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ON THE HIGHER ORDER POINCARÉ-CARTAN FORMS

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This paper is intended to clarify some basic geometric structures related with the higher order Hamilton formalism in arbitrary fibred manifolds. Using a suitable generalization of the formal exterior differentiation, we show that any r-th order Lagrangian determines a family of the Poincaré-Cartan forms, which is reduced to a single form only if $r \leqq 2$ or the base manifold X is one-dimensional. For $\dim X = 1$, we then derive a relation generalizing the so-called basic theorem of the first order Hamilton formalism in fibred manifolds deduced by Goldschmidt and Sternberg, [2]. — Our consideration is in the category C^∞.

1. A general form of the variational formula. Given any fibred manifold $\pi\colon Y \to X$, we denote by $\pi_r\colon J^rY \to X$ its r-th jet prolongation and by $\pi_r^s\colon J^rY \to J^sY$, $0 \leqq s < r$, $(J^0Y = Y)$ the jet projections. All morphisms are assumed to be base-preserving.

For any morphism $\varphi\colon J^rY \to \bigwedge^k T^*X$, one defines its formal exterior differential $D\varphi\colon J^{r+1}Y \to \bigwedge^{k+1}T^*X$ by

$$D\varphi(j^{r+1}s) = d(\varphi \circ j^rs) \tag{1}$$

for every section s of Y, [3], [8]. If

$$x^i,\ y^p,\ i, j, \ldots = 1, \ldots, n = \dim X\,, \quad p = 1, \ldots, m = \dim Y - \dim X\,,$$

are some local fibre coordinates on Y, $y^p_j, \ldots, y^p_{j_1\ldots j_r}$ are the induced coordinates on J^rY and the coordinate expression of φ is

$$\varphi \equiv a_{i_1\ldots i_k}(x^i, y^p, y^p_j, \ldots, y^p_{j_1\ldots j_r})\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}\,, \tag{2}$$

then

$$D\varphi \equiv D_l a_{i_1\ldots i_k}\, dx^l \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}\,, \tag{3}$$

provided

$$D_l f = \partial_l f + (\partial_p f)\, y^p_l + \ldots + (\partial_p^{j_1\ldots j_r} f)\, y^p_{j_1\ldots j_r l} \tag{4}$$

with $\partial_l = \partial/\partial x^l$, $\partial_p = \partial/\partial y^p, \ldots, \partial_p^{j_1\ldots j_r} = \partial/\partial y^p_{j_1\ldots j_r}$, means the formal (or total) derivative of a function $f\colon J^rY \to \mathbf{R}$. Clearly, $DD\varphi = 0$. It is well-known that

$J^{r+1}Y \to J^rY$ is an affine bundle, whose associated vector bundle is the pullback of $VY \otimes S^{r+1}T^*X$ over J^rY, where VY means the vertical tangent bundle of Y and S denotes the symmetric tensor product. By (3) and (4), $\mathrm{D}\varphi\colon J^{r+1}Y \to \bigwedge^{k+1}T^*X$ is an affine morphism for every φ.

Given a projectable vector field η on Y, we denote by $J^r\eta$ the vector field on J^rY induced by means of flows

$$\exp(tJ^r\eta) = J^r(\operatorname{expt}\eta)\,. \tag{5}$$

If $\eta \equiv \eta^p(x, y)\,\partial_p$ is a vertical vector field, then the coordinate expression of $J^r\eta$ is

$$J^r\eta \equiv \eta^p\,\partial_p + \mathrm{D}_j\eta^p\,\partial_p^j + \ldots + \mathrm{D}_{j_1\ldots j_r}\eta^p\,\partial_p^{j_1\ldots j_r}\,. \tag{6}$$

Using (3) and (6), one verifies easily

Lemma 1. *For every morphism* $A\colon J^rY \to V^*J^qY \otimes \bigwedge^kT^*X$ *over the identity of* J^qY, $q \leqq r$, *there exists a unique morphism* $\mathrm{D}A\colon J^{r+1}Y \to V^*J^{q+1}Y \otimes \bigwedge^{k+1}T^*X$ *satisfying*

$$\langle \mathrm{D}A, J^{q+1}\eta\rangle = \mathrm{D}(\langle A, J^q\eta\rangle) \tag{7}$$

for every vertical vector field η *on* Y.

