Czechoslovak Mathematical Journal

Ladislav Bican

The splitting of the tensor product of two mixed abelian groups of rank one

Czechoslovak Mathematical Journal, Vol. 33 (1983), No. 2, 193–211

Persistent URL: http://dml.cz/dmlcz/101872

Czechoslovak Mathematical Journal, 33 (108) 1983, Praha

THE SPLITTING OF THE TENSOR PRODUCT OF TWO MIXED ABELIAN GROUPS OF RANK ONE

LADISLAV BICAN, Praha

(Received May 28, 1980)

Irwin, Khabbaz and Rayna [7] have studied the splitting properties of the tensor product of mixed abelian groups. They defined the splitting length of a mixed group G as the infimum of the set of all positive integers n such that the n-th tensor power $G^n = G \otimes G \otimes \overset{n\text{-times}}{\cdots} \otimes G$ splits and they constructed a mixed group of rank one having the splitting length n for every positive integer n. In my previous paper [3] I have characterized the mixed abelian groups of rank one having the splitting length n. The purpose of the present paper is to give a characterization of all pairs A, B of mixed abelian groups of rank one having the property that the tensor product $A \otimes B$ splits. Thus, the paper is devoted to the proof of the following result.

Theorem. *The following three conditions are equivalent for mixed groups A, B of rank one:*

a) *Any two elements $a \in A \setminus T(A)$, $b \in B \setminus T(B)$ have non-zero multiples ma, nb having the p-property for each prime p.*

b) *There exist elements $a \in A \setminus T(A)$ and $b \in B \setminus T(B)$ having the p-property for each prime p.*

c) *The tensor product $A \otimes B$ splits.*

By the word "group" we shall always mean an additively written abelian group. As in [1], we use the notions "characteristic" and "type" in the broad meaning, i.e. we deal with these notions in mixed groups. The symbols $h_p^A(a)$, $\tau^A(a)$ and $\hat{\tau}^A(a)$ denote respectively the p-height, the characteristic and the type of the element a in the group A. π will denote the set of all primes. If T is a torsion group, then T_p is the p-primary component of T and similarly, if $\Pi' \subseteq \Pi$ then $T_{\Pi'}$ is defined by $T_{\Pi'} = \sum_{p\in\Pi'}^{\oplus} T_p$. The torsion part of a mixed group A is denoted by $T(A)$. If $\Pi' \subseteq \Pi$ and if A is a mixed group with $T(A)_{\Pi'} = 0$ then for each subset $S \subseteq A$ the symbol $\langle S \rangle_{\Pi'}^A$ denotes the Π'-pure closure of S in A, the existence of which is easily seen.

For a mixed group A we denote by $\bar{A}$ the factor group $A/T(A)$ and for $a \in A$ $\bar{a}$ is the element $a + T(A)$ of $\bar{A}$. The symbol $|a|$ means the order of the element $a \in A$.

The rank of a mixed group A is that of $\bar{A}$. The set of all positive integers is denoted by $\mathbb{N}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The other notation will be essentially the same as in [4].

It was proved in [1; Theorem 2] that a mixed group A of rank one splits if and only if each element $a \in A \setminus T(A)$ has a non-zero multiple ma such that $\hat{\tau}^A(ma) = \hat{\tau}^{\bar{A}}(\bar{a})$ and ma has a p-sequence whenever $h_p^{\bar{A}}(\bar{a}) = \infty$ (i.e. there exist elements $h_0^{(p)} = ma$, $h_1^{(p)}, \ldots$ such that $ph_{n+1}^{(p)} = h_n^{(p)}$, $n = 0, 1, \ldots$). Recall [3] that the p-height sequence of an element $a \in A$ is the double sequence $\{k_i, l_i\}_{i=0}^{\infty}$ of elements of $\mathbb{N}_0 \cup \{\infty\}$ defined inductively in the following way: Put $k_1 = k_0 = l_0 = 0$ and $l_1 = h_p^A(a)$. If k_i, l_i are defined and either $h_p^A(p^{k_i}a) = l_i = \infty$, or $l_i < \infty$ and $h_p^A(p^{k_i+k}a) = l_i + k$ for all $k \in \mathbb{N}$ then put $k_{i+1} = k_i$ and $l_{i+1} = l_i$. If $l_i < \infty$ and there are $k \in \mathbb{N}$ with $h_p^A(p^{k_i+k}a) > l_i + k$ then let k_{i+1} be the smallest positive integer for which $h_p^A(p^{k_{i+1}}a) = l_{i+1} > l_i + k_{i+1} - k_i$.

Definition. Let A, B be mixed groups, $a \in A$, $b \in B$ be elements of infinite orders. Further, let p be a prime, $h_p^{\bar{A}}(\bar{a}) = l$, $h_p^{\bar{B}}(\bar{b}) = s$ and let $\{k_i, l_i\}_{i=0}^{\infty}$, $\{r_i, s_i\}_{i=0}^{\infty}$ be the p-height sequences of a and b in A and B, respectively. If there is a sequence $\{i_t\}_{t=1}^{\infty}$ of positive integers such that $i_1 = 1$, the subsequences $\{i_{2t}\}_{t=1}^{\infty}$, $\{i_{2t-1}\}_{t=1}^{\infty}$ are nondecreasing,

$$\lim_{t\to\infty} k_{i_{2t-1}} = \lim_{t\to\infty} k_t\,, \quad \lim_{t\to\infty} r_{i_{2t}} = \lim_{t\to\infty} r_t\,, \tag{1}$$

and the sequence $\{\alpha_t\}_{t=1}^{\infty}$, where

$$\begin{aligned} \alpha_{2t-1} &= l_{i_{2t-1}} - k_{i_{2t-1}} - r_{i_{2t}}\,, \\ \alpha_{2t} &= s_{i_{2t}} - r_{i_{2t}} - k_{i_{2t+1}}\,, \quad t = 1, 2, \ldots, \end{aligned} \tag{2}$$

has non-negative terms, then we say that the elements a, b have the *weak p-property*.

If, moreover, one of the conditions

(i) $l < \infty$, $s < \infty$,
(ii) $l < \infty$, $s = \infty$ and $p^l b$ has a p-sequence in B,
(iii) $l = \infty$, $s < \infty$ and $p^s a$ has a p-sequence in A,
(iv) $l = s = \infty$ and $\lim_{t\to\infty} \alpha_t = \infty$.

is satisfied then we say that the elements a, b have the *p-property*.

Since the exponents are sometimes rather complicated we shall frequently denote the k-th power of p by $[p : k]$.

We start our investigations with some preliminary lemmas.

Lemma 1. *Let p be a prime and let A be a mixed group. If $a \in A \setminus T(A)$ is an arbitrary element, $h_p^{\bar{A}}(\bar{a}) = l < \infty$ and if $\{k_i, l_i\}_{i=0}^{\infty}$ is the p-height sequence of a in A then there is an integer n such that $k_n = k_{n+1} = \ldots$, $l_n = l_{n+1} = \ldots$ and $l_n - k_n = l$.*

Proof. From $h_p^{\bar{A}}(\bar{a}) = l < \infty$ it follows that there exists an element $t \in T(A)$ with

$h_p^A(a + t) = l$. Writing t in the form $t = t_1 + t_2$ where $|t_1| = p^k$ and $(|t_2|, p) = 1$, we have $h_p^A(p^k a) = h_p^A(p^k(a + t)) \geqq k + 1 = h_p^{\bar{A}}(p^k \bar{a}) \geqq h_p^A(p^k a)$, so that $h_p^A(p^k a) - k = l$ and the assertion follows easily.

Lemma 2. *Let p be a prime, A, B mixed groups, $a \in A \setminus T(A)$, $b \in B \setminus T(B)$. If the elements a, b have the weak p-property then for all $t = 1, 2, \ldots$ with $\alpha_1, \alpha_2, \ldots, \alpha_{2t}, \alpha_{2t+1} < \infty$ we have*

$$a \otimes b = [p : \alpha_{2t-1} + s_{i_{2t}}] (a_{i_{2t-1}} \otimes b_{i_{2t}}) \tag{3}$$

and

$$a \otimes b = [p : \alpha_{2t} + l_{i_{2t+1}}] (a_{i_{2t+1}} \otimes b_{i_{2t}}), \tag{4}$$

where $p^{l_i} a_i = p^{k_i} a$, $p^{s_i} b_i = p^{r_i} b$, $i = 1, 2, \ldots$.

Proof. Obviously, $a \otimes b = p^{l_1}(a_1 \otimes b) = [p : l_{i_1} - \alpha_{i_1}] (a_{i_1} \otimes b) = [p : \alpha_1 + r_{i_2}] (a_{i_1} \otimes b) = [p : \alpha_1 + s_{i_2}] (a_{i_1} \otimes b_{i_2})$.

Using the induction principle let us assume that (3) holds for some $t \geqq 1$. Then the hypothesis $\alpha_{2t} = s_{i_{2t}} - r_{i_{2t}} - k_{i_{2t+1}} \geqq 0$ yields $s_{i_{2t}} - r_{i_{2t}} \geqq k_{i_{2t+1}} \geqq k_{i_{2t-1}}$ and by the induction hypothesis we have $a \otimes b = [p : l_{i_{2t-1}} - k_{i_{2t-1}} - r_{i_{2t}} + s_{i_{2t}}] \cdot (a_{i_{2t-1}} \otimes b_{i_{2t}}) = [p : s_{i_{2t}} - r_{i_{2t}}] (a \otimes b_{i_{2t}}) = [p : \alpha_{2t} + l_{i_{2t+1}}] (a_{i_{2t+1}} \otimes b_{i_{2t}})$. Similarly, if we assume, that (4) holds for some $t \geqq 1$ then the hypothesis $\alpha_{2t+1} = l_{i_{2t+1}} - k_{i_{2t+1}} - r_{i_{2t+2}} \geqq 0$ yields $l_{i_{2t+1}} - k_{i_{2t+1}} \geqq r_{i_{2t+2}} \geqq r_{i_{2t}}$ and by the induction hypothesis we have $a \otimes b = [p : s_{i_{2t}} - r_{i_{2t}} - k_{i_{2t+1}} + l_{i_{2t+1}}] (a_{i_{2t+1}} \otimes b_{i_{2t}}) = [p : l_{i_{2t+1}} - k_{i_{2t+1}}] (a_{i_{2t+1}} \otimes b) = [p : \alpha_{2t+1} + s_{i_{2t+2}}] (a_{i_{2t+1}} \otimes b_{i_{2t+2}})$.

Lemma 3. *Let p be a prime, A, B mixed groups, $a \in A \setminus T(A)$, $b \in B \setminus T(B)$. If the elements a, b have the weak p-property, $s_{i_{2t-2}} < s_n = s_{i_{2t}} = \infty$ and $\alpha_{i_{2t-1}} < \infty$ then $a \otimes b = p^{l_m - k_m}(a_m \otimes b)$ for each $m \geqq i_{2t-1}$, $l_m - k_m < \infty$, $p^{l_m} a_m = p^{k_m} a$.*

Proof. By the hypothesis there exists an element $b' \in B$ with $[p : k_m + r_n] b' = [p : r_n] b$ and by Lemma 2 we have $a \otimes b = [p : \alpha_{2t-2} + l_{i_{2t-1}}] (a_{i_{2t-1}} \otimes b_{i_{2t-2}}) = [p : l_{i_{2t-1}} - k_{i_{2t-1}}] (a_{i_{2t-1}} \otimes b) = [p : l_{i_{2t-1}} - k_{i_{2t-1}} - r_n] (a_{i_{2t-1}} \otimes [p : r_n] b) = [p : l_{i_{2t-1}} - k_{i_{2t-1}} + k_m] (a_{i_{2t-1}} \otimes b') = [p : k_m] (a \otimes b') = [p : l_m] (a_m \otimes b') = [p : l_m - k_m - r_n] (a_m \otimes [p : r_n] b) = [p : l_m - k_m] \cdot (a_m \otimes b)$ since $l_m - k_m - r_n \geqq l_{i_{2t-1}} - k_{i_{2t-1}} - r_{i_{2t}} = \alpha_{2t-1} \geqq 0$.

