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CONGRUENCE EXTENSION FROM A SEMILATTICE TO THE FREELY GENERATED DISTRIBUTIVE LATTICE

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The author has shown that every congruence for the $\wedge$-operation of a semilattice which is also compatible for any set of distributive $\vee$'s is induced by a lattice congruence on the distributive lattice freely generated with preservation of these sups [F, Theorem DL]. Cornish and Hickman [CH], and in more detail Hickman [H], have addressed the question of uniqueness of the inducing congruence, basing themselves on a preliminary analysis of the structure of these congruences. It will be pointed out here that their main results may be obtained more simply, and in a considerably more general setting, by a systematic use of the author's result; moreover an explicit form for the smallest inducing congruence may in their setting be deduced from MacNeille's construction [M] of the freely generated lattice – a construction to which they have inadvertently had recourse for their study of congruences.

For example, the main result of [CH], that uniqueness obtains in the presence of "the upper bound property", may be seen rather quickly by remembering that the lattice L consists of $\vee$'s of families in the semilattice S. Then if one attempts to identify $\bigvee a_i$ with $\bigvee b_j$ in L one will be forced to identify a_i with $a_i \wedge \bigvee b_j = \bigvee_j a_i \wedge \wedge b_j$ which, being bounded by $a_i \in S$, is $= c_i \in S$; similarly $b_j \sim \bigvee_i a_i \wedge b_j = d_j \in \in S$; and of course $\bigvee c_i = \bigvee d_j$ in L. But conversely these conditions require the identification of $\bigvee a_i$ with $\bigvee b_j$ by any lattice congruence inducing $\sim$ in S, and therefore such a congruence is uniquely determined. Unlike the development in [CH], the $\vee$'s occurring here are not limited to be finite, nor has it been necessary to postulate that all existing (finite) $\vee$'s in S be distributive, as will be elaborated more fully below.

Again, the (at first sight unexpected) implication i) $\rightarrow$ ii) of [H, Theorem 2.4], to the effect that non-uniqueness of the inducing lattice congruence entails that restriction to S does not preserve the join of congruences in L (whose proof in [H] takes up two and a half pages) drops out as follows: By dividing modulo a congruence on S which extends non-uniquely to L one reduces to a non-identical congruence $\sim$ on L inducing the identity on S (this type of reduction will also be detailed more fully); say it identifies the pair $\bigvee a_i < \bigvee b_j$ where some $b_j = b \nleq \bigvee a_i$. According to the form of the principal congruence generated by a comparable pair in a distributive lattice [G, Theorem 3 p. 74], $\theta(a \wedge b, \bigvee a_i)$ for any $a = a_i$ fails to identify a and b while $\theta(\bigvee a_i, \bigvee b_j) \subset \sim$ induces the identity on S, whence the join of their restrictions fails to identify a and b – but their join in L, $\theta(a \wedge b, \bigvee b_j)$, does. This, too, will go through in a more general setting; we turn now to spelling out this generalization.

Recall that a (not necessarily finite) sup $\bigvee a_i$ in a semilattice S is *distributive* if for each a the sup of the translate by a of the a_i, $\bigvee a \wedge a_i$, also exists and $= a \wedge \bigvee a_i$.[1]) MacNeille [M] has shown (an outline is sketched in [F′]) that every $\bigvee$-semilattice can be embedded in a complete, completely join-distributive lattice with preservation of all its distributive $\vee$'s. It follows that for any collection D of subsets having distributive $\vee$'s, the universal morphism or reflection (into the subcategory of complete $\vee$-preserving morphisms) which sends S into the complete such lattice L required to preserve only the $\vee$'s in D, is also an embedding. Note that the complete join-distributivity of L entails that it consists of the $\vee$'s (in L) of subsets of S (i.e. S is $\vee$-generating or "join-dense"), the pairwise $\wedge$ being calculated as $(\bigvee a_i) \wedge (\bigvee b_j) = \bigvee(a_i \wedge b_j)$. S is a fortiori join-dense in every intermediate $\wedge$-semilattice $S \subset F \subset L$; and the inclusion into F is universal among semilattice morphisms S' from S which preserve the $\vee$'s in D as well as send some of its subsets, whose $\vee$'s include the elements of F, (these subsets may be taken closed for pariwise $\wedge$'s) to subsets having distributive $\vee$'s: Indeed for any such morphism the image semilattice S' may be embedded into a complete L' with preservation of the $\vee$'s of both the images of the subsets in D and of any of those whose $\vee$'s include all the elements of F; and then the given semilattice morphism, which may even be extended to send L into L' in complete $\vee$-preserving fashion,[2]) will have a restriction to F which sends it into S'.

It was shown in [F, Theorem DL] that every congruence for $\wedge$ compatible also for the $\vee$'s in a translation-closed class D (in the "weak" sense that from $a_i \sim a_i'$ and the presence of both subsets in D follows $\bigvee a_i \sim \bigvee a_i'$) is the restriction of a lattice congruence on F — this was done there for the case that D consists only of pairwise $\vee$'s in S and F of the $\vee$'s of its finite subsets in L, but the result holds in general: the quotient map modulo the given congruence $\sim$ sends the $\vee$'s in D to distributive $\vee$'s in the quotient semilattice[3]) whence its composition with the embedding of the quotient into the complete lattice universal for preserving these image $\vee$'s extends to a complete $\vee$-preserving morphism on L whose kernel meets any intermediate F in a congruence extending $\sim$. It is in fact the smallest on F, compatible for the $\vee$'s

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[1]) This identity for any (partial) operation $\vee \geqq$ its arguments already shows it to be sup: for if $a_i = a \wedge a_i$ then $\bigvee a_i = a \wedge \bigvee a_i$.