In coordinates, if

$$A \equiv (a_{pi_1\ldots i_k}\,\mathrm{d}y^p + \ldots + a_{pi_1\ldots i_k}^{j_1\ldots j_q}\,\mathrm{d}y^p_{j_1\ldots j_q}) \otimes \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}\,, \tag{8}$$

then

$$\mathrm{D}A = (\mathrm{D}_l a_{pi_1\ldots i_k}\,\mathrm{d}y^p + a_{pi_1\ldots i_k}\,\mathrm{d}y^p_l + \ldots + \mathrm{D}_l a_{pi_1\ldots i_k}^{j_1\ldots j_q}\,\mathrm{d}y^p_{j_1\ldots j_q} + a_{pi_1\ldots i_k}^{j_1\ldots j_q}\,\mathrm{d}y^p_{j_1\ldots j_q l}) \otimes \mathrm{d}x^l \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x_{i_k}\,. \tag{9}$$

Obviously, $\mathrm{D}A$ is an affine morphism and it holds $\mathrm{DD}A = 0$.

Define K^s_q by an exact sequence

$$0 \to K^s_q \to VJ^qY \xrightarrow{V\pi_q{}^s} VJ^sY \to 0\,, \quad s \leqq q\,, \tag{10}$$

where $V\pi^s_q$ means the vertical tangent map to π^s_q. It is well-known that K^{q-1}_q is the pullback of $VY \otimes S^qT^*X$ over J^qY. We have a sequence of inclusions

$$K^{q-1}_q \to K^{q-2}_q \to \ldots \to K^0_q \to VJ^qY\,, \tag{11}$$

which induces the dual sequence of epimorphisms

$$V^*J^qY \to K^{0*}_q \to \ldots \to K^{q-2*}_q \to K^{q-1*}_q\,. \tag{12}$$

A morphism $A\colon J^rY \to V^*J^qY \otimes \bigwedge^kT^*X$ will be called graded if there exist $A_1, \ldots, A_q$ such that the following diagram commutes

$$
\begin{array}{ccc}
J^rY & \xrightarrow{A} & V^*J^qY \otimes \bigwedge^k T^*X \\
\downarrow \pi_r^{r-1} & & \downarrow \\
J^{r-1}Y & \xrightarrow{A_1} & K_q^{0*} \otimes \bigwedge^k T^*X \\
\downarrow & & \downarrow \\
\vdots & \vdots & \vdots \\
\downarrow & & \downarrow \\
J^{r-q-1}Y & \xrightarrow{A_{q-1}} & K_q^{q-2*} \otimes \bigwedge^k T^*X \\
\downarrow \pi_{r-q-1}^{r-q} & & \downarrow \\
J^{r-q}Y & \xrightarrow{A_q} & K_q^{q-1*} \otimes \bigwedge^k T^*X
\end{array}
\tag{13}
$$

where the arrows in the second column are the tensor products of (12) and the identity of $\bigwedge^k T^*X$. Clearly, if A is graded, then $\mathrm{D}A$ is also graded.

We define an r-th order Lagrangian λ on Y as a morphism $\lambda\colon J^rY \to \bigwedge^n T^*X$, [3], [4], [8]. In coordinates,

$$\lambda \equiv L(x^i, y^p, y^p_j, \ldots, y^p_{j_1 \ldots j_r})\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n .$$

Its vertical differential $VJ^rY \to \bigwedge^n T^*X$ will be interpreted as a map $\delta\lambda\colon J^rY \to$ $\to V^*J^rY \otimes \bigwedge^n T^*X$.

Proposition 1. *For any r-th order Lagrangian λ on Y, there exist a graded morphism $M\colon J^{2r-1}Y \to V^*J^{r-1}Y \otimes \bigwedge^{n-1} T^*X$ and a unique Euler morphism $E\colon J^{2r}Y \to$ $\to V^*Y \otimes \bigwedge^n T^*X$ such that*

$$\delta\lambda = \mathrm{D}M + E . \tag{14}$$

For $r = 1$ or $n = 1$, M is uniquely determined. If $r \geqq 2$ and $n \geqq 2$, any other morphism with this property is of the form $M + \mathrm{D}C$, where C is any graded morphism $C\colon J^{2r-2}Y \to V^*J^{r-2}Y \otimes \bigwedge^{n-2} T^*X$.

