Czechoslovak Mathematical Journal

Václav Koubek

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Czechoslovak Mathematical Journal, Vol. 34 (1984), No. 4, 506–527

Persistent URL: http://dml.cz/dmlcz/101978

Czechoslovak Mathematical Journal, 34 (109) 1984, Praha

LARGE SYSTEMS OF INDEPENDENT OBJECTS IN CONCRETE CATEGORIES I

VÁCLAV KOUBEK, Praha

(Received September 2, 1982)

0. INTRODUCTION

One of the basic ideas in combinatorics is that of independence of certain parts of a structure, intuitively described as absence of links or overlapping between any two of them. To express it formally, we have to define, on the collection of the parts in question, *adjacency* represented by a collection of adjacent pairs. A collection is then called *independent* if no two distinct members of it form an adjacent pair. Relevant questions are those concerning large independent collections of specified parts of a given structure.

In a graph G, two distinct points p and q are adjacent if they form a line $\{p, q\}$ of G; two distinct lines of G are adjacent if they have a point in common; two subsets of points of G are adjacent if either $A \cap B \neq \emptyset$ or there is a line $\{a, b\}$ with $a \in A$, $b \in B$. Finite combinatorics was concerned with point and line independencies in graphs in late twenties and early thirties, when the by now classical results were obtained by Menger, König, Dilworth, Hall and others.

A typical and interesting phenomenon in the infinite combinatorics, unparalleled in the finite theory, is represented by infinite graphs G containing independent subsets of points of the same cardinality as the whole set of points of G. We shall call such a subset a *span* of G, a graph G having a span will be said to be *spanned* (by any one of its spans).

In an arbitrary category $\mathscr{K}$, the notion of independence of objects arises naturally from that of adjacency of a pair $\{A, B\}$ of objects established by the existence of a morphism either from A to B or from B to A. Two classes $\mathscr{A}, \mathscr{B}$ of objects of $\mathscr{K}$ are adjacent if either $\mathscr{A} \cap \mathscr{B} \neq \emptyset$ or there is an adjacent pair $\{A, B\}$ with $A \in \mathscr{A}$, $B \in \mathscr{B}$. Thus we can speak about independent families of classes of objects of $\mathscr{K}$. To be able to speak of spans of $\mathscr{K}$, we have to express the size of $\mathscr{K}^{\mathrm{ob}}$ and of its subclasses. To this end, we add to the class Card of all cardinals in Bernays-Gödel set theory a class cardinal c to follow all cardinals. If we further assume the strong axiom of choice, then we can assign to $\mathscr{K}$ a cardinal card $\mathscr{K}$ equal to the cardinal of a maximal class of pairwise unisomorphic objects of $\mathscr{K}$.

A typical independence question to be dealt with in this categorial setting is e.g. that of S. Ulam who asks whether there exist $2^{\aleph_0}$ countable graphs such that there is no morphism from one to another. Put otherwise, is the category of countable graphs spanned?

Let us call a category $\mathscr{C}$ self-independent if the class $\mathscr{C}^{\text{ob}}$ of all its objects is independent. Since an independent class of objects of a category $\mathscr{K}$ determines a full self-independent subcategory of $\mathscr{K}$, the problem of finding large independent classes of objects in $\mathscr{K}$ amounts to that of full embeddings of large self-independent categories into $\mathscr{K}$. The theory of full embeddings, developed in Prague as part of infinite combinatorics rather than of abstract algebra, possesses methods, results, and techniques powerful enough to deal with independence questions in concrete categories.