[2]) Thus L is also the universal, complete $\vee$-preserving, completion of F for preservation of D augmented with such a collection of (L—) $\vee$'s of S-subsets (they are distributive in F).

[3]) Use footnote 1; the structure of a translation-closed class of $\vee$'s is preserved by homomorphic image, product and subsemilattice.

of any translation-closed class including D and yielding all of F, which induces $\sim$ on S, as follows from the universality noted on the preceding page of the inclusion of $S/\sim$ into the image of F.

An explicit form for this smallest congruence could therefore be derived by carrying back to F a description of equality in such complete join-distributive lattices universal over the quotient semilattice. Now such a description appears in MacNeille [M] at the beginning of Section 12 p. 446 (one may also consult [F']) for the case D *all* distributive $\vee$'s in S and $F = L$; but analysis of his argument shows that it remains valid, thus furnishing the desired description, if only D is translation closed, includes every subset having a greatest element, and satisfies the following "Fubini" condition: if a doubly-indexed a_{ij} has $\bigvee_j a_{ij}$, $\bigvee_i a_{ij}$ and $\bigvee_j(\bigvee_i a_{ij})$ all in D then it also has $\bigvee_i(\bigvee_j a_{ij})$ [4]. Specifically, let such a collection D of subsets having distributive $\vee$'s in S be specified. Then on the power set of S (construed as indexed subsets of S) the relation: for every j, $\{a_i \wedge b_j\}_i \in D$ and $\bigvee_i a_i \wedge b_j = b_j$, is reflexive and transitive. Modulo the equivalence making this a partial order the image of every subset is the join of its singleton subset images, whence the quotient is a complete lattice. The images of singletons are an order-isomorphic copy of S, which permits writing every element of the lattice (not necessarily uniquely) as $\bigvee a_i$, inasmuch as the $\vee$'s in D are preserved by the embedding of S (in fact they are the only singleton $\vee$'s of singletons). The finite $\vee$'s in S are preserved by virtue of $(\bigvee a_i) \wedge (\bigvee b_j) = \bigvee_{ij} a_i \wedge b_j$, from which also complete join-distributivity follows.

It does not appear to be particularly easy to assure the Fubini condition in a quotient. It would pass to the images, from a class of distributive $\vee$'s in which it holds, modulo an $\bigwedge$-congruence compatible with them in the "strong" sense that from $a_i \sim a_i'$ and the presence of one of the subsets in D follows that of the other (in addition to $\bigvee a_i \sim \bigvee a_i'$). It also holds in the class of all subsets less numerous than some regular cardinal which have distributive $\vee$'s in S, and so with D just these subsets augumented with the subsets having greatest elements, the above sufficient criterion for the description of equality in L is fulfilled. The augmentation is not needed for the description of equality in the sublattice F of $\vee$'s of such cardinal-limited subsets of S: This F realizes the lattice over S universal for preserving all the $\vee$'s in D and in which all limited subsets have distributive $\vee$'s. This is the situation treated by [CH] when the limited subsets are the finite ones (under the additional requirement that all existent finite $\vee$'s are distributive) and by [FS] more generally when they are those bounded by a regular cardinal – both treatments retrace MacNeille's.

Now when the images of the designated class D of subsets with distributive $\vee$'s in S fulfill the above criterion modulo $\sim$, an $\wedge$-congruence compatible for these $\vee$'s,

[4]) Only the presence in D of $\{\bigvee_j a_{ij}\}$ as an i-indexed subset is at issue: its $\vee$, $\bigvee_{ij} a_{ij}$, exists in S and (as a $\vee$ which becomes distributive on decomposing its terms into distributive $\vee$'s) is distributive. Also, it is enough to have this for $a_{ij} = a_i \wedge b_j$.

then MacNeille's description of equality for $\vee$'s of subsets of $S/\sim$ in the distributive lattice completion universal for preserving all these image $\vee$'s, yields that the smallest lattice congruence extending $\sim$ to any distributive lattice universal for preserving the $\bigvee$'s in D identifies $\bigvee a_i$ with $\bigvee b_j$ (recall that this universal lattice, as a sublattice of L, consists of $\vee$'s of subsets of S) just when $a_i \wedge b_j \sim c_{ij} \in S$ with $a_i \sim \bigvee_j c_{ij} \in D$ (whence also $\sim \bigvee_j a_i \wedge c_{ij} \in D$) and symmetrically $b_j \sim \bigvee_i b_i \wedge d_{ji}$.

Some supplementary remarks: One can of course always reduce $\bigvee a_i \sim \bigvee b_j$ to $a_i \sim \bigvee a_i \wedge b_j$ (and symmetrically) by using that $\sim$ is an $\wedge$-congruence and that all $\vee$'s in F are distributive, but the $\vee$ on the right is in general only in F and so this does not, like MacNeille's, provide a description internal to S – unless one makes an additional postulate such as the "upper bound property" of [CH]: Cf the analysis of their theorem given at the beginning.

They have couched their discussion in terms of the operation of restriction of congruences from F to S (a surjection by the author's cited Theorem DL): this preserves arbitrary $\cap$ (and so in particular there is a smallest inducing congruence) whence the lack of lattice preservation touches only the join of congruences. A dual discussion would assign to each congruence on S its smallest inducing congruence on F and lead to a complete join-preserving injection of the congruence lattice of S onto the subsemilattice of smallest inducing congruences of F – but now it is presumably $\cap$ which is not preserved. (The smallest congruence inducing $\sim$ must identify $\bigvee a_i$ with $\bigvee a'_i$ whenever $a_i \sim a'_i$: this is already reflexive symmetric and substitutional but taking the transitive closure cannot be expected to commute with $\cap$.)

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