Proof. Write $\omega_i = (\partial/\partial x^i) \lrcorner\, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ and

$$M = (b^i_p\, \mathrm{d}y^p + b^{ji}_p\, \mathrm{d}y^p_j + \ldots + b^{j_1 \ldots j_{r-1} i}_p\, \mathrm{d}y^p_{j_1 \ldots j_{r-1}}) \otimes \omega_i , \tag{15}$$

$$E = (e_p\, \mathrm{d}y^p) \otimes \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n , \tag{16}$$

so that (14) is equivalent to

$$
\begin{aligned}
\partial^{j_1 \ldots j_r}_p L &= b^{(j_1 \ldots j_r)}_p \\
&\vdots \\
\partial^{j_1 \ldots j_k}_p L &= \mathrm{D}_j b^{j_1 \ldots j_k j}_p + b^{(j_1 \ldots j_k)}_p \\
&\vdots \\
\partial_p L &= \mathrm{D}_j b^j_p + e_p
\end{aligned}
$$

where the round bracket denotes symmetrization. Therefore,

$$
\begin{aligned}
b_p^{j_1\dots j_r} &= \partial_p^{j_1\dots j_r}L + c_p^{j_1\dots j_r}\,, \quad c_p^{j_1\dots(j_{r-1}j_r)} = 0\,,\\
&\vdots\\
b_p^{j_1\dots j_k} &= \partial_p^{j_1\dots j_k}L - \mathrm{D}_j b_p^{j_1\dots j_k j} + c_p^{j_1\dots j_k}\,, \quad c^{j_1\dots(j_{k-1}j_k)} = 0\,,\\
&\vdots\\
e_p &= \partial_p L - \mathrm{D}_j b_p^j\,,
\end{aligned}
$$

where $c_p^{j_1\dots j_k}$ are any functions on $J^{2r-k}Y$ antisymmetric in j_{k-1} and j_k. Hence $\mathrm{D}_{ij}c_p^{j_1\dots j_{k-2}ij} = 0$, which implies that

$$
(17) \qquad e_p = \partial_p L - \mathrm{D}_j\, \partial_p^j L + \dots + (-1)^r\, \mathrm{D}_{j_1\dots j_r}\, \partial_p^{j_1\dots j_r} L
$$

is uniquely determined. Further, the space of all $c_p^{j_1\dots j_r}$ is the pullback over J^rY of the following vector bundle

$$
(18) \qquad V^*Y \otimes (S^{q-1}TX \otimes TX \cap S^{q-2}TX \otimes \Lambda^2 TX) \otimes \Lambda^n T^*X
$$

with $q = r$. Take a global section c_r of the latter vector bundle and apply induction. By the induction hypothesis, for any $i = 0, 1, \dots, r-k-1$ we have considered an affine subbundle of the pullback of $(K_r^{r-i-1})^* \otimes \Lambda^n T^*X$ over $J^{r+i}Y$, the associated vector bundle being the pullback of (18) with $q = r - i$ over $J^{r+i}Y$, and we have constructed a section c_{r-i} of the latter bundle. In this situation, the space of all $c_p^{j_1\dots j_k}$ is an affine subbundle of the pullback of $(K_r^{k-1})^* \otimes \Lambda^n T^*X$ over $J^{r+k}Y$, the associated vector bundle being the pullback of (18) with $q = k$ over $J^{r+k}Y$. Indeed, since the values of $b_p^{j_1\dots j_r}, \dots, b_p^{j_1\dots j_{k+1}}$ are already fixed by means of $c_r, \dots, c_{k+1}$, the differences $c_p^{j_1\dots j_k}$ lie in the subspace $(K_k^{k-1})^* \otimes \Lambda^n T^*X \subset (K_r^{k-1})^* \otimes \Lambda^n T^*X$ determined by means of the dual map to the epimorphism $V\pi_r^k\colon VJ^rY \to VJ^kY$. Hence we can construct a global section c_k of the affine bundle in question and continue in our induction procedure. Finally, we obtain a graded morphism $M\colon J^{2r-1}Y \to V^*J^{r-1}Y \otimes \Lambda^{n-1}T^*X$ satisfying (14). Analyzing this construction, we find easily that any other graded morphism with the same property is of the form $M + \mathrm{D}C$ mentioned in our Proposition, QED.