Indeed, the affirmative answer to Ulam's question follows immediately from results of Hedrlín and Pultr [8]. Moreover, one can restrict oneself to graphs with special properties (symmetry, fixed chromatic number $\geqq 3$, [18], [19]) and the answer remains affirmative. Analogous results hold for various types of algebras instead of graphs (unary algebras with at least two operations, groupoids, or more specially semigroups, integral domains, etc., [7], [10], [3]). It is essential that all these categories are *binding*, i.e. receiving full embeddings of any concrete categories. A remarkable result, pertaining to Ulam's question but obtained differently, by Katětov (in an unpublished paper) and independently by Gavalec and Jakubíková [4], says that there exist $2^{\aleph_0}$ independent countable unary algebras with one operation. Here the technique of full embeddings does not work, since the category of monounary algebras is not binding. Also, a negative answer has been obtained in this case for cardinals bigger than $2^{\aleph_0}$ [1], [4].

It is reasonable, when dealing with a concrete category $(\mathscr{K}, U)$, not to look for spans in the whole category $\mathscr{K}$ but for the so called α-spans defined as spans in its full subcategories $\mathscr{K}_\alpha$, $\alpha \in \text{Card}$, $\mathscr{K}_\alpha^{\text{ob}} = \{A \in \mathscr{K}^{\text{ob}};\ \text{card}\, U(A) \leqq \alpha\}$. To make it more difficult, we can prescribe endomorphism monoids of the objects of a span of $\mathscr{K}_\alpha$.

Definition 0,1. Let $(\mathscr{K}, U)$ be a concrete category, let $\mathscr{M}$ be a monoid. A full subcategory $\mathscr{C}$ of $\mathscr{K}_\alpha$, $\alpha \in \text{Card}$, is called an *α-span* of $\mathscr{K}$ if

(1) $\mathscr{C}$ is self-independent,

(2) $\text{card}\, \mathscr{C} = \text{card}\, \mathscr{K}_\alpha$.

If, moreover,

(3) $\text{End}\,(A) \cong \mathscr{M}$ for every $A \in \mathscr{C}^{\text{ob}}$,

than we call $\mathscr{C}$ an *$(\mathscr{M}, \alpha)$-span* of $\mathscr{K}$.

A category $(\mathscr{K}, U)$ is said to be *α-spanned* (*$(\mathscr{M}, \alpha)$-spanned*) if it has an α-span ($(\mathscr{M}, \alpha)$-span).

If $\mathcal{K}$ is α-spanned ($(\mathcal{M}, \alpha)$-spanned) for all $\alpha \geqq \beta$, $\beta \in \mathrm{Card}$, then it is said to be *spanned* (*$\mathcal{M}$-spanned*) *from β upwards.*

If $\mathcal{K}$ is spanned ($\mathcal{M}$-spanned) from some β upwards, then it is said to be *ultimately spanned* (*ultimately $\mathcal{M}$-spanned*).

In the case of trivial $\mathcal{M}$ we shall call an $(\mathcal{M}, \alpha)$-span also a *discrete α-span*, and an $(\mathcal{M}, \alpha)$-spanned category $\mathcal{K}$ also *discretely α-spanned.*

If a category $(\mathcal{K}, U)$ is both $(\mathcal{M}, \alpha)$- and $(\mathcal{M}', \alpha)$-spanned, for two non-isomorphic monoids $\mathcal{M}$ and $\mathcal{M}'$, we may be interested in finding in $\mathcal{K}$ the corresponding $(\mathcal{M}, \alpha)$-span $\mathcal{C}$ and $(\mathcal{M}', \alpha)$-span $\mathcal{C}'$ in such a way that $\mathcal{C}$ and $\mathcal{C}'$ are not adjacent. More generally, we have

Definition 0,2. Let $(\mathcal{K}, U)$ be a concrete category, let $\mathfrak{M} = (\mathcal{M}_i)_{i \in I}$ be a family (possibly a big one) of monoids. An $(\mathfrak{M}, \alpha)$-*span* of $\mathcal{K}$, $\alpha \in \mathrm{Card}$, is an independent family $(\mathcal{C}_i)_{i \in I}$, where $\mathcal{C}_i$ is an $(\mathcal{M}_i, \alpha)$-span of $\mathcal{K}$ for every $i \in I$.

