y_3) = (1, 0, 0)$ and $(v_1, v_2, v_3) = (0, 1, 0)$ in (1.2). Then, with the help of (3.3) we have

$$F(1, 0) + F(0, 1) = 0 .$$

Since $F(1, 0) = 0$, we have

$$F(0, 1) = 0 \quad \text{and} \quad F(p, 0) = F(0, p) = 0 , \quad p \in [0, 1] .$$

Theorem 3.1. *If $F : [0, 1] \times [0, 1] \to R$ (Reals) satisfies the functional equation (1.2) with $F(1, 0) = 0$, and F is measurable (or continuous at a point, or bounded in a small interval) in each of its variables. Then*

(3.12)
$$F(p, q) = C[p^\alpha q^\beta - p^r q^\delta], \quad \text{where} \quad \alpha, \beta, r, \delta > 0 \ (\alpha - r)(\delta - \beta) < 0$$
where C is an arbitrary constant.

Proof. From Lemma 3.4, it is clear that
$$A_{xu}(1, 0) = 0 = A_{xu}(0, 1).$$
Hence from (3.7), we have
$$A_{xu}(p, q) = 0$$
which when combined with (3.5) gives

(3.13)
$$F(xp, uq) + F((1 - x) p, (1 - u) q) =$$
$$= F(p, q) . [x^\alpha u^\beta + (1 - x)^\alpha (1 - u)^\beta] + p^r q^\delta [F(x, u) + F(1 - x, 1 - u)].$$

But we know from Lemma 3.1 that

(3.14)
$$F(1 - p, 1 - q) + F(xp, uq) + F((1 - x) p, (1 - u) q) =$$
$$= C[(1 - p)^\alpha (1 - q)^\beta - (1 - p)^r (1 - q)^\delta + (xp)^\alpha (uq)^\beta - (xp)^r (uq)^\delta +$$
$$+ ((1 - x) p)^\alpha ((1 - u) q)^\beta - ((1 - x) p)^r ((1 - u) q)^\delta].$$

Now, substracting (3.13) from (3.14) and then using (3.4) and substituting $x = u = \frac{1}{2}$, we have

(3.15)
$$F(1 - p, 1 - q) = 2^{1-\alpha-\beta}[Cp^\alpha q^\beta - Cp^r q^\delta - F(p, q)] +$$
$$+ C . [(1 - p)^\alpha (1 - q)^\beta - (1 - p)^r (1 - q)^\delta].$$

From (3.15) and (3.4), we get

(3.16)
$$F(p, q) = C[p^\alpha q^\beta - p^r q^\delta] \quad \text{for} \quad \alpha + \beta \neq 1, \quad p, q \in [0, 1].$$

When $\alpha + \beta = 1$, Lemma 3.1 gives

(3.17)
$$\sum_{i=1}^{n} F(x_i, u_i) = C \sum_{i=1}^{n} (x_i^\alpha u_i^{1-\alpha} - x_i^r u_i^\delta), \quad n = 2, 3.$$

For $x_i = u_i$, $i = 1, \ldots, m$; $y_j = v_j$, $j = 1, \ldots, n$ and $\alpha + \beta = 1$ the equation (1.2) reduces to the equation (1.1). Hence using theorem 2.1, we have

(3.18)
$$F(x, x) = C[x - x^{r+\delta}] \quad r + \delta \neq 1.$$

Without loss of generality, suppose $p < q$, then (3.17) yields,

(3.19)
$$F(p, q) + F(1 - q, 1 - q) + F(q - p, 0) =$$
$$= C[p^\alpha q^{1-\alpha} - p^r q^\delta + (1 - q) - (1 - q)^{r+\delta}].$$

The equation (3.19) on using (3.18) and (3.8) gives

(3.20)
$$F(p, q) = C[p^\alpha q^{1-\alpha} - p^r q^\delta], \quad r + \delta \neq 1, \quad p, q \in [0, 1].$$

Thus (3.16) and (3.20) prove theorem 3.5.

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