Any morphism $M\colon J^{2r-1}Y \to V^*J^{r-1}Y \otimes \Lambda^{n-1}T^*X$ such that there is an E satisfying (14) will be called a morphism associated to λ. From the proof of Proposition 1 we obtain a somewhat stronger result: if M_i, $i = 1, 2$ are two morphisms associated to λ (not necessarily graded) and if E_i satisfies (14), then $E_1 = E_2$. In other words, the Euler morphism is uniquely determined even if M is not graded.

Remark 1. If we take any vertical vector field η on Y, $\langle \delta\lambda, J^r\eta \rangle =: \delta_\eta \lambda$ is the classical variation of λ with respect to η. Then (14) implies

$$
(19) \qquad \delta_\eta \lambda = \mathrm{D}\langle M, J^{r-1}\eta \rangle + \langle E, \eta \rangle\,,
$$

which is a variational formula of classical type.

2. Poincaré-Cartan morphisms. We shall distinguish a special class of the associated morphisms. For this purpose, we need a modification of our operation D.

Consider a morphism $A\colon J^rY \to (K^s_q)^* \otimes \bigwedge^k T^*X$ with a coordinate expression

$$A \equiv \left(a^{j_1 \ldots j_{s+1}}_{pi_1 \ldots i_k} \mathrm{d}y^p_{j_1 \ldots j_{s+1}} + \ldots + a^{j_1 \ldots j_q}_{pi_1 \ldots i_k} \mathrm{d}y^p_{j_1 \ldots j_q}\right) \otimes \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}.$$

As we have no inclusion $(K^s_q)^* \subset V^*J^qY$, we cannot construct $\mathrm{D}A$. However, one can consider a map $B\colon J^rY \to V^*J^qY \otimes \bigwedge^k T^*X$ covering A,

$$B \equiv \left(b_{pi_1 \ldots i_k} \mathrm{d}y^p + \ldots + b^{j_1 \ldots j_s}_{pi_1 \ldots i_k} \mathrm{d}y^p_{j_1 \ldots j_s} + a^{j_1 \ldots j_{s+1}}_{pi_1 \ldots i_k} \mathrm{d}y^p_{j_1 \ldots j_{s+1}} + \right.$$
$$\left. + a^{j_1 \ldots j_q}_{pi_1 \ldots i_k} \mathrm{d}y^p_{j_1 \ldots j_q}\right) \otimes \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}.$$

Denote by $(\mathrm{D}B)_s$ the canonical projection of $\mathrm{D}B$ into $(K^s_{q+1})^* \otimes \bigwedge^{k+1} T^*X$, so that

$$(20) \qquad (\mathrm{D}B)_s \equiv \left(b^{j_1 \ldots j_s}_{pi_1 \ldots i_k} \mathrm{d}y^p_{j_1 \ldots j_s l} + \mathrm{D}_l a^{j_1 \ldots j_{s+1}}_{pi_1 \ldots i_k} \mathrm{d}y^p_{j_1 \ldots j_{s+1}} + \ldots \right.$$
$$\left. \ldots + a^{j_1 \ldots j_q}_{pi_1 \ldots i_k} \mathrm{d}y^p_{j_1 \ldots j_q l}\right) \mathrm{d}x^l \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}.$$

This depends on the choice of B only by the first term. But (20) is an affine map and $b^{j_1 \ldots j_s}_{pi_1 \ldots i_k}$ belong to its "absolute" part, so that the linear map associated to (20) is completely determined by A. The latter map will be denoted by $\bar{\mathrm{D}}A$.