$\mathcal{K}$ is $(\mathfrak{M}, \alpha)$-*spanned* if it has an $(\mathfrak{M}, \alpha)$-span.

$\mathcal{K}$ is *$\mathfrak{M}$-spanned from β upwards* if it has an $(\mathfrak{M}, \alpha)$-span for all $\alpha \geqq \beta$.

$\mathcal{K}$ is *ultimately $\mathfrak{M}$-spanned* if it is $\mathfrak{M}$-spanned from some β upwards.

Calling two distinct full embeddings $\Phi, \Phi' : \mathcal{K} \to \mathcal{L}$ adjacent if their images are adjacent in $\mathcal{L}$, we can speak about independent families of full embeddings of $\mathcal{K}$ into $\mathcal{L}$.

A full embedding $\Phi : (\mathcal{K}, U) \to (\mathcal{L}, V)$ is called *strong* (in [19]) if there exists a covariant set functor F such that the following diagram commutes. Any such F is called an *underlying functor of Φ.*

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\;\Phi\;} & \mathcal{L} \\ {\scriptstyle U}\big\downarrow & & \big\downarrow{\scriptstyle V} \\ \mathrm{Set} & \xrightarrow[\;F\;]{} & \mathrm{Set} \end{array}$$

Definition 0,3. Let $(\mathcal{K}, U)$ and $(\mathcal{L}, V)$ be concrete categories. For $\alpha \in \mathrm{Card}$, a family $\mathcal{F} = (\Phi_i)_{i \in I}$ of (strong) full embeddings of $\mathcal{K}$ into $\mathcal{L}$ is called a (*strong*) $(\mathcal{K}, \alpha)$-*span* of $\mathcal{L}$ if

(1) $\mathcal{F}$ is independent,

(2) $\operatorname{card} I = \operatorname{card} \mathcal{L}_\alpha$,

(3) $\forall A \in \mathcal{K}^{\mathrm{ob}}\ \forall i \in I (\operatorname{card} V\Phi_i A \leqq \max(\operatorname{card} UA, \alpha))$.

$\mathcal{L}$ is (*strongly*) $(\mathcal{K}, \alpha)$-*spanned* if it has a (strong) $(\mathcal{K}, \alpha)$-span.

$\mathcal{L}$ is (*strongly*) *$\mathcal{K}$-spanned from β upwards* if it is (strongly) $(\mathcal{K}, \alpha)$-spanned for all $\alpha \geqq \beta$.

$\mathcal{L}$ is *ultimately* (*strongly*) *$\mathcal{K}$-spanned* if it is (strongly) $\mathcal{K}$-spanned from some β upwards.

We shall study independence questions in an important class, introduced in [8], of concrete categories $S(F)$ determined by set functors F. The objects of $S(F)$ are the pairs (X, R), where X is a set and $R \subseteq FX$, and the morphisms from (X, R) to (Y, S) are the mappings f fulfilling $Ff(R) \subseteq S$ $(Ff(S) \subseteq R)$ if F is covariant (contravariant). The underlying set functor from $S(F)$ to Set is defined in the natural way.

Most day-life categories are "nice" (i.e. reflective or coreflective) full subcategories of suitable $S(F)$'s. The categories of n-ary relations, symmetric n-ary relations, and hypergraphs actually are categories $S(F)$. Moreover, Kučera [16] proved the existence of functors F for which every concrete category can be fully embedded into $S(F)$. Kučera and Pultr [17] showed that every "reasonable" concrete category even has a realization (i.e. a strong full embedding with identity underlying functor) in $S(F)$ for a suitable F. These facts clearly demonstrate the importance of the categories $S(F)$.