Lemma 4 (See [3; Lemma 1].) *Let p a prime, A a mixed group and let $a_i \in A \setminus T(A)$, $i = 0, 1, \ldots$, be such elements that $p^{r_i} a_i = p^{s_{i-1}} a_{i-1}$, $i = 1, 2, \ldots, s_0 = 0$. If $\sum_{i=1}^{\infty} (r_i - s_i)$ has non-negative partial sums and $\sum_{i=1}^{\infty} (r_i - s_i) = \infty$ then a_0 has a p-sequence in A.*

Proof. Since $\liminf_{n \to \infty} \{ \sum_{i=1}^{n} (r_i - s_i) \} = \infty$ there exists the greatest integer k_1 such that $\gamma_1 = \sum_{i=1}^{k_1 - 1} (r_i - s_i) = \inf \{ \sum_{i=1}^{n} (r_i - s_i) \mid n = 1, 2, \ldots \}$. If the non-negative integers $k_1, k_2, \ldots, k_j, \gamma_1, \gamma_2, \ldots, \gamma_j$ are defined then let k_{j+1} be the greatest positive integer such that $\gamma_{j+1} = \sum_{i=1}^{k_{j+1}-1} (r_i - s_i) = \inf\{\sum_{i=1}^{n}(r_i - s_i) \mid n \geqq k_j\} > \gamma_j$. For every $j = 1, 2, \ldots$ and every $k_j \leqq m < k_{j+1}$ we have $\sum_{i=k_j}^{m} (r_i - s_i) = \sum_{i=1}^{m}(r_i - s_i) - \sum_{i=1}^{k_j-1} (r_i - s_i) \geqq \gamma_{j+1} - \gamma_j$ and consequently $\sum_{i=k_j}^{m} (r_i - s_i) - (\gamma_{j+1} - \gamma_j) \geqq 0$. In particular, for $m = k_{j+1} - 1$ we have $\sum_{i=k_j}^{k_{j+1}-1} (r_i - s_i) - (\gamma_{j+1} - \gamma_j) = 0$. Hence we obtain $[p : r_{k_j} - (\gamma_{j+1} - \gamma_j)]\, a_{k_j} = [p : s_{k_j} + (r_{k_j} - s_{k_j}) - (\gamma_{j+1} - \gamma_j)]\, a_{k_j} = [p : r_{k_j+1} + (r_{k_j} - s_{k_j}) - (\gamma_{j+1} - \gamma_j)]\, a_{k_j+1} = [p : s_{k_j+1} + (r_{k_j} - s_{k_j}) + (r_{k_j+1} - s_{k_j+1}) - (\gamma_{j+1} - \gamma_j)]\, a_{k_j+1} = \ldots = [p : s_{k_{j+1}-1} + \sum_{i=k_j}^{k_{j+1}-1} (r_i - s_i) - (\gamma_{j+1} - \gamma_j)]\, a_{k_{j+1}-1} = [p : r_{k_{j+1}}]\, a_{k_{j+1}}$. Moreover, for each $1 \leqq m < k_1$ we have $\sum_{i=1}^{m}(r_i - s_i) \geqq \gamma_1$, so that $a_0 = [p : r_1]\, a_1 = [p : s_1 + (r_1 - s_1)]\, a_1 = [p : r_2 + (r_1 - s_1)]\, a_2 = \ldots = [p : s_{k_1-1} + \sum_{i=1}^{k_1-1} (r_i - s_i)]\, a_{k_1-1} = [p : r_{k_1} + \gamma_1]\, a_{k_1}$. Now it is easy to see that $a_0 = [p : r_{k_1} + \gamma_1]\, a_{k_1}, [p : r_{k_1} + \gamma_1 - 1]\, a_{k_1}, \ldots, [p : r_{k_1} - (\gamma_2 - \gamma_1)]\, a_{k_1} = [p : r_{k_2}]\, a_{k_2}, [p : r_{k_2} - 1]\, a_{k_2}, \ldots, [p : r_{k_2} - (\gamma_3 - \gamma_2)]\, a_{k_2} = [p : r_{k_3}]\, a_{k_3}, \ldots, [p : r_{k_j}]\, a_{k_j}, [p : r_{k_j} - 1]\, a_{k_j}, \ldots, [p : r_{k_j} - (\gamma_{j+1} - \gamma_j)]\, a_{k_j} = [p : r_{k_{j+1}}]\, a_{k_{j+1}}, \ldots$ is a p-sequence of the element a_0 in A.

Lemma 5. *Let p be a prime, A, B mixed groups, $a \in A \setminus T(A)$, $b \in B \setminus T(B)$. If $\{k_i, l_i\}_{i=0}^{\infty}$ and $\{r_i, s_i\}_{i=0}^{\infty}$ are the p-height sequences of the elements a and b in the groups A and B, respectively, and if $l_1 = \infty$, $s_{m-1} < s_m = \infty$ then the element $a \otimes b$ has a p-sequence in $A \otimes B$.*

Proof. By the hypothesis there are elements $a_1, a_2, \ldots \in A$, $b_1, b_2, \ldots \in B$ such that $p^{i+r_m}a_i = a$, $p^{i+r_m}b_i = p^{r_m}b$, $i = 1, 2, \ldots$. Obviously, $p^{r_m+2}(a_1 \otimes b_1) = p^{r_m+1}(a_1 \otimes b) = a \otimes b$ and $p^{i+1+r_m}(a_i \otimes b_i) = pa \otimes b_i = p^{i+r_m}(a_{i-1} \otimes b_i) = a_{i-1} \otimes p^{r_m}b = p^{i-1+r_m}(a_{i-1} \otimes b_{i-1})$ for each $i = 2, 3, \ldots$. Now $r_m + 2 - (1 + r_m) + \sum_{i=2}^{n}((i + 1 + r_m) - (i + r_m)) = n$ and the element $a \otimes b$ has a p-sequence in $A \otimes B$ by Lemma 4.

Lemma 6. *Let p be a prime, A, B mixed groups, $a \in A \setminus T(A)$, $b \in B \setminus T(B)$. If $\{k_i, l_i\}_{i=0}^{\infty}$ and $\{r_i, s_i\}_{i=0}^{\infty}$ are the p-height sequences of the elements a and b in the groups A and B, respectively, and if $s = h_p^{\bar{B}}(\bar{b}) = l_1 = \infty$, $s_m < \infty, m = 1, 2, \ldots$, then the element $a \otimes b$ has a p-sequence in $A \otimes B$.*

Proof. By the hypothesis there are elements $a_1, a_2, \ldots \in A$, $b_1, b_2, \ldots \in B$ such that $p^{s_{i+1}}a_i = a$, $p^{s_i}b_i = p^{r_i}b$, $i = 1, 2, \ldots$. Then $p^{s_i}b_i = p^{r_i}b = p^{r_i - r_{i-1} + s_{i-1}}b_{i-1}$ for each $i = 1, 2, \ldots$ and so $a \otimes b = p^{s_1+s_2}(a_1 \otimes b_1)$ and $p^{r_i - r_{i-1} + s_{i-1}}(a_{i-1} \otimes b_{i-1}) = p^{s_i}(a_{i-1} \otimes b_i) = p^{s_{i+1}}(a_i \otimes b_i)$. However, $s_1 + s_2 - (r_2 - r_1 + s_1) +$

$+ \sum_{i=2}^{n} (s_{i+1} - (r_{i+1} - r_i + s_i)) = s_{n+1} - r_{n+1}$ and it suffices to use Lemma 4 owing to the fact that $\lim_{n\to\infty} (s_{n+1} - r_{n+1}) = s = \infty$.

Lemma 7. *Let p be a prime, A, B mixed groups, $a \in A \setminus T(A)$, $b \in B \setminus T(B)$. If $\{k_i, l_i\}_{i=0}^{\infty}$ and $\{r_i, s_i\}_{i=0}^{\infty}$ are the p-height sequences of the elements a and b in the groups A and B, respectively, the elements a, b have the p-property and if $h_p^{\bar{A}}(\bar{a}) = l = s = h_p^{\bar{B}}(\bar{b}) = \infty$, $l_n < \infty$, $s_n < \infty$, $n = 1, 2, \ldots$, then the element $a \otimes b$ has a p-sequence in $A \otimes B$.*

Proof. Put $y_1 = l_{i_1} + s_{i_2} - r_{i_2}$ and $x_{2t-1} = l_{i_{2t-1}} + k_{i_{2t+1}} - k_{i_{2t-1}}$, $x_{2t} = s_{i_{2t}} + r_{i_{2t+2}} - r_{i_{2t}}$, $y_{2t} = l_{i_{2t+1}}$, $y_{2t+1} = s_{i_{2t+2}}$ for each $t = 1, 2, \ldots$. Then $[p : y_1] \cdot (a_{i_1} \otimes b_{i_2}) = [p : l_{i_1}] (a_{i_1} \otimes b) = a \otimes b$ (since $\alpha_1 = l_{i_1} - r_{i_2} \geqq 0$) and $[p : y_{2t}] \cdot (a_{i_{2t+1}} \otimes b_{i_{2t}}) = [p : k_{i_{2t+1}}] (a \otimes b_{i_{2t}}) = [p : x_{2t-1}] (a_{i_{2t-1}} \otimes b_{i_{2t}})$, $[p : y_{2t+1}] \cdot (a_{i_{2t+1}} \otimes b_{i_{2t+2}}) = [p : r_{i_{2t+2}}] (a_{i_{2t+1}} \otimes b) = [p : x_{2t}] (a_{i_{2t+1}} \otimes b_{i_{2t}})$ for each $t = 1, 2, \ldots$. Now $\sum_{t=1}^{2n} (y_t - x_t) = l_{i_1} + s_{i_2} - r_{i_2} - (l_{i_1} + k_{i_3} - k_{i_1}) + \sum_{t=1}^{n} (y_{2t} - x_{2t}) + \sum_{t=1}^{n-1} (y_{2t+1} - x_{2t+1}) = s_{i_2} - r_{i_2} - k_{i_3} + \sum_{t=1}^{n} (l_{i_{2t+1}} - s_{i_{2t}} - r_{i_{2t+2}} + r_{i_{2t}}) + \sum_{t=1}^{n-1} (s_{i_{2t+2}} - l_{i_{2t+1}} - k_{i_{2t+3}} + k_{i_{2t+1}}) = l_{i_{2n+1}} - k_{i_{2n+1}} - r_{i_{2n+2}} = \alpha_{2n+1}$, $\sum_{t=1}^{2n-1} (y_t - x_t) = l_{i_1} + s_{i_2} - r_{i_2} - (l_{i_1} + k_{i_3} - k_{i_1}) + \sum_{t=1}^{n-1} (y_{2t} - x_{2t}) + \sum_{t=1}^{n-1} (y_{2t+1} - x_{2t+1}) = s_{i_2} - r_{i_2} - k_{i_3} + \sum_{t=1}^{n-1} (l_{i_{2t+1}} - s_{i_{2t}} - r_{i_{2t+2}} + r_{i_{2t}}) + \sum_{t=1}^{n-1} (s_{i_{2t+2}} - l_{i_{2t+1}} - k_{i_{2t+3}} + k_{i_{2t+1}}) = s_{i_{2n}} - r_{i_{2n}} - k_{i_{2n+1}} = \alpha_{2n}$ for each $n = 1, 2, \ldots$ and the element $a \otimes b$ has a p-sequence in $A \otimes B$ by the hypothesis and Lemma 4.