Consider a graded morphism M associated to λ. Since M is graded, the values of $\bar{\mathrm{D}}(M_k)$ lie in the subspace $(K^{k-1}_k)^* \otimes \bigwedge^n T^*X = V^*Y \otimes S^kTX \otimes \bigwedge^n T^*X$. Let $\lrcorner : S^qTX \otimes \bigwedge^k T^*X \to S^{q-1}TX \otimes \bigwedge^{k-1} T^*X$ be the standard map, [7]. We shall say that M is a Poincaré-Cartan (in short: P.-C.) morphism of λ, if it holds

(i) $(\mathrm{id} \otimes \lrcorner) \circ M_{r-1} = 0$,

(ii) M_{k-1} is affine and its associated linear morphism coincides with $-(\mathrm{id} \otimes \lrcorner) \circ$ $\circ\, \bar{\mathrm{D}}(M_k)$ for all $k = r - 1, \ldots, 2$.

The coordinate meaning of these additional conditions is $c^{j_1 \ldots j_r}_p = 0$ and $c^{j_1 \ldots j_k}_p$ are some functions defined on $J^{2r-k-1}Y$ (and not on $J^{2r-k}Y$ as in the general case). Quite similarly to Proposition 1, one deduces

Proposition 2. *For any r-th order Lagrangian λ on Y there exists a P.-C. morphism M. This morphism is unique if $r = 1, 2$ or $n = 1$. If $r \geqq 3$ and $n \geqq 2$, any other P.-C. morphism of λ is of the form $M + \mathrm{D}C$, where C is any graded morphism $C\colon J^{2r-3}Y \to V^*J^{r-3}Y \otimes \bigwedge^{n-2} T^*X$.*

Remark 2. If the base manifold X is an affine space, we have a map $a\colon VJ^{r-1}Y \otimes$ $\otimes\, T^*X \to VJ^rY$ defined as follows. Any $b \in T^*_xX$, $x \in X$, determines a unique affine map $f\colon X \to R$ such that $b = j^1_x f$. Using the well-known identification $VJ^{r-1}Y \approx$ $\approx J^{r-1}VY$, [2], we can express any $u \in (VJ^{r-1}Y)_x$ as $u = j^{r-1}_x \sigma$, where σ is a local section of VY. Then $j^r_x(f \,.\, \sigma) \in J^rVY \approx VJ^rY$ is completely determined by b and u. This map is bilinear and induces a. Having any manifold Q and any map $F\colon Q \to$

$\to V^*J^rY \otimes \bigwedge^{n-1}T^*X$, we can now construct the following maps

$$\begin{array}{c} Q \to^F V^*J^rY \otimes \bigwedge^{n-1}T^*X \\ \downarrow \quad a^* \otimes \mathrm{id} \\ V^*J^{r-1}Y \otimes TX \otimes \bigwedge^{n-1}T^*X \\ \downarrow \quad \mathrm{id} \otimes \lrcorner \\ V^*J^{r-1}Y \otimes \bigwedge^{n-2}T^*X \, . \end{array}$$

If the resulting map vanishes, F will be called quasi-symmetric. Having a coordinate expression of F of the from (15), F is quasi-symmetric iff all b's are symmetric in all superscripts. Analyzing the proof of Proposition 1, we deduce easily that there is a unique quasi-symmetric P.-C. morphism of λ, which will be called the affine P.-C. morphism of λ. Its coordinate expression is

$$(21) \qquad [(\partial_p^{j_1 \ldots j_{r-1} l} L) \, \mathrm{d}y^p_{j_1 \ldots j_{r-1}} + (\partial_p^{j_1 \ldots j_{r-2} l} L - \mathrm{D}_j \, \partial_p^{j_1 \ldots j_{r-2} j l} L) \, \mathrm{d}y^p_{j_1 \ldots j_{r-2}} + \ldots$$
$$\ldots + (\partial^l_p L - \mathrm{D}_j \, \partial^{jl}_p L + \ldots + (-1)^{r-1} \, \mathrm{D}_{j_1 \ldots j_{r-1}} \, \partial_p^{j_1 \ldots j_{r-1} l} L) \, \mathrm{d}y^p] \otimes \omega_l \, .$$

3. Transfer to exterior forms. We recall that an exterior k-form ω on J^rY is called contact, if $(j^rs)^* \omega = 0$ for every section of Y. A contact k-form ω is said to be 2-contact, if $\zeta \lrcorner \omega$ is a contact $(k-1)$-form for every vertical vector field ζ on J^rY.

