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BOLZANO'S INFINITESIMAL NUMBERS

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In his posthumously published manuscripts on numbers and magnitudes Bolzano gave an early theory of real numbers based on the rationals. As a tool for this construction he introduced expressions which define infinitesimal and infinitely large numbers as legitimate entities in mathematics. Whereas for calculating all these expressions are admitted, he selected the class of measurable expressions for the procedure of measuring quantities or magnitudes. Two measurable expressions define the same measurable (i.e., real) number iff their difference is infinitesimal. He proves the properties of a completely ordered field. The recent edition [1] has been used.

This paper contains some supplements to the invited address at TOPOSYM V given by D. Kurepa. The material is based on the edition [1] of Bolzano's "Reine Zahlenlehre" which is more comprehensive than the first pioneer publication [6] was. In particular, the editor Jan Berg considered three versions from Bolzano's manuscripts which were written between 1830 and his death in 1848, and it is now clear that Bolzano himself corrected several mistakes of the earlier versions, or at least indicated his ideas how to overcome the difficulties.

Bolzano proceeds in four steps: He defines and considers "*infinite number expressions*", selects some of them which are called *measurable*, and gives an equivalence relation between these measurable expressions which leads to "*measurable*" (i.e., real) "*numbers*". He proves, in modern terminology, that the measurable numbers form a completely ordered field. I shall describe these steps.

In his first step Bolzano introduced *infinite number expressions* (unendliche Größenausdrücke) formed by natural numbers connected by (finitely or) infinitely many rational operations. Examples are

(a) $$1 + 2 + 3 + 4 + \ldots \quad \text{in inf.}\,,$$

which is infinitely large,

(b) $$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots \quad \text{in inf.}\,,$$

(c) $$\frac{1}{1 + 1 + 1 + \ldots \quad \text{in inf.}}\,,$$

which is positive and infinitesimal, and

(d) $$1 - 1 + 1 - 1 + \dots \quad \text{in inf.}$$

Bolzano says that the latter expression "represents a number", and I agree with Berg [1, p. 115]: "Bolzano scheint vorauszusetzen, daß jedem Zahlenausdruck eine Zahl entspricht." Moreover.

(e) $$1 - 2 + 3 - 4 + 5 - \dots \quad \text{in inf.} \quad [1, \text{p. } 137]$$

is "an expression, of which we are neither entitled to say that it be positive nor that it be negative nor that it be nought or infinitely small".

One possibility to understand the concept of a number expression is the representation by sequences of rational numbers, in the example (c) by

$$\left(1, \frac{1}{1+1}, \frac{1}{1+1+1}, \dots\right).$$

In [6], [5], [3], [1] the sequential interpretation was shown to be in agreement with the actual use of the number expressions by Bolzano in [1] and [2]: If the expressions A, B are represented by (a_n), (b_n), then $A \pm B$, $A \,.\, B$ are represented by $(a_n \pm b_n)$, $(a_n \,.\, b_n)$. $A > B$ holds if $a_n > b_n$ for sufficiently large n.

In the second step Bolzano defines what it means that an expression is *measurable* (meßbar): For each natural number q there exists an integer $p = p(q)$ such that

$$\frac{p}{q} \leqq S < \frac{p+1}{q};$$

p/q is called a *measuring fraction* (messender Bruch). This leads to a particular type of what was later called nested intervals.

An expression is a *positive infinitesimal* iff all its measuring fractions are nought. Infinitely large expressions are not measurable, but there are expressions like (d) which are finite without being measurable. Expressions like (e) are neither finite nor infinitely large.

In this connection it should be noted that Bolzano makes a clear distinction between the two mathematical activities, calculating and measuring. Though all number expressions are admitted for arithmetical calculations only some of them make sense for measuring of magnitude. Thus, Bolzano's conceptual background was much broader than that of the mathematicians of the second half of the 19th century who restricted the concept of number to the reals (and, of course, complex numbers, that being no essential enrichment from our point of view). In 1817 or even earlier when Bolzano began to feel the need for a foundation of the concept of a "real" number he was much closer to the views of later mathematicians like Weierstrass. Deeper reflections as expressed in the manuscripts considered here did not mingle the fundamenatal concepts of arithmetics, i.e. number expressions, with those of the sciences which need measuring numbers, take geometry and mechanics as examples. This is a continuation of traditions founded by Eudoxos, Leibniz, Euler, and others.

Bolzano's third step is the following definition: Two measurable expressions are identified if they give the same results with respect to the action of measuring. We might call this an equivalence relation, but if taken literally this is not a congruence relation: A negative infinitesimal like

$$\frac{-1}{1 + 1 + 1 + \text{in inf.}}$$

will have $-1/q$ and not $0/q$ as its measuring fractions, and will not be equivalent to 0 or any positive infinitesimal. Thus it may happen that A and A' are equivalent, B an B' are equivalent, but $A - B$ and $A' - B'$ are not equivalent. As an example take $A = A' = 1$ and $B = 0$,

$$B' = \frac{1}{1 + 1 + 1 + \text{in inf.}}$$

It may even happen that $A - B$ is not measurable though A and B are. Examples can be found in [6], [5], [3] and in footnote 60 of [1, p. 128].

In [3] I indicated modifications of Bolzano's definitions, regarding the partial publication [6]. It was a surprise to see from [1] that Bolzano himself had discovered the difficulties, and that he proposed modifications on sheets in his own shorthand writing which was deciphered by Jan Berg, who reads [1, p. 130]: "A and B heißen hier einander gleich in der Hinsicht, daß beide dieselben Beschaffenheiten haben, daß ihr Unterschied ... absolut betrachtet die gleichen Merkmale bei dem Geschäfte des Messens darbietet wie Null." (I decided to cite in modern German, suppressing brackets which Berg has to use to indicate his supplements to Bolzano's shorthand). In other words, $A \approx B$ iff $|A - B|$ is an infinitesimal. All of Bolzano's theorems become true with this definition. He proves that the equivalence classes of measurable expressions, which are called *measurable numbers*, have the properties of an ordered field. He also gives a proof of what we now call *completeness*: Every bounded non-empty set has a least upper bound, precisely [1, p. 156]: "Wenn wir von einer gewissen Beschaffenheit B bloß wissen, daß sie nicht allen Werten einer veränderlichen meßbaren Zahl X, die größer, wohl aber allen, die kleiner als eine gewisse U sind, zukomme: so können wir mit Gewißheit behaupten, daß es eine meßbare Zahl A gibt, welche die größte derjenigen ist, von denen gesagt werden kann, daß alle kleineren X die Beschaffenheit B haben; wobei noch unentschieden bleibt, ob der Wert $X = A$ auch selbst diese Beschaffenheit habe."

At the end of the manuscript [1, p. 168] there is a remark which has been read by Berg as follows: "Zur Lehre von den meßbaren Zahlen. Sollte die Lehre von den meßbaren Zahlen nicht vielleicht vereinfacht werden können, wenn man die Erklärung derselben so errichtet, daß A meßbar heißt, wenn man 2 Gleichungen von der Form

$$A = \frac{p}{q} + P = \frac{p+n}{q} - P$$

hat, wo bei einerlei n, q ins Unendliche zunehmen kann?" Actually, the capital P is always standing for a positive number, such that the equations can be translated into

$$\frac{p}{q} < A < \frac{p+n}{q}.$$

As was shown in [3], $n = 1$ will suffice if the "limit" of the sequence belonging to A is irrational, and $n = 2$ in the rational case.

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