Lemma 8. *Let p be a prime and A a mixed group with a p-primary torsion part T. Further, let $a \in A \setminus T$ be an arbitrary element and let $\{k_i, l_i\}_{i=0}^{\infty}$ be its p-height sequence. If n is a positive integer such that $l_1 < l_2 < \ldots < l_n < \infty$, $p^{l_i} a_i = p^{k_i} a$, $i = 1, 2, \ldots, n$, and if $U = \langle t_2, \ldots, t_n \rangle$ where*

$$t_i = p^{l_i - k_i - l_{i-1} + k_{i-1}} a_i - a_{i-1}, \quad i = 2, \ldots, n, \tag{5}$$

then

$$U = \sum_{i=2}^{n}{}^{\oplus} \langle t_i \rangle \quad \text{and} \quad T = U \oplus V \tag{6}$$

for a suitable subgroup V of T.

Proof. In the proof of [3; Lemma 4] it has been proved that

$$|t_i| = p^{l_{i-1} + k_i - k_{i-1}}, \quad i = 2, \ldots, n, \tag{7}$$

and

$$h_p^A(p^j t_i) = j, \quad i = 2, \ldots, n, \quad 0 \leqq j < |t_i|. \tag{8}$$

If $n = 2$ then U is a bounded pure subgroup of T and we are through. With respect to the induction principle we can suppose that $\langle t_2, \ldots, t_{n-1}\rangle = \sum_{i=2}^{n-1}{}^{\oplus} \langle t_i\rangle$ is a direct summand of T and it suffices to show that $\langle t_2, \ldots, t_{n-1}\rangle \cap \langle t_n\rangle = 0$ and that U is pure in T. If $0 \neq p^j t_n = \sum_{i=2}^{n-1} \lambda_i t_i$ then by (8) and the induction hypothesis we have $\lambda_i = p^j \mu_i$ for suitable integers μ_i, $i = 2, \ldots, n-1$. Further, by (7) we have $0 \neq$ $\neq p^{l_{n-1}+k_n-k_{n-1}-1} t_n = p^{l_{n-1}+k_n-k_{n-1}-1-j} \sum_{i=2}^{n-1} \lambda_i t_i = \sum_{i=2}^{n-1} p^{l_{n-1}+k_n-k_{n-1}-1} \mu_i t_i = 0$ (since by the definition of the p-height sequence, $l_{n-1} + k_n - k_{n-1} - 1 \geqq l_{n-1} >$ $> l_{i-1} + k_i - k_{i-1}$ for each $i = 2, \ldots, n-1$) — a contradiction showing that $\langle t_2, \ldots, t_{n-1}\rangle \cap \langle t_n\rangle = 0$. Now let the equation $p^j x = \sum_{i=2}^{n} \lambda_i t_i$ be solvable in T. If $\lambda_n t_n = 0$ then $p^j \mid \lambda_i$, $i = 2, \ldots, n-1$, by the induction hypothesis. If $\lambda_n t_n \neq 0$, $\lambda_n = p^m \mu_n$, $(\mu_n, p) = 1$, then for $m \geqq j$, $p^j \mid \lambda_i$, $i = 2, \ldots, n-1$, by the induction hypothesis. The case $m < j$ is impossible, since then $p^m \mid \lambda_i$, $i = 2, \ldots, n-1$, and $p^{l_{n-2}+k_{n-1}-k_{n-2}-m+j} x = p^{l_{n-2}+k_{n-1}-k_{n-2}} \mu_n t_n$ together with (8) gives $j \leqq m$ and we are through.

Lemma 9. *Let p be a prime and A a mixed group with a p-primary torsion part T. Further, let $a \in A \setminus T$ be an arbitrary element and let $\{k_i, l_i\}_{i=0}^{\infty}$ be its p-height sequence such that $l_{m-1} < l_m = \infty$ for some $m \in \mathbb{N}$. If $p^{l_i} a_i = p^{k_i} a$, $i = 1, 2, \ldots, n-1$, then there are elements $a_m, a_{m+1}, \ldots$ in $G \setminus T$ such that*:

(i) *If t_i, $i = 1, 2, \ldots, m-1$ are elements* (5),

$$t_m = p^{l_{m-1}+k_m-k_{m-1}} a_m - a_{m-1} \tag{9}$$

and $U = \langle t_2, \ldots, t_m\rangle$ then

$$U = \sum_{i=2}^{m}{}^{\oplus} \langle t_i\rangle \quad \text{and} \quad T = U \oplus V \tag{10}$$

where V is a suitable subgroup of T such that

$$\langle p^{l_{m-1}+k_m-k_{m-1}} a_{m+i+1} - a_{m+i} \mid i \in \mathbb{N}_0\rangle \subseteq V. \tag{11}$$

(ii) *If A is of rank one and if we denote $H = \langle V \cup \{a_m, a_{m+1}, \ldots\}\rangle^A_{\pi\setminus\{p\}}$ then*

$$A = U \oplus H. \tag{12}$$

Proof. With respect to [3; Lemmas 6, 7] and their proofs it remains to show that $U = \sum_{i=2}^{m}{}^{\oplus} \langle t_i\rangle$. By the preceding Lemma we have $\tilde{U} = \langle t_2, \ldots, t_{m-1}\rangle = \sum_{i=2}^{m-1}{}^{\oplus} \langle t_i\rangle$. The hypothesis $h_p^A(p^{k_m} a) = \infty$ yields the existence of an element $a'_m \in A \setminus T$ such that $p^{2(l_{m-1}+k_m-k_{m-1})} a'_m = p^{k_m} a = p^{l_{m-1}+k_m-k_{m-1}} a_{m-1}$. Put $t_m = p^{l_{m-1}+k_m-k_{m-1}} a'_m - a_{m-1}$. If $p^j t_m = 0$ for some $j < l_{m-1} + k_m - k_{m-1}$ then we can clearly assume that $j \geqq l_{m-1}$.

No we have $p^{l_{m-1}+k_m-k_{m-1}+j}a'_m = p^j t_m + p^j a_{m-1} = p^{j-l_{m-1}+k_{m-1}} a$ which contradicts the definition of the p-height sequence. Thus

$$|t_m| = p^{l_{m-1}+k_m-k_{m-1}}. \tag{13}$$

Suppose now that for some $0 \leqq j < l_{m-1} + k_m - k_{m-1}$ we have $h_p^A(p^j t_m) > j$. Without loss of generality we can assume that $j \geqq l_{m-1}$. Then $p^{l_{m-1}+k_m-k_{m-1}+j}a'_m - p^j t_m = p^j a_{m-1} = p^{j-l_{m-1}+k_{m-1}}a$, which contradicts the definition of the p-height sequence and consequently $h_p^A(p^j t_m) = j$, $0 \leqq j < l_{m-1} + k_m - k_{m-1}$. Suppose now that $0 \neq p^j t_m = \sum_{i=2}^{m-1} \lambda_i t_i$. Then there are integers μ_i with $\lambda_i = p^j \mu_i$, $i = 2, \ldots, m-1$, $\tilde{U}$ being a direct summand of T. By the definition of the p-height sequence and by (13) and (8) we have $0 \neq p^{l_{m-1}+k_m-k_{m-1}-1} t_m = p^{l_{m-1}+k_m-k_{m-1}-j-1} \sum_{i=2}^{m-1} \lambda_i t_i = \sum_{i=2}^{m-1} p^{l_{m-1}+k_m-k_{m-1}-1} \mu_i t_i = 0$ – a contradiction showing that $U = \sum_{i=2}^{m}{}^{\oplus} \langle t_i \rangle$ is a subgroup of T. With respect to the proof of [3; Lemma 6] we have $t_m = p^{l_{m-1}+k_m-k_{m-1}} . a_m - a_{m-1}$ and the proof is complete.

Lemma 10. *Let p be a prime and A a mixed group of rank one with a p-primary torsion part T. Further, let $a \in A \setminus T$ be an arbitrary element and let $\{k_i, l_i\}_{i=0}^{\infty}$ be its p-height sequence, $l_i < \infty$, $p^{l_i} a_i = p^{k_i} a$, $i = 0, 1, \ldots$. If $n \geqq 2$ is positive integer, t_i, $i = 2, \ldots$, are the elements (5), $U = \sum_{i=2}^{n}{}^{\oplus} \langle t_i \rangle$ and $A \doteq \langle a_0, a_1, \ldots \rangle_{\pi \setminus \{p\}}^A$ then*

$$A = U \oplus \langle a_n, a_{n+1}, \ldots \rangle_{\pi \setminus \{p\}}^A. \tag{14}$$

Proof. For the sake of brevity we shall use the notations $C = \langle a_n, a_{n+1}, \ldots \rangle_{\pi \setminus \{p\}}^A$ and $D = \langle a_n, a_{n+1}, \ldots \rangle$. If $c \in U \cap C$ is an arbitrary element then $\varrho c \in U \cap D$ for some integer ϱ with $(\varrho, p) = 1$. Hence $\varrho c = \sum_{i=2}^{n} \lambda_i t_i = \sum_{i=n}^{m} \mu_i a_i$ for some $m \geqq n$. Multiplying by p^{l_m} we get $p^{l_m} \sum_{i=n}^{m} \mu_i a_i = (p^{k_m} \mu_m + p^{l_m - l_{m-1} + k_{m-1}} \mu_{m-1} + \ldots + p^{l_m - l_n + k_n} \mu_n) a$ and $p^{l_m} \sum_{i=2}^{n} \lambda_i t_i = 0$ owing to the fact that $l_m \geqq l_i > l_{i-1} + k_i - k_{i-1} = |t_i|$, $i = 2, \ldots, n$. Thus $p^{l_m} \varrho c = 0$ and $|a| = \infty$ yields $p^{k_m} \mu_m + p^{l_m - l_{m-1} + k_{m-1}} \mu_{m-1} + \ldots + p^{l_m - l_n + k_n} \mu_n = 0$. However, for each $n \leqq i < m$ we have $l_m - l_i + k_i \geqq l_m - l_{m-1} + k_{m-1} > k_m$ and consequently $\mu_m = p^{l_m - l_{m-1} + k_{m-1} - k_m} \nu_m$ for some integer ν_m. So, $\varrho c = \nu_m (p^{l_m - l_{m-1} + k_{m-1} - k_m} a_m - a_{m-1}) + \sum_{i=n}^{m-2} \mu_i a_i + (\mu_{m-1} + \nu_m) a_{m-1} = \nu_m t_m + \sum_{i=2}^{m-2} \mu_i a_i + (\mu_{m-1} + \nu_m) a_{m-1}$. Using the induction principle we easily obtain the equality $\varrho c = \sum_{i=n+1}^{m} \nu_i t_i + \nu_n a_n$ for suitable integers $\nu_n, \nu_{n+1}, \ldots, \nu_m$. However, $\nu_n a_n = \varrho c - \sum_{i=n+1}^{m} \nu_i t_i \in T \cap \langle a_n \rangle = 0$, thus $\nu_n = 0$ and $\varrho c \in U \cap \sum_{i=n+1}^{m} \langle t_i \rangle = 0$. Consequently, by Lemma 8, $c = 0$,