Any morphism $\varphi \colon J^rY \to \bigwedge^k T^*X$ can be canonically interpreted as an exterior k-form on J^rY, which will be denoted by $\tilde{\varphi}$. By the very definition of the formal exterior differentiation,

$$(22) \qquad \Delta\varphi := \mathrm{d}(\tilde{\varphi}) - (\mathrm{D}\varphi)^\sim$$

is a contact $(k+1)$-form on $J^{r+1}Y$. In particular, any function $f \colon J^rY \to \mathbf{R}$ determines a contact 1-form Δf on $J^{r+1}Y$.

Lemma 2. *For every morphism* $A \colon J^rY \to V^*J^qY \otimes \bigwedge^k T^*X$ *and every vertical vector field* ζ *on* J^qY,

$$(23) \qquad (\mathrm{D}\langle A, \zeta\rangle)^\sim - (\mathrm{d}(\langle A, \zeta\rangle))^\sim$$

is a contact form.

Proof. If $\zeta^p \, \partial_p + \ldots + \zeta^p_{j_1 \ldots j_q} \, \partial_p^{j_1 \ldots j_q}$ is the coordinate expression of ζ, then the coordinate expression of (23) is

$$[(\Delta a_{p i_1 \ldots i_k}) \, \zeta^p + a_{p i_1 \ldots i_k} \, \Delta\zeta^p + \ldots + (\Delta a^{j_1 \ldots j_q}_{p i_1 \ldots i_k}) \, \zeta^p_{j_1 \ldots j_q} +$$
$$+ a^{j_1 \ldots j_q}_{p i_1 \ldots i_k} \, \Delta\zeta^p_{j_1 \ldots j_q}] \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k} \, .$$

Consider the canonical map $\psi_q \colon TJ^{q+1}Y \to VJ^qY$ (called the structure form of $TJ^{q+1}Y$), [1], [2]. The coordinate expression of ψ_q is

$$(\Delta y^p) \, \partial_p + \ldots + (\Delta y^p_{j_1 \ldots j_q}) \, \partial_p^{j_1 \ldots j_q} \, .$$

For every $A \colon J^rY \to V^*J^qY \otimes \bigwedge^k T^*X$, $r > q$, we define a $(k+1)$-form $\psi_q \barwedge A$ on J^rY by the natural combination of the contraction with respect to VJ^qY and alternation. The coordinate expression of $\psi_q \barwedge A$ is

$$(a_{pi_1\dots i_k} \Delta y^p + \dots + a^{j_1\dots j_q}_{pi_1\dots i_k} \Delta y^p_{j_1\dots j_q}) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} . \tag{24}$$

Obviously, it holds

$$\zeta \lrcorner (\psi_q \barwedge A) = (\langle A, \zeta \rangle)^{\sim} \tag{25}$$

for every vertical vector field ζ on $J^q Y$. (As $\psi_q \barwedge A$ is a π^q_r-horizontal form, $\zeta \lrcorner$ $\lrcorner$ $(\psi_q \barwedge A)$ has a well-defined meaning even though ζ is a vector field on $J^q Y$ and not on $J^r Y$.)

Lemma 3. *The following form is 2-contact*

$$(\psi_{q+1} \barwedge DA) + d(\psi_q \barwedge A) . \tag{26}$$

Proof. The coordinate expression of (26) is

$$(\Delta y^p \wedge \Delta a_{pi_1\dots i_k} + \dots + \Delta y^p_{j_1\dots j_q} \wedge \Delta a^{j_1\dots j_q}_{pi_1\dots i_k}) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k} .$$

Set $\varepsilon = \psi_1 \barwedge E$, which is an $(n + 1)$-form on $J^{2r} Y$.