T being p-primary and ϱ relatively prime to p. We have shown that $U \cap C = 0$ and we proceed to $U \vee C = A$. Let $b \in A$ be arbitrary. By the hypothesis we have $\varrho b = \sum_{i=0}^{m} \lambda_i a_i$ for some integer ϱ, $(\varrho, p) = 1$, and without loss of generality we can assume that $m \geqq n$. Putting $\varrho_i = l_i - k_i$, $i = 0, 1, \ldots, m$, we have $\varrho b = \sum_{i=0}^{m} \lambda_i a_i = \sum_{i=0}^{n-1} \lambda_i a_i + \sum_{i=n}^{m} \lambda_i a_i = \sum_{i=0}^{n-1} \lambda_i (p^{\varrho_n - \varrho_i} a_n - \sum_{j=i}^{n-1} (p^{\varrho_{j+1} - \varrho_i} a_{j+1} - p^{\varrho_j - \varrho_i} a_j)) + \sum_{i=n}^{m} \lambda_i a_i = \sum_{i=0}^{n-1} \lambda_i (p^{\varrho_n - \varrho_i} a_n - \sum_{j=i}^{n-1} p^{\varrho_j - \varrho_i} (p^{\varrho_{j+1} - \varrho_j} a_{j+1} - a_j)) + \sum_{i=n}^{m} \lambda_i a_i = - \sum_{i=0}^{n-1} \sum_{j=i}^{n-1} p^{\varrho_j - \varrho_i} \lambda_i t_{j+1} + \sum_{i=0}^{n} p^{\varrho_n - \varrho_i} \lambda_i a_n + \sum_{i=n+1}^{m} \lambda_i a_i = t + c$, $t \in U$, $c \in C$. However, t is divisible by ϱ, U being p-primary, and the assertion follows easily.

Lemma 11. *Let p be a prime and A, B mixed groups of rank one with p-primary torsion parts T, S, respectively. Suppose that $a \in A \setminus T$, $b \in B \setminus S$ are arbitrary elements, $\{k_i, l_i\}_{i=0}^{\infty}$, $\{r_i, s_i\}_{i=0}^{\infty}$ are the p-height sequences of the elements a, b in A, B, respectively, and $k_{m-1} < k_m = k_{m+1} = \ldots$, $r_{n-1} < r_n = r_{n+1} = \ldots$ for some $m, n \in \mathbb{N}_0$. If $h_p^{A \otimes B}(a \otimes b) = h_p^{\overline{A \otimes B}}(\overline{a \otimes b})$ then for each $i = 2, \ldots, m$, $j = 2, \ldots, n$ at least one of the following two inequalities is satisfied*:

$$s_{j-1} - r_{j-1} - k_i \geqq 0\,, \tag{15}$$

$$l_{i-1} - k_{i-1} - r_j \geqq 0\,. \tag{16}$$

Moreover, for $i = m + 1$, $j = n + 1$ both these inequalities hold.

Proof. Assume first, that $l_m < \infty$, $s_n < \infty$. By Lemma 8 and [3; Lemma 8] we have $A = U \oplus V \oplus \langle a_m \rangle^A_{\pi \setminus \{p\}}$, $B = X \oplus Y \oplus \langle b_n \rangle^B_{\pi \setminus \{p\}}$, $U \oplus V = T$, $X \oplus Y = S$, $U = \sum_{i=2}^{m}{}^{\oplus} \langle u_i \rangle$, $X = \sum_{j=2}^{n}{}^{\oplus} \langle x_j \rangle$, $u_i = p^{l_i - k_i - l_{i-1} + k_{i-1}} a_i - a_{i-1}$, $x_j = p^{s_j - r_j - s_{j-1} + r_{j-1}} b_j - b_{j-1}$, $p^{l_i} a_i = p^{k_i} a$, $i = 1, 2, \ldots, m$, $p^{s_j} b_j = p^{r_j} b$, $j = 1, 2, \ldots, n$. It is easy to see that $a = p^{l_m - k_m} a_m - \sum_{i=2}^{m} p^{l_{i-1} - k_{i-1}} u_i$ and $b = p^{s_n - r_n} b_n - \sum_{j=2}^{n} p^{s_{j-1} - r_{j-1}} x_j$ where $l_m - k_m = l = h_p^{\bar{A}}(\bar{a})$ and $s_n - r_n = s = h_p^{\bar{B}}(\bar{b})$. Now for each $i = 2, \ldots, m$, $j = 2, \ldots, n$ the element $g_{ij} = p^{l_{i-1} - k_{i-1} + s_{j-1} - r_{j-1}} u_i \otimes x_j$ lies in a direct summand of $A \otimes B$ and from the equality $a \otimes b = p^{l+s} a_m \otimes b_n - \sum_{j=2}^{n} p^{l + s_{j-1} - r_{j-1}} (a_m \otimes x_j) - \sum_{i=2}^{m} p^{l_{i-1} - k_{i-1} + s} u_i \otimes b_n + \sum_{i=2}^{m} \sum_{j=2}^{n} g_{ij}$ it follows that $h_p^{A \otimes B}(g_{ij}) \geqq l + s = h_p^{A \otimes B}(a \otimes b)$. However, if $g_{ij} \neq 0$ then the p-height of the element g_{ij} is obviously $l_{i-1} - k_{i-1} + s_{j-1} - r_{j-1} < l + s$ and so necessarily $g_{ij} = 0$. Since (by (7)) $|u_i \otimes x_j| = \min \{|u_i|, |x_j|\} = \min \{p^{l_{i-1} + k_i - k_{i-1}}, p^{s_{j-1} + r_j - r_{j-1}}\}$, we get the desired result for each $i = 2, \ldots, m$, $j = 2, \ldots, n$. Consider now the elements $p^{l + s_{n-1} - r_{n-1}} (a_m \otimes x_n)$ and $p^{l_{m-1} - k_{m-1} + s} (u_m \otimes b_n)$. Suppose that $h_p^A(p^j a_m) = k > j$ for a positive integer j. Then $h_p^A(p^{j + l_m} a_m) \geqq k + l_m > j +$

$+ l_m$. On the other hand, $h_p^A(p^{j+l_m}a_m) = h_p^A(p^{j+k_m}a) = j + l_m$ by the definition of the p-height sequence — a contradiction showing the p-purity of $\langle a_m \rangle$ in A. By [5; Corollary 60.5] the natural mapping $\langle a_m \rangle \otimes \langle x_n \rangle \to^{\alpha} A \otimes \langle x_n \rangle$ is monic and by [5; Theorem 60.4] the natural mapping $A \otimes \langle x_n \rangle \to^{\beta} A \otimes B$ is also monic ($\langle x_n \rangle$ is pure in B). Composing these monomorphisms with the natural isomorphisms $\langle x_n \rangle \cong \langle a_m \rangle \otimes \langle x_n \rangle$ we see that $|a_m \otimes x_n| = |x_n| = p^{s_{n-1}+r_n-r_{n-1}}$ from which it similarly as above follows the inequality $l_m - k_m - r_n \geqq 0$. The inequality (15) for $j = n + 1$, $i = m$, is proved similarly.

Assume now that $l_m < \infty$, $s_n = \infty$. By Lemma 9 we have $B = X \oplus \langle Y \cup \cup \{b_n, b_{n+1}, \ldots\} \rangle^B_{\pi \setminus \{p\}}$, $X \oplus Y = S$, $X = \sum_{j=2}^{n\oplus} \langle x_j \rangle$, $x_j = p^{s_j - r_j - s_{j-1} + r_{j-1}} b_j - b_{j-1}$, $j = 2, \ldots, n-1$, $x_n = p^{s_{n-1}+r_n-r_{n-1}} b_n - b_{n-1}$, $|x_n| = p^{s_{n-1}+r_n-r_{n-1}}$. It is easy to see that $b = p^{2(s_{n-1}-r_{n-1})+r_n} b_n - \sum_{j=2}^{n} p^{s_{j-1}-r_{j-1}} x_j$ and the same arguments as in the preceding case yield the desired result for each $i = 2, \ldots, m$, $j = 2, \ldots, n$ and the validity of (16) for $i = m + 1, j = n$. The inequality (15) for $j = n + 1$ is trivial.

Finally, if $l_m = r_n = \infty$ then similar treatments as above yield the result.

Lemma 12. *Let A be a mixed group of rank one with a p-primary torsion part T and let B be a mixed group with a p-primary torsion part S and $\bar{B}$ p-divisible. Further, let $\{k_i, l_i\}_{i=0}^{\infty}$ be the p-height sequence of an element $a_0 \in A \setminus T$ such that $l_i < \infty$, $i = 1, 2, \ldots$. If $p^{l_i} a_i = p^{k_i} a_0$, $i = 1, 2, \ldots$, and $A \otimes B$ splits then $\langle a_0, a_1, \ldots \rangle^A_{\pi \setminus \{p\}} \otimes \otimes B$ splits as well.*

Proof. By Lemma 8 and [3; Lemmas 4, 5] there exists a basic subgroup P of T such that $P = U \oplus V$, $H = \langle P \cup \{a_0, a_1, \ldots,\} \rangle^A_{\pi \setminus \{p\}}$, $H \cap T = P$ and $H = V \oplus \oplus \langle a_0, a_1, \ldots \rangle^A_{\pi \setminus \{p\}}$. It is easy to see that for each $g \in A$ there are integers ϱ, σ, m such that $\varrho \bar{g} = \sigma \bar{a}_m$, $(\varrho, p) = 1$. Then $\varrho g = \sigma a_m + t$, $t \in T$, and so $A = H \vee T$. Hence $A/T = H \vee T/T \cong H/H \cap T = H/P$.

The sequences $0 \to P \to T \to T/P \to 0$ and $0 \to S \to B \to B/S \to 0$ are pure exact, so that by [5; Theorem 60.4] we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P \otimes S & \longrightarrow & T \otimes S & \longrightarrow & T/P \otimes S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P \otimes B & \longrightarrow & T \otimes B & \longrightarrow & T/P \otimes B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & P \otimes B/S & \to & T \otimes B/S & \to & T/P \otimes B/S & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with exact rows and columns and natural homomorphisms. By the hypothesis, S and T/P are p-primary and T/P, B/S are p-divisible, hence $T/P \otimes S = T/P \otimes B/S = 0$ and consequently $T/B \otimes B = 0$. Thus $P \otimes B \cong T \otimes B$. Further, in the commutative diagram

$$\begin{array}{ccccccccc} 0 \to & P \otimes B & \to & H \otimes B & \to & H/P \otimes B & \to 0 \\ & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & \\ 0 \to & T \otimes B & \to & A \otimes B & \to & A/T \otimes B & \to 0 \end{array}$$

with exact rows and natural homomorphisms, the homomorphisms α and γ are isomorphisms by the preceding part and β is therefore an isomorphism by "Five Lemma". We see that $H \otimes B = (V \otimes B) \oplus (\langle a_0, a_1, \ldots \rangle^A_{\pi \setminus \{p\}} \otimes B$ splits and the assertion immediately follows.