Proposition 3. *For any morphism M associated to λ, any vertical vector field η on Y and any section s of Y, it holds*

$$(j^r s)^* (J^r \eta \lrcorner d\tilde{\lambda}) = (j^{2r-1} s)^* d(J^{r-1} \eta \lrcorner (\psi_{r-1} \barwedge M)) + + (j^{2r} s)^* (\eta \lrcorner \varepsilon) . \tag{27}$$

Proof. Let us start from (19). Obviously, $\langle \delta\lambda, J^r \eta \rangle = J^r \eta \lrcorner d\tilde{\lambda}$ and $(\langle E, \eta \rangle)^{\sim} = \eta \lrcorner \varepsilon$. By Lemma 2, $(j^{2r-1} s)^* (D\langle M, J^{r-1} \eta \rangle)^{\sim} = (j^{2r-1} s)^* (d\langle M, J^{r-1} \eta \rangle)^{\sim}$. Using (25), we obtain $(\langle M, J^{r-1} \eta \rangle)^{\sim} = J^{r-1} \eta \lrcorner (\psi_{r-1} \barwedge M)$, which proves (27).

For any morphism M associated to λ, the n-form $\Theta = \tilde{\lambda} + \psi_{r-1} \barwedge M$ will be called a form associated to λ.

Proposition 4. *Let Θ be a form associated to λ. Then $\zeta \lrcorner d\Theta$ is a contact form for any π^0_{2r-1}-vertical vector field ζ on $J^{2r-1} Y$.*

Proof. By (14), $\psi_r \barwedge \delta\lambda = \psi_r \barwedge DM + \psi_r \barwedge E$. Hence $(\zeta \lrcorner \delta\lambda)^{\sim} - (\zeta \lrcorner DM)^{\sim} = (\zeta \lrcorner E)^{\sim} = 0$ as ζ is π^0_{2r-1}-vertical. By Lemma 3, the form $\psi_r \barwedge DM + d(\psi_{r-1} \barwedge \barwedge M)$ is 2-contact, so that $(\zeta \lrcorner DM)^{\sim} + \zeta \lrcorner d(\psi_{r-1} \barwedge M)$ is a contact form. Since $(\zeta \lrcorner \delta\lambda)^{\sim} = \zeta \lrcorner d\tilde{\lambda}$, the form $\zeta \lrcorner d\Theta$ is also contact, QED.

If M is a P.-C. morphism of λ, then the corresponding form $\Theta = \tilde{\lambda} + \psi_{r-1} \barwedge M$ will be called a P.-C. form of λ. If the base manifold is an affine space, then the P.-C. form corresponding to the affine P.-.C. morphism will be also said to be affine. Such a form was considered by Krupka, [5]. However, we remark that for $r \geqq 3$ and $n \geqq 2$, the coordinate expressions corresponding to (21), which are used in [5], have no intrinsic meaning in the case of an arbitrary fibred manifold.

4. On the higher order Hamilton formalism. Let $L_r(Y)$ denote the pullback of $V^*Y \otimes S^rTX \otimes \bigwedge^n T^*X$ over $J^{r-1}Y$. This vector bundle will be said to be the r-th Legendre bundle of Y. The restriction of $\delta\lambda$ to K_r^{r-1} can be interpreted as a map $\sigma : J^rY \to L_r(Y)$, which will be called the Legendre transformation of λ. If $w_p^{j_1 \dots j_r}$ are the natural fibre coordinates on $L_r(Y)$, then the equations of σ are

$$w_p^{j_1 \dots j_r} = \partial_p^{j_1 \dots j_r} L .$$

A Lagrangian will be called regular, if its Legendre transformation is a local diffeomorphism. (The general idea of a regular higher order Lagrangian is due to D. Krupka and M. Francaviglia, [9].) Hence λ is regular iff $\partial_p^{j_1 \dots j_r} \partial_q^{k_1 \dots k_r} L$ is a regular matrix.

Lemma 4. *For* $\dim X = 1$, *if* Θ *is the P.-C. form of a regular Lagrangian and a section* $u: X \to J^{2r-1}Y$ *satisfies*

$$u^*(\zeta \lrcorner \, \mathrm{d}\Theta) = 0 \tag{28}$$

for every π^0_{2r-1}*-vertical vector field* ζ *on* $J^{2r-1}Y$, *then* u *is holonomic, i.e. there is a section* $s: X \to Y$ *such that* $u = j^{2r-1}s$. *If* $r = 1$ *the same result holds for any* $\dim X$.