Lemma 13. *Let p be a prime and let A, B be mixed groups of rank one with p-primary torsion parts T, S, respectively, $\bar{A}$, $\bar{B}$ p-divisible. Further, let $\{k_i, l_i\}_{i=0}^{\infty}$, $\{r_i, s_i\}_{i=0}^{\infty}$ be the p-height sequences of elements $a_0 \in A \setminus T$, $b_0 \in B \setminus S$, respectively, and $l_i < \infty$, $i = 1, 2, \ldots$, $s_{n-1} < s_n = \infty$ for some $n \in \mathbb{N}$. If $A = \langle a_0, a_1, \ldots \rangle^A_{\pi \setminus \{p\}}$, $p^{l_i} a_i = p^{k_i} a_0$, $i = 1, 2, \ldots$, $h_p^{A \otimes B}(a \otimes b) = \infty$ and $m \in \mathbb{N}$ is such that $l_m - k_m - r_n \geqq 0$ then for each $i = 2, \ldots, m$, $j = 2, \ldots, n$, at least one of the inequalities* (15), (16) *is satisfied.*

Proof. By Lemma 10 we have $A = \sum_{i=2}^{m}{}^{\oplus} \langle u_i \rangle \oplus \langle a_m, a_{m+1}, \ldots \rangle^A_{\pi \setminus \{p\}}$ where u_i, $i = 2, \ldots, m$, are the elements (5) corresponding to A and by Lemma 9 we have $B = \sum_{i=2}^{n}{}^{\oplus} \langle x_i \rangle \oplus \langle Y \cup \{b_n, b_{n+1}, \ldots\} \rangle^B_{\pi \setminus \{p\}}$, where $Y \subseteq S$, x_i, $i = 2, \ldots, n-1$, are the elements (5) corresponding to B and $x_n = p^{s_{n-1} + r_n - r_{n-1}} b_n - b_{n-1}$. Now the proof runs along the same lines as that of Lemma 11.

Lemma 14. *Let p be a prime, A a mixed group with a p-primary torsion part T and let $\{k_i, l_i\}_{i=0}^{\infty}$ be the p-height sequence of an element $a_0 \in A \setminus T$. If $p^{l_i} a_i = p^{k_i} a_0$, $l_i < l_{i+1} < \infty$, $i = 1, 2, \ldots$, and $A = \langle a_0, a_1, \ldots \rangle^A_{\pi \setminus \{p\}}$ then $T = \sum_{i=2}^{\infty}{}^{\oplus} \langle t_i \rangle$, where t_i are the elements* (5).

Proof. With respect to Lemma 8 and [3; Lemma 4] it suffices to show that $T = \sum_{i=2}^{\infty} \langle t_i \rangle$. If $t \in T$ is an arbitrary element then $mt = \sum_{i=0}^{n} \lambda_i a_i$ for some $m \in \mathbb{N}$, $(m, p) = 1$. For $n = 0$ we have $\lambda_0 a_0 \in T$, hence $\lambda_0 = 0$ and $t = 0 \in \sum_{i=2}^{\infty} \langle t_i \rangle$. For $n > 0$ it is $p^{l_n} mt = (\sum_{i=0}^{n} p^{l_n - l_i + k_i} \lambda_i)\, a_0 \in T$, so that $\sum_{i=0}^{n} p^{l_n - l_i + k_i} \lambda_i = 0$ and $\lambda_n = p^{l_n - l_{n-1} - k_n + k_{n-1}} \cdot \lambda'_n$ for a suitable integer λ'_n. Thus $mt = \lambda'_n t_n + (\lambda'_n + \lambda_{n-1})\, a_{n-1} + \sum_{i=0}^{n-2} \lambda_i a_i$ and the assertion follows by induction.

Lemma 15. *Let p be a prime and A, B mixed groups of rank one with p-primary torsion parts T, S, respectively, $\bar{A}$, $\bar{B}$ p-divisible. Further, let $\{k_i, l_i\}_{i=0}^{\infty}$, $\{r_i, s_i\}_{i=0}^{\infty}$ be the p-height sequences of the elements $a_0 \in A \setminus T$, $b_0 \in B \setminus S$, respectively, and*

$l_i < \infty$, $s_i < \infty$, $i = 1, 2, \ldots$. *If* $A = \langle a_0, a_1, \ldots \rangle^A_{\pi \setminus \{p\}}$, $p^{l_i} a_i = p^{k_i} a_0$, $i = 1, 2, \ldots$, $B = \langle b_0, b_1, \ldots \rangle^B_{\pi \setminus \{p\}}$, $p^{s_i} b_i = p^{r_i} b_0$, $i = 1, 2, \ldots$, *and* $h_p^{A \otimes B}(a \otimes b) = \infty$ *then for each* $i = 2, 3, \ldots$, $j = 2, 3, \ldots$ *we have*

$$s_{j-1} - r_{j-1} - k_i \geqq l_{i-1} - k_{i-1} - r_j \tag{17}$$

and

$$s_{j-1} - r_{j-1} - k_i \geqq 0 \tag{18}$$

provided $|u_i| \leqq |x_j|$ *and*

$$l_{i-1} - k_{i-1} - r_j \geqq s_{j-1} - r_{j-1} - k_i \tag{19}$$

and

$$l_{i-1} - k_{i-1} - r_j \geqq 0 \tag{20}$$

provided $|x_j| \leqq |u_i|$, *where* u_i, x_j *are the elements* (5) *corresponding to the groups* A, B, *respectively.*

Proof. If $|u_i| = l_{i-1} - k_{i-1} + k_i \leqq s_{j-1} - r_{j-1} + r_j$ then the inequality (17) is obvious. Further, by Lemma 10 we have $A = \sum_{m=2}^{i}{}^{\oplus} \langle u_m \rangle \oplus \langle a_i, a_{i+1}, \ldots \rangle^A_{\pi \setminus \{p\}}$ and $B = \sum_{n=2}^{j}{}^{\oplus} \langle x_n \rangle \oplus \langle b_j, b_{j+1}, \ldots \rangle^B_{\pi \setminus \{p\}}$. Continuing as in the proof of Lemma 11 we see that $p^{l_{i-1} - k_{i-1} + s_{j-1} - r_{j-1}} u_i \otimes x_j = 0$, and this, for $|u_i| \leqq |x_j|$, yields the inequality (18). The inequalities (19) and (20) are proved dually.

Lemma 16. *Let p be a prime and A a mixed group with a p-primary torsion part T. Suppose that A contains elements $a_0, a_1, \ldots \in A \setminus T$ such that* $A = \langle a_0, a_1, \ldots \rangle^A_{\pi \setminus \{p\}}$, $p^{r_i} a_i = p^{s_{i-1}} a_{i-1}$, $r_i > s_{i-1}$, $s_0 = 0$, $|p^{r_i - s_{i-1}} a_i - a_{i-1}| = p^{s_{i-1}}$, $i = 1, 2, \ldots$, *and* $T = \sum_{i=2}^{\infty}{}^{\oplus} \langle p^{r_i - s_{i-1}} a_i - a_{i-1} \rangle$. *Further, let* $U = \sum_{i=0}^{\infty}{}^{\oplus} U_i$, *where* U_i, $i = 0, 1, \ldots$, *be a p-reduced torsionfree group of rank one, the p-divisible closure of which is isomorphic to $\bar{A}$. Then for each $i = 0, 1, \ldots$ there exists an element $c_i \in U_i$ such that $A \cong U/V$, where* $V = \langle p^{r_i} c_i - p^{s_{i-1}} c_{i-1} \mid i \in \mathbb{N} \rangle^U_{\pi \setminus \{p\}}$ *and the element a_0 is mapped onto $c_0 + V$.*

Proof. For each $i = 0, 1, \ldots$ choose an element $c_i \in U_i$ such that $h_p^U(c_i) = 0$ and $h_q^U(c_i) = h_q^A(a_i)$ for each prime $q \neq p$. Now it is easy to see that there exists a homomorphism $\varphi : U \to A$ with $\varphi(c_i) = a_i$, $i = 0, 1, \ldots$. If $a \in A$ is an arbitrary element then $r\bar{a} = s\bar{a}_0$ for some integers r, s, $(r, s) = 1$. If $r = p^k r'$, $(r', p) = 1$, then from $r_i > s_{i-1}$ it easily follows the existence of $j, l \in \mathbb{N}$ with $\bar{a}_0 = p^{k+1} \bar{a}_j$. Clearly, $r'a = p^l s a_j + r't$ and so $r'y = p^l s c_j$ for some $y \in U_j$. If $t = \sum_{i=2}^{n} \lambda_i p^{r_i - s_{i-1}} a_i - a_{i-1})$ then $\varphi(y + \sum_{i=2}^{n} \lambda_i (p^{r_i - s_{i-1}} c_i - c_{i-1}) = a$, φ is an epimorphism and obviously $V \subseteq \operatorname{Ker} \varphi$.

Show that each element $a = \sum_{i=0}^{n} \lambda_i a_i \in T$ can be written in the form

(21) $$a = \sum_{i=2}^{n} \mu_i (p^{r_i - s_{i-1}} a_i - a_{i-1})$$

where

(22) $$\lambda_n = p^{r_n - s_{n-1}} \mu_n \,.$$

If $n \geqq 1$ then for sufficiently large $r \in \mathbb{N}$ we have $[p : r + r_n + \sum_{j=1}^{n-1} (r_j - s_j)] \sum_{i=0}^{n} \lambda_i a_i = (\sum_{i=0}^{n} \lambda_i [p : r_n + \sum_{j=1}^{n-1} (r_j - s_j) - r_i - \sum_{j=1}^{i-1} (r_j - s_j)]) \, p^r a_0 \in T$, from which the validity of (22) immediately follows owing to the fact that $r_i > s_{i-1}$, $i = 1, 2, \ldots, n$. Thus $a = \mu_n (p^{r_n - s_{n-1}} a_n - a_{n-1}) + (\lambda_n + \mu_n) \, a_{n-1} + \sum_{i=0}^{n-2} \lambda_i a_i$ and we can use the induction principle, the case $n = 0$ being trivial.

If $u \in \operatorname{Ker} \varphi$ then $mu = \sum_{i=0}^{n} \lambda_i c_i$ for some $m \in \mathbb{N}$, $(m, p) = 1$. So, $\varphi(mu) = \sum_{i=0}^{n} \lambda_i a_i = 0 \in T$, hence $0 = \sum_{i=0}^{n} \lambda_i a_i = \sum_{i=2}^{n} \mu_i (p^{r_i - s_{i-1}} a_i - a_{i-1})$ by (21) and $p^{s_{n-1}} v_n = \mu_n$ owing to the hypothesis $T = \sum_{i=2}^{\infty}{}^{\oplus} \langle p^{r_i - s_{i-1}} a_i - a_{i-1} \rangle$ and $|p^{r_i - s_{i-1}} a_i - a_{i-1}| = p^{s_{i-1}}$. With respect to (22) we now have $mu = v_n (p^{r_n} c_n - p^{s_{n-1}} c_{n-1}) + (\lambda_{n-1} + v_n p^{s_{n-1}}) \, c_{n-1} + \sum_{i=0}^{n-2} \lambda_i c_i$ and we can continue by induction.

Proof of Theorem. a) implies b) trivially.

b) implies c). With respect to [1; Theorem 2] it suffices to show that $\tau^{A \otimes B}(a \otimes b) = \tau^{\overline{A \otimes B}}(\overline{a \otimes b})$ and that $a \otimes b$ has a p-sequence in $A \otimes B$ whenever $h_p^{A \otimes B}(a \otimes b) = \infty$.

Assume (i). By Lemma 1 there exists an integer n such that $k_n = k_{n+1} = \ldots$, $l_n = l_{n+1} = \ldots$, $r_n = r_{n+1} = \ldots$, $s_n = s_{n+1} = \ldots$ and $l = l_n - k_n$, $s = s_n - r_n$. From the definition of the p-height sequence we obtain the existence of elements $a_1, a_2, \ldots, a_n \in A$, $b_1, b_2, \ldots, b_n \in B$ such that $p^{l_i} a_i = p^{k_i} a$, $p^{s_i} b_i = p^{r_i} b$, $i = 1, 2, \ldots, n$. Further, the relations (1) yield the existence of an integer t with $k_{i_{2t+1}} = k_n$ and $r_{i_{2t}} = r_n$. By Lemma 2 we now have $a \otimes b = [p : \alpha_{2t} + l_{i_{2t+1}}] \, (a_{i_{2t+1}} \otimes b_{i_{2t}}) = p^{l+s} (a_{i_{2t+1}} \otimes b_{i_{2t}})$ and consequently $l + s = h_p^{\overline{A \otimes B}}(\overline{a \otimes b}) \geqq h_p^{A \otimes B}(a \otimes b) \geqq l + s$.

Assume (ii). By Lemma 1 there exists an integer m such that $k_m = k_{m+1} = \ldots$, $l_m = l_{m+1} = \ldots$ and $l = l_m - k_m$. By hypothesis, $h_p^B(p^l b) = \infty$ and hence there exists an integer n with $l \geqq r_n$ and $r_n = r_{n+1} = \ldots$, $s_n = s_{n+1} = \ldots = \infty$. By (1) there exists an integer t with $r_{i_{2t-2}} < r_{i_{2t}} = r_n$. Now by Lemma 3 we have $a \otimes b = a_m \otimes p^l b$ and the element $a \otimes b$ has a p-sequence in $A \otimes B$ by the hypothesis.

Assume (iii). The proof is similar as in the preceding part.

Assume (iv). If $l_n < \infty$, $s_n < \infty$ for each $n = 1, 2, \ldots$ then it suffices to use Lemma 7.

Suppose now that $s_n = \infty$ for an integer n. With respect to Lemmas 5 and 6 we can suppose that $n > 1$. The relations (1) yield the existence of an integer t such that $s_{i_{2t-2}} < s_{i_{2t}} = s_n$. If $\alpha_{2t-1} < \infty$ then, by Lemma 3, $a \otimes b = [p : l_{i_{2t-1}} - k_{i_{2t-1}}] \cdot (a_{i_{2t-1}} \otimes b)$. However, $\alpha_{2t-1} \geqq 0$, $a \otimes b = [p : \alpha_{2t-1}] (a_{i_{2t-1}} \otimes p^{r_n} b)$, the element $a_{i_{2t-1}} \otimes p^{r_n} b$ has a p-sequence in $A \otimes B$ by Lemma 5 or 6 and consequently $a \otimes b$ has a p-sequence in $A \otimes B$.

It remains now to consider the case $\alpha_{2t-1} = \infty = l_{i_{2t-1}}$. If $l_1 = \infty$ then it suffices to use Lemma 5. If $l_1 < \infty$ then the relations (1) yield the existence of an integer $n < t$ such that $l_{i_{2n-1}} < l_{i_{2n+1}} = \infty$. In this case we have $0 \leqq \alpha_{2n} < \infty$ so that Lemma 3 gives $a \otimes b = [p : s_{i_{2n}} - r_{i_{2n}}] (a \otimes b_{i_{2n}}) = p^{\alpha_{2n}}([p : k_{i_{2n+1}}] a \otimes b_{i_{2n}})$ and it suffices to use Lemma 5.

c) implies a). Assume that the tensor product $A \otimes B$ splits and let $a' \in A \setminus T(A)$, $b' \in B \setminus T(B)$ be arbitrary elements. By [1; Theorem 2] and [1; Lemma 3] there are non-zero integers m, n such that for the element $a \otimes b = ma' \otimes nb'$ we have $\tau^{A \otimes B}(a \otimes \otimes b) = \tau^{\overline{A \otimes B}}(\overline{a \otimes b})$ and $a \otimes b$ has a p-sequence in $A \otimes B$ for every prime p with $\overline{A \otimes B}$ p-divisible.

Let p be a prime. Denote $T' = T(A)_{\pi \setminus \{p\}}$, $S' = T(B)_{\pi \setminus \{p\}}$ and let $\alpha : A \to A/T'$, $\beta : B - B/S'$ be the canonical projections. By [5; Corollary 60.3] $\operatorname{Ker} \alpha \otimes \beta$ is a homomorphic image of $(T' \otimes B) \oplus (A \otimes S')$ and it is consequently a torsion group. So $A/T' \otimes B/S'$ splits and we can assume that $T(A)$ and $T(B)$ are p-primary groups.

(i) We shall assume that $l < \infty$, $s < \infty$ and we shall construct inductively the sequence $\{i_t\}_{t=1}^{\infty}$ satisfying conditions (1) and (2). By Lemma 1 there are $m, n \in \mathbb{N}$ such that $k_{m-1} < k_m = k_{m+1} = \ldots$, $r_{n-1} < r_n = r_{n+1} = \ldots$, $l_m - k_m = l$, $s_n - - r_n = s$. If $m = 1$ then we put $i_1 = i_3 = \ldots = 1$ and $i_2 = i_4 = \ldots = n$. In this case (1) is obviously satisfied and $\alpha_{2t-1} = l_m - k_m - r_n \geqq 0$, $\alpha_{2t} = s_n - r_n - k_m \geqq \geqq 0$ for each $t = 1, 2, \ldots$ by Lemma 11.

For $m > 1$ we put $i_1 = 1$. Suppose now that we have constructed the integers $i_1, i_2, \ldots, i_{2t-1}$, $t \geqq 1$, in such a way that $i_1 < i_3 < \ldots < i_{2t-1} < m$, $i_2 < i_4 < \ldots \ldots < i_{2t-2} < n$, $\alpha_j \geqq 0$ for each $j = 1, 2, \ldots, 2t - 2$ and $l_{i_{2j-1}} - k_{i_{2j-1}} - r_{i_{2j}+1} < < 0$, $s_{i_{2j}} - r_{i_{2j}} - k_{i_{2j+1}} < 0$ for each $j = 1, 2, \ldots, t - 1$. From Lemma 11 and $s_{i_{2t-2}} - r_{i_{2t-2}} - k_{i_{2t-1}+1} < 0$ it follows that $l_{i_{2t-1}} - k_{i_{2t-1}} - r_{i_{2t-2}+1} \geqq 0$ so that there exists an integer $i_{2t} > i_{2t-2}$ such that $\alpha_{2t-1} = l_{i_{2t-1}} - k_{i_{2t-1}} - r_{i_{2t}} \geqq 0$ and either $i_{2t} = m$ or $l_{i_{2t-1}} - k_{i_{2t-1}} - r_{i_{2t}+1} < 0$. Similarly, let us suppose that we have constructed the integers $i_1, i_2, \ldots, i_{2t}$, $t \geqq 1$, in such a way that $i_1 < i_3 < \ldots \ldots < i_{2t-1} < m$, $i_2 < i_4 < \ldots < i_{2t} < n$, $\alpha_j \geqq 0$ for each $j = 1, 2, \ldots, 2t - 1$ and $l_{i_{2j-1}} - k_{i_{2j-1}} - r_{i_{2j}+1} < 0$, $s_{i_{2j}} - r_{i_{2j}} - k_{i_{2j+1}+1} < 0$ for each $j = 1, 2, \ldots \ldots, t - 1$, $l_{i_{2t-1}} - k_{i_{2t-1}} - r_{i_{2t}+1} < 0$. By Lemma 11 we have $s_{i_{2t}} - r_{i_{2t}} - k_{i_{2t-1}+1} \geqq \geqq 0$ so that there exists an integer $i_{2t+1} > i_{2t-1}$ such that $\alpha_{2t} = s_{i_{2t}} - r_{i_{2t}} - - k_{i_{2t+1}} \geqq 0$ and either $i_{2t+1} = m$ or $s_{i_{2t}} - r_{i_{2t}} - k_{i_{2t+1}+1} < 0$.

It is easy to see that there exists an integer t such that either $i_{2t+1} = m$ or $i_{2t} = n$. In the former case we put $i_{2t+1} = i_{2t+3} = \ldots = m$, $i_{2t+2} = i_{2t+4} = \ldots = n$ and by

Lemma 11 we obtain $\alpha_{2j+1} = l_m - k_m - r_n \geqq 0$, $\alpha_{2j+2} = s_n - r_n - k_m \geqq 0$ for each $j = t, t+1, \ldots$. In the latter we put $i_{2t} = i_{2t+2} = \ldots = n$, $i_{2t+1} = i_{2t+3} = \ldots = m$ and by Lemma 11 we again get $\alpha_{2j+1} = l_m - k_m - r_n \geqq 0$, $\alpha_{2j} = s_n - r_n - k_m \geqq 0$ for each $j = t, t+1, \ldots$.

We have shown that in this case the elements a, b have the p-property.

(ii) Assume now that $l < \infty$, $s = \infty$ and show that the elements a, b have the p-property. By Lemma 8 and [3; Lemma 8] we have $A = U \oplus V \oplus \langle a_m \rangle^A_{\pi \setminus \{p\}}$, $U \oplus V = T(A)$, $U = \sum_{i=2}^{n}{}^{\oplus} \langle u_i \rangle$, $u_i = p^{l_i - k_i - l_{i-1} + k_{i-1}} a_i - a_{i-1}$, $p^{l_i} a_i = p^{k_i} a$, $i = 1, 2, \ldots, m$, and $l = l_m - k_m > l_{m-1} - k_{m-1}$. It has been mentioned in the proof of Lemma 11 that $\langle a_m \rangle$ is a p-pure subgroup of A. Thus the exact sequence $0 \to \langle a_m \rangle \to A \to A/\langle a_m \rangle \to 0$ is p-pure and the exact sequence $0 \to S \to B \to B/S \to 0$, where $S = T(B)$, is pure. Let us consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \langle a_m \rangle \otimes S & \xrightarrow{\alpha} & A \otimes S & \longrightarrow & A/\langle a_m \rangle \otimes S & \longrightarrow & 0 \\
 & & \delta\downarrow & & \downarrow\beta & & \downarrow & & \\
0 & \to & \langle a_m \rangle \otimes B & \xrightarrow{\gamma} & A \otimes B & \longrightarrow & A/\langle a_m \rangle \otimes B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \langle a_m \rangle \otimes B/S & \to & A \otimes B/S & \to & A/\langle a_m \rangle \otimes B/S & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{23}$$

with natural homomorphisms, where all three columns are exact by [5; Theorem 60.4], the first row is exact by [5; Cotollary 60.5] and the third row is exact by [5; Theorem 60.6]. Using [5; Theorem 60.2] one easily obtain the exactness of the second row.