Proof. We shall use local coordinates. For $\dim X = 1$, we shall write $y^p_{j_1 \dots j_r} = y^p_{(r)}$ and $L_{pq} = \partial_p^{j_1 \dots j_r} \partial_q^{k_1 \dots k_r} L$. Let $\zeta = \zeta^p_{(1)} \partial_p^{(1)} + \dots + \zeta^p_{(2r-1)} \partial_p^{(2r-1)}$. Then (28) leads to the following sequence of equations

$$u^*(L_{pq}\zeta^p_{(2r-1)} \, \Delta y^q) = 0 , \quad \text{which gives} \quad u^* \, \Delta y^q = 0 ,$$
$$\vdots$$
$$u^*(L_{pq}\zeta^p_{(1)} \, \Delta y^q_{(2r-2)}) = 0 , \quad \text{which gives} \quad u^* \, \Delta y^q_{(2r-2)} = 0 .$$

Thus, u is holonomic. If $r = 1$, the same result is deduced for any $\dim X$ in [3], QED.

Lemma 5. *For any contact* k*-form* ω *on* J^rY, *any projectable vector field* η *on* Y *and any section* s *of* Y, *it holds*

$$(j^rs)^* \, \mathrm{d}(J^r\eta \lrcorner \, \omega) = -(j^rs)^* \, (J^r\eta \lrcorner \, \mathrm{d}\omega) . \tag{29}$$

Proof. Since ω vanishes on the r-jet prolongation of any section of Y and the flow of $J^r\eta$ transforms any r-jet prolongation of a section into an r-jet prolongation of a section, the Lie derivative $L_{J^r\eta}\omega$ is also a contact form. Using the standard formula for the Lie derivative

$$L_{J^r\eta}\omega = J^r\eta \lrcorner \, \mathrm{d}\omega + \mathrm{d}(J^r\eta \lrcorner \, \omega) ,$$

we find (29), QED.

Proposition 5. *If* Θ *is any form associated to* λ *and* $s: X \to Y$ *is a section, then the* *condition* $(j^{2r-1}s)^*(\zeta \lrcorner \mathrm{d}\Theta) = 0$ *for any* π_{2r-1}*-vertical vector field* ζ *on* $J^{2r-1}Y$ *is equivalent to* $(j^{2r}s)^* E = 0$.

Proof. Consider first a section s of Y and set $\zeta_s = \zeta|_{j^{2r-1}s}$. Then the projection $\eta_s = T\pi^0_{2r-1}(\zeta_s)$ can be extended to a vertical vector field η on Y. By construction, $(J^{2r-1}\eta - \zeta)|_{j^{2r-1}s}$ is π^0_{2r-1}-vertical, so that $(j^{2r-1}s)^*(\zeta \lrcorner \mathrm{d}\Theta) = (j^{2r-1}s)^* \,.\, (J^{2r-1}\eta \lrcorner \mathrm{d}\Theta)$ by Proposition 4. According to Lemma 5 and Proposition 3, $(j^{2r-1}s)^*(J^{2r-1}\eta \lrcorner \mathrm{d}\Theta) = (j^{2r}s)^*(\eta \lrcorner \varepsilon)$. Hence $(j^{2r}s)^* E = 0$ is equivalent to $(j^{2r-1}s)^*(\zeta \lrcorner \mathrm{d}\Theta) = 0$, QED.

The following result generalizes the so-called basic theorem of the first order Hamilton formalism [2], [3], (Another geometrical treatment of the first order case can be found in a paper by Ragionieri and Ricci, [6].)

Theorem. *For* $\dim X = 1$, *let* Θ *be the Poincaré-Cartan form of a regular r-th order Lagrangian on* Y *and* $u: X \to J^{2r-1}Y$ *any section. Then the equation* $u^*(\zeta \lrcorner \mathrm{d}\Theta) = 0$ *for every* π_{2r-1}*-vertical vector field* ζ *on* $J^{2r-1}Y$ *is equivalent to a pair of conditions*

$$u = j^{2r-1}s \quad \text{for a section } s \text{ of } Y \text{ and} \quad (j^{2r}s)^* E = 0\,.$$

The same result holds for $r = 1$ *and any* $\dim X$.

Proof. By Lemma 4, $u^*(\zeta \lrcorner \mathrm{d}\Theta)$ implies $u = j^{2r-1}s$. Then our Theorem follows from Proposition 5, QED.

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