Since A is of rank one and $\langle a_m \rangle$ is p-pure in A, the factor-group $A/\langle a_m \rangle$ is $(\pi \setminus \{p\})$-primary. Further, S is p-primary by the hypothesis, so that $A/\langle a_m \rangle \otimes S = 0$ and α is an isomorphism. If we denote $T = T(A)$ then the sequence $0 \to T \otimes B/S \to A \otimes B/S \to A/T \otimes B/S \to 0$ is exact by [5; Theorem 60.4] and $T \otimes B/S = 0$, T being p-primary and B/S being p-divisible. Thus $A \otimes B/S \cong A/T \otimes B/S$ is torsion-free, hence $\operatorname{Im} \beta = T(A \otimes B)$ and the middle column splits. If $\varepsilon : A \otimes B \to A \otimes S$ is the splitting map, $\varepsilon\beta = 1_{A \otimes S}$, then for $\eta = \alpha^{-1}\varepsilon\gamma : \langle a_m \rangle \otimes B \to \langle a_m \rangle \otimes S$ we have $\eta\delta = \alpha^{-1}\varepsilon\gamma\delta = \alpha^{-1}\varepsilon\beta\alpha = 1_{\langle a_m \rangle \otimes S}$ showing that η is the splitting map for the first column. Consequently, $B \cong \langle a_m \rangle \otimes B$ splits. By [1; Theorem 2], b has a multiple $p^{l'} b$ having a p-sequence in B, S being p-primary. Thus $s_{n-1} < s_n = \infty$ for some integer n.

By Lemma 9 we now have $B = X \oplus \langle Y \cup \{b_n, b_{n+1}, \ldots\} \rangle^B_{\pi \setminus \{p\}}$, $X \oplus Y = T(B)$, $X = \sum_{j=2}^{n}{}^{\oplus} \langle x_j \rangle$, $x_j = p^{s_j - r_j - s_{j-1} + r_{j-1}} b_j - b_{j-1}$, $j = 2, \ldots, n-1$, $x_n = p^{s_{n-1} + r_n - r_{n-1}} b_n - b_{n-1}$, $p^{s_j} b_j = p^{r_j} b$, $j = 1, 2, \ldots, n$, $|x_n| = p^{s_{n-1} + r_n - r_{n-1}}$. It is easy to see that $a = p^{l_m - k_m} a_m - \sum_{i=2}^{m} p^{l_{i-1}-k_{i-1}} u_i$ and $b = p^{2(s_{n-1}-r_{n-1})+r_n} b_n - \sum_{j=2}^{n} p^{s_{j-1}-r_{j-1}} x_j$.

Using Lemma 11 and the method from part (i) we can construct the sequence $\{i_t\}_{t=1}^{\infty}$ such that the elements a, b have the weak p-property. Now it remains to show that $p^l b$ has a p-sequence in B. The factor-group $A/\langle a_m \rangle$ is $(T \setminus \{p\})$-primary, $\langle a_m \rangle$ being p-pure in A and A being of rank one. Consequently, in the middle row of the diagram (23) the group $A/\langle a_m \rangle \otimes B$ has zero p-primary part. Moreover, $a \otimes b = p^l(a_m \otimes b) \in \langle a_m \rangle \otimes B$ by Lemma 3, from which it easily follows that the element $a \otimes b$ has a p-sequence in $\langle a_m \rangle \otimes B$ ($a \otimes b$ has a p-sequence in $A \otimes B$ by the hypothesis). Thus, in view of the natural isomorphism $B \cong \langle a_m \rangle \otimes B$, the element $p^l b$ has a p-sequence in B.

(iii) The case $l = \infty$, $s < \infty$ is similar to the preceding one.

(iv) Assume, finally, that $l = s = \infty$. We shal distinguish four cases.

α) Suppose that $l_{m-1} < l_m = \infty$, $s_{n-1} < s_n = \infty$ for some $m, n \in \mathbb{N}$. Using Lemma 11 and the method from part (i) we can construct the sequence $\{i_t\}_{t=1}^{\infty}$ such that the elements a, b have the weak p-property. However, in this case, the elements a, b have in fact the-p-property.

β) Suppose now that $l_i < \infty$, $i = 1, 2, \ldots$, and $s_{n-1} < s_n = \infty$ for some $n \in \mathbb{N}$. With respect to Lemma 12 we can restrict ourselves to the case $A = \langle a_0, a_1, \ldots \rangle^{A}_{\pi \setminus \{p\}}$. Obviously, there exists $m \in \mathbb{N}$ such that $l_m - k_m \geqq r_n$. If u_i, $i = 2, \ldots$, are elements (5) corresponding to A and $U = \sum_{i=2}^{m}{}^{\oplus} \langle u_i \rangle$ then $A = U \oplus \langle a_m, a_{m+1}, \ldots \rangle^{A}_{\pi \setminus \{p\}}$ by Lemma 10. By Lemma 13 and the method used in part (i) one can construct the integers $i_1, i_2, \ldots, i_{2t}, i_{2t+1}$ such that $i_1 < i_3 < \ldots < i_{2t-1}$, $i_2 < i_4 < \ldots < i_{2t}$, $\alpha_j \geqq 0$, $j = 1, 2, \ldots, 2t + 1$, and either $i_{2t-1} < i_{2t+1} = m$, $i_{2t} = n$, or $i_{2t-1} = i_{2t+1} = m$, $i_{2t} = n$. In both cases we put $i_{2t} = i_{2t+2} = \ldots = n$ and $i_{2(t+i)+1} = m + i$, $i = 0, 1, \ldots$. Then $\alpha_{2j} = s_n - r_n - k_m = \infty$, $\alpha_{2j+1} = l_{m+j-t} - k_{m+j-t} - r_n$, $j = t, t + 1, \ldots$, $\lim_{j \to \infty} \alpha_j = \infty$ and the elements a, b have the p-property.

γ) The case $l_{m-1} < l_m = \infty$ for some $m \in \mathbb{N}$ and $s_j < \infty$, $j = 1, 2, \ldots$, is similar to the preceding one.

δ) Finally, let us suppose that $l_i < \infty$, $s_i < \infty$, $i = 1, 2, \ldots$. Using Lemma 12 twice we can suppose that $A = \langle a_0, a_1, \ldots \rangle^{A}_{\pi \setminus \{p\}}$, $B = \langle b_0, b_1, \ldots \rangle^{B}_{\pi \setminus \{p\}}$, $a_0 = a$, $b_0 = b$.

During this part of the proof we shall use the notation $\varrho_i = l_i - k_i$, $\sigma_i = s_i - r_i$, $i = 0, 1, \ldots,$. Let u_i, x_i, $i = 2, 3, \ldots$, be the elements (5) corresponding to A, B, respectively. Put $i_1 = 1$. If $i_1, i_2, \ldots, i_{2t-1}$ are constructed let i_{2t} be the smallest positive integer such that $|u_{i_{2t-1}+1}| < |x_{i_{2t}+1}|$ and i_{2t+1} be the smallest positive integer such that $|x_{i_{2t}+1}| < |u_{i_{2t+1}+1}|$. The sequence $\{i_t\}_{t=1}^{\infty}$ obviously satisfies relations (1). Further, $|x_{i_{2t}}| \leqq |u_{i_{2t-1}+1}|$, $|u_{i_{2t+1}}| \leqq |x_{i_{2t}+1}|$ so that Lemma 15 gives

$\alpha_{2t-1} = \varrho_{i_{2t-1}} - r_{i_{2t}} \geqq 0$, $\alpha_{2t} = \sigma_{i_{2t}} - k_{i_{2t+1}} \geqq 0$, $t = 1, 2, \dots$, and the elements a, b have the weak p-property. It remains to show that $\lim_{t \to \infty} \alpha_t = \infty$.

Let $g \in A$, $h \in B$ be arbitrary elements. By the hypothesis there are $\varrho, \sigma, m, n \in \mathbb{N}$ and integers $\lambda_1, \lambda_2, \dots, \lambda_m$, $\mu_1, \mu_2, \dots, \mu_n$ such that $(\varrho, p) = (\sigma, p) = 1$ and $\varrho g = \sum_{i=0}^{m} \lambda_i a_i$, $\sigma h = \sum_{j=0}^{n} \mu_j b_j$. Then $\varrho \sigma a \otimes b = \sum_{i=0}^{m} \sum_{j=0}^{n} \lambda_i \mu_j (a_i \otimes b_j)$ and so

$$A \otimes B = \langle a_i \otimes b_j \mid i, j \in \mathbb{N}_0 \rangle_{\pi \setminus \{p\}}^{A \otimes B} . \tag{24}$$

By [3; Lemma 11(ii)] we have $T(A \otimes B) = T \otimes S$ and therefore Lemma 14 yields

$$T(A \otimes B) = T \otimes S = \sum_{i=2}^{\infty}{}^{\oplus} \sum_{j=2}^{\infty}{}^{\oplus} \langle u_i \rangle \otimes \langle x_j \rangle . \tag{25}$$

Let $M = \sum_{t=1}^{\infty}{}^{\oplus} ((\langle u_{i_{2t-1}+1} \rangle \otimes \langle x_{i_{2t}} \rangle) \oplus (\langle u_{i_{2t+1}} \rangle \otimes \langle x_{i_{2t}+1} \rangle))$ and let L be the complementary direct summand of $T(A \otimes B)$ from the decomposition (25).

Now for each $t = 1, 2, \dots$ put $z_{2t-1} = a_{i_{2t-1}} \otimes b_{i_{2t}-1}$, $z_{2t} = a_{i_{2t+1}-1} \otimes b_{i_{2t}}$ and show that

$$(A \otimes B)/L = \langle z_t + L \mid t \in \mathbb{N}_0 \rangle_{\pi \setminus \{p\}}^{(A \otimes B)/L} \tag{26}$$

where $z_0 = a \otimes b$. For each $i, j \in \mathbb{N}_0$ choose an integer t such that $i \leqq i_{2t+1} - 1$, $j \leqq i_{2t}$. Then

$$[p : \varrho_{i_{2t+1}-1} - \varrho_i + \sigma_{i_{2t}} - \sigma_j]\, z_{2t} - a_i \otimes b_j = m + l\,, \quad m \in M\,, \quad l \in L, \tag{27}$$

owing to the fact that $[p : l_{i_{2t+1}-1}]\, a_{i_{2t+1}-1} = [p : k_{i_{2t+1}-1} + l_i - k_i]\, a_i$ and $[p : s_{i_{2t}}]\, b_{i_{2t}} = [p : r_{i_{2t}} + s_j - r_j]\, b_j$. Now we set

$$\begin{aligned} R_{2t-1} &= \sigma_{i_{2t}-1} - \sigma_{i_{2t-2}} + \varrho_{i_{2t-1}} + k_{i_{2t-1}}\,, \\ S_{2t-1} &= r_{i_{2t}} + \sigma_{i_{2t}-1}\,, \\ R_{2t} &= \varrho_{i_{2t+1}-1} - \varrho_{i_{2t-1}} + \sigma_{i_{2t}} + r_{i_{2t}}\,, \\ S_{2t} &= k_{i_{2t+1}} + \varrho_{i_{2t+1}-1} \end{aligned} \tag{28}$$

and we are going to show that

$$[p : R_t]\, z_t + L = [p : S_{t-1}]\, z_{t-1} + L \tag{29}$$

for each $t = 1, 2, \dots$ $(S_0 = 0)$. Since $a = p^{\varrho_1} a_1$, $b = [p : \sigma_{i_2-1}]\, b_{i_2-1} - \sum_{k=2}^{i_2-1} [p : \sigma_{k-1}]\, x_k$, we have $a \otimes b = p^{R_1} z_1 - \sum_{k=2}^{i_2-1} p^{\varrho_1 + \sigma_{k-1}} a_1 \otimes x_k$. However, Lemma 15 and $|x_k| < |u_{i_1+1}| = |u_2|$, $k = 2, \dots, i_2 - 1$, yield $\varrho_1 \geqq k_2$. Hence $\sum_{k=2}^{i_2-1} p^{\varrho_1 + \sigma_{k-1}} a_1 \otimes x_k = 0$ and (29) holds for $t = 1$. Further, $a_{i_{2t-1}} = [p : \varrho_{i_{2t+1}-1} - \varrho_{i_{2t-1}}]\, a_{i_{2t+1}-1} - \sum_{j=i_{2t-1}+1}^{i_{2t+1}-1} [p : \varrho_{j-1} - \varrho_{i_{2t-1}}]\, u_j$, $b_{i_{2t}-1} = [p : \sigma_{i_{2t}} - \sigma_{i_{2t}-1}]\, b_{i_{2t}} -$

$- x_{i_{2t}}$, hence

$$(30) \qquad z_{2t-1} = [p : \varrho_{i_{2t+1}-1} - \varrho_{i_{2t-1}} + \sigma_{i_{2t}} - \sigma_{i_{2t}-1}]\, z_{2t} - - [p : \varrho_{i_{2t+1}-1} - \varrho_{i_{2t-1}}]\, a_{i_{2t+1}-1} \otimes x_{i_{2t}} - \sum_{j=i_{2t-1}+1}^{i_{2t+1}-1} [p : \varrho_{j-1} - \varrho_{i_{2t-1}} + \sigma_{i_{2t}} - \sigma_{i_{2t}-1}]\, u_j \otimes b_{i_{2t}} + \sum_{j=i_{2t-1}+1}^{i_{2t+1}-1} [p : \varrho_{j-1} - \varrho_{i_{2t-1}}]\, u_j \otimes x_{i_{2t}}.$$

Choosing integers j_1, j_2 such that $\varrho_{j_1} - \varrho_{i_{2t-1}} \geqq |x_{i_{2t}}|$ and $\sigma_{j_2} - \sigma_{i_{2t}-1} \geqq |u_{i_{2t+1}-1}|$ we get $[p : \varrho_{i_{2t+1}-1} - \varrho_{i_{2t-1}}]\, a_{i_{2t+1}-1} \otimes x_{i_{2t}} = ([p : \varrho_{j_1} - \varrho_{i_{2t-1}}]\, a_{j_1} - - \sum_{r=i_{2t}+1}^{j_1} [p : \varrho_{r-1} - \varrho_{i_{2t-1}}]\, u_r \otimes x_{i_{2t}} = - \sum_{r=i_{2t}+1}^{j_1} [p : \varrho_{r-1} - \varrho_{i_{2t-1}}]\, u_r \otimes x_{i_{2t}} \in L$ and $[p : \varrho_{j-1} - \varrho_{i_{2t-1}} + \sigma_{i_{2t}} - \sigma_{i_{2t}-1}]\, u_j \otimes b_{i_{2t}} = [p : \varrho_{j-1} - \varrho_{i_{2t-1}} + \sigma_{i_{2t}} - - \sigma_{i_{2t}-1}]\, u_j \otimes [p : \sigma_{j_2} - \sigma_{i_{2t}}]\, b_{j_2} - \sum_{r=i_{2t}+1}^{j_2} [p : \sigma_{r-1} - \sigma_{i_{2t}}]\, x_r) = = - \sum_{r=i_{2t}+1}^{j_2} [p : \varrho_{j-1} - \varrho_{i_{2t-1}} + \sigma_{r-1} - \sigma_{i_{2t}-1}]\, u_j \otimes x_r \in L$. From this and from (30) we easily get

$$(31) \qquad z_{2t-1} + L = [p : \varrho_{i_{2t+1}-1} - \varrho_{i_{2t-1}} + \sigma_{i_{2t}} - \sigma_{i_{2t}-1}]\, z_{2t} + + u_{i_{2t-1}+1} \otimes x_{i_{2t}} + L.$$

Finally, $z_{2t} = ([p : \varrho_{i_{2t+1}} - \varrho_{i_{2t+1}-1}]\, a_{i_{2t+1}} - u_{i_{2t+1}}) \otimes ([p : \sigma_{i_{2t+2}-1} - - \sigma_{i_{2t}}]\, b_{i_{2t+2}-1} - \sum_{k=i_{2t}+1}^{i_{2t+2}-1} [p : \sigma_{k-1} - \sigma_{i_{2t}}]\, x_k)$, from which it similarly as above follows that

$$(32) \qquad z_{2t} + L = [p : \varrho_{i_{2t+1}} - \varrho_{i_{2t+1}-1} + \sigma_{i_{2t+2}-1} - \sigma_{i_{2t}}]\, z_{2t+1} + + u_{i_{2t+1}} \otimes x_{i_{2t}+1} + L.$$

The inequalities $|x_{i_{2t}}| \leqq |u_{i_{2t-1}+1}|$ and $|u_{i_{2t+1}}| \leqq |x_{i_{2t}+1}|$ together with (31) and (32) prove the validity of (29) for each $t = 1, 2, \ldots$. Moreover, the formulas (31) and (32) together with (27) prove (26).

By (26), (29), (31), (32) and Lemma 16 the factor-group $(A \otimes B)/L$ can be represented as U/V. Since $a \otimes b + L$ is mapped onto $c_0 + V$, the element $c_0 + V$ has a p-sequence in U/V and consequently the series $\sum_{i=1}^{\infty}(R_i - S_i)$ has nonnegative partial sums and $\sum_{i=1}^{\infty}(R_i - S_i) = \infty$ by [3; Lemma 16]. However, $\sum_{i=1}^{2n}(R_i - S_i) = = \sum_{t=1}^{n}(R_{2t} - S_{2t} + R_{2t-1} - S_{2t-1}) = \sum_{t=1}^{n}(\sigma_{i_{2t}} - \sigma_{i_{2t-2}} - k_{i_{2t+1}} + k_{i_{2t-1}}) = = \sigma_{i_{2n}} - k_{i_{2n+1}} = \alpha_{2n}$, $\sum_{i=1}^{2n+1}(R_i - S_i) = \sum_{t=1}^{n}(R_{2t+1} - S_{2t+1} - R_{2t} - S_{2t}) + R_1 - - S_1 = \sum_{t=1}^{n}(\varrho_{i_{2t+1}} - \varrho_{i_{2t-1}} - r_{i_{2t+2}} + r_{i_{2t}}) + \varrho_{i_1} + k_{i_1} - r_{i_2} = \varrho_{i_{2n+1}} - r_{i_{2n+2}} = = \alpha_{2n+1}$ and the proof is complete.

Corollary 1. *Let A, B be mixed groups of rank one and let P, Q be non-torsion pure subgroups of the groups A, B, respectively. Then $P \otimes Q$ splits if and only if $A \otimes B$ splits.*

Proof. Each element $a \in P \setminus T(P)$ has in P the same p-height sequence as in A and it suffices to apply Theorem.

Corollary 2. *Let P, Q be pure subgroups of a splitting mixed group A of rank one. Then $P \otimes Q$ splits. In particular, each pure subgroup of a splitting mixed group of rank one has the splitting length at most 2.*

Proof. It follows immediately from Corollary 1.

Corollary 3. *Let A be a torsionfree group of rank one and B a mixed group of rank one. Then $A \otimes B$ splits if and only if for each $0 \neq a \in A$ there exists $b \in B \setminus T(B)$ with the p-height sequence $\{r_i, s_i\}_{i=0}^{\infty}$ such that for each prime p with A p-reduced we have $h_p^A(a) \geqq r_n = r_{n+1} = \dots$ for some $n \in \mathbb{N}$ and $[p : h_p^A(a)]\, b$ has a p-sequence in B whenever $s_n = \infty$.*

Proof. If p is any prime and $\{k_i, l_i\}_{i=0}^{\infty}$ is the p-height sequence of a in A then $l_1 = l_2 = \dots = h_p^A(a)$, $k_1 = k_2 = \dots = 0$ and it suffices to apply Theorem.

As a final application of our results we shall present a new proof of a special case of [3; Theorem] characterizing mixed abelian groups of rank one having the splitting length 2.

Corollary 4. *A non-splitting mixed abelian group A of rank one has the splitting length 2 if and only if it contains an element $a \in A \setminus T(A)$ such that for each prime p the p-height sequence $\{k_i, l_i\}_{i=0}^{\infty}$ of a has the following two properties:*

$$l_i - k_i - k_{i+1} \geqq 0\,, \quad i = 0, 1, \dots, \tag{33}$$

$$\lim_{i \to \infty} (l_i - k_i - k_{i+1}) = 2h_p^{\bar{A}}(\bar{a}) - \lim_{i \to \infty} l_i\,, \tag{34}$$

where we put $\infty - m = \infty$ for every $m \in \mathbb{N}_0 \cup \{\infty\}$.

Proof. Assume first that $A^2 = A \otimes A$ splits. If p is a prime and $\bar{A}$ is p-reduced then $l_i - k_i - k_{i+1} \geqq 0$, $i = 0, 1, \dots$, by Lemma 11 and (34) obviously holds by Lemma 1. If $\bar{A}$ is p-divisible and $l_n = \infty$ for some $n \in \mathbb{N}$ then Lemma 11 proves (33) while (34) is obvious. Finally, if $\bar{A}$ is p-divisible and $l_i < \infty$, $i = 1, 2, \dots$, then Lemma 15 proves (33) and (34) is true by the proof of Theorem, since in this case $i_t = t$, $t = 1, 2, \dots$.

Conversely, if the conditions (33) and (34) are satisfied then the elements a, a have the p-propety for $i_t = t$ and A^2 splits by Theorem.

References

[1] *L. Bican:* Mixed abelian groups of torsionfree rank one, Czech. Math. J. *20* (*95*), (1970), 232—242.

[2] *L. Bican:* A note on mixed abelian groups, Czech. Math. J. *21* (*96*), (1971), 413—417.

[3] *L. Bican:* The splitting length of mixed abelian groups of rank one, Czech. Math. J. *27* (*102*), (1977), 144—154.

[4] *L. Fuchs:* Abelian groups, Budapest, 1958.

[5] *L. Fuchs:* Infinite abelian groups I, Academic Press, New York and London, 1970.

[6] *L. Fuchs:* Infinite abelian groups II, Academic Press, New York and London, 1973.

[7] *I. M. Irwin, S. A. Khabbaz, G. Rayna:* The role of the tensor product in the splitting of abelian groups, J. Algebra *14*, (1970), 423—442.

Author's address: 186 00 Praha 8 - Karlín, Sokolovská 83, ČSSR (Matematicko-fyzikální fakulta UK).