The present paper is the first one of a series of papers devoted to the study of spans in the categories $S(F)$. Its objective is to settle the covariant case by

Main Theorem. *Let $\mathscr{D}$ be the category of directed graphs. For a covariant set functor F the following are equivalent:*

(1) *$S(F)$ is strongly ultimately $\mathscr{D}$-spanned,*
(2) *$S(F)$ is ultimately discretely spanned,*
(3) *there exist sets A, B such that A is finite, and $F(A \cup B) \neq \operatorname{Im} Fi \cup \operatorname{Im} Fj$, where $i: A \to A \cup B$, $j: B \to A \cup B$ are the inclusions.*

In the next paper we shall prove an analogous theorem for F contravariant, with (1) and (2) the same and (3) replaced by a statement of a corresponding equivalent property of F. Further papers will be devoted to α-spans in the categories $S(F)$. For example, by additional restrictions on cardinalities we strengthen a result of Hedrlín and Sichler [11] who proved, under the assumption that measurable cardinals form a set, that $\mathscr{D}$ is $\mathscr{D}$-spanned.

I want to express my thanks to A. Pultr who called my attention to these interesting problems, and to P. Goralčík and the referee who helped me to clarify the exposition.

I

This section has an auxiliary character and is devoted to the investigation of set functors. It is based on notions and propositions in [21] and [12].

For a filter $\mathscr{F}$ on a set X let us denote

$$\|\mathscr{F}\| = \min \{\text{card } Z;\ Z \in \mathscr{F}\},$$
$$\bigcap\mathscr{F} = \bigcap\{Z;\ Z \in \mathscr{F}\}.$$
$$\|\bigcap\mathscr{F}\| = \text{card} \bigcap\mathscr{F},$$
$$\mathrm{d}\mathscr{F} = \{V;\ \exists Z \in \mathscr{F},\ V \supset Z - \bigcap\mathscr{F}\}.$$

For every filter $\mathscr{F}$ we have $\bigcap(\mathrm{d}\mathscr{F}) = \emptyset$, and the relation $\mathrm{d}\mathscr{F} = \exp X = \{Z; Z \subset X\}$ is valid iff $\mathscr{F} = \{Z; X \supset Z \supset \bigcap\mathscr{F}\}$.

For a mapping $f: X \to Y$ and a filter $\mathscr{F}$, $f(\mathscr{F}) = \{V; \exists Z \in \mathscr{F}, V \supset f(Z)\}$ is a filter, too. In particular, we shall consider $\exp X$ as a filter (on X), i.e., a filter on X is a family of subsets of X closed under meets. The following is proved in [21]:

Proposition 1,1. *Let F be a covariant set functor, $x \in FX$. If Y, Z are subsets of X with the inclusions $i: Y \to X$, $j: Z \to X$ such that $x \in \operatorname{Im} Fi \cap \operatorname{Im} Fj$ then either $\emptyset = Y \cap Z$ or $x \in \operatorname{Im} Fk$ where $k: Y \cap Z \to X$ is the inclusion.*

According to [12], for a covariant set functor F and $x \in FX$ put $\mathscr{F}^X_F(x) = \{Z \subset X;$ $x \in \operatorname{Im} Fi$, $i: Z \to X$ is the inclusion$\}$. Then $\mathscr{F}^X_F(x)$ is a filter or $\mathscr{F}^X_F(x) = \exp X - \{\emptyset\}$ by the foregoing proposition. In the latter case we add to $\mathscr{F}^X_F(x)$ the empty set, then $\mathscr{F}^X_F(x)$ is always a filter and is called a filter of the point x.

The following proposition characterizing the relation of a filter of a point and a mapping is simple but very useful for a description of the behaviour of a set functor.

Proposition 1,2 [12]. *Let $f: X \to Y$ be a mapping, let F be a covariant set functor, $x \in FX$. Then $f(\mathscr{F}^X_F(x)) \subseteq \mathscr{F}^Y_F(F f(x))$.*

If for a set $Z \in \mathscr{F}^X_F(x)$, f is one-to-one onto Z, then $f(\mathscr{F}^X_F(x)) = \mathscr{F}^Y_F(Ff(x))$.

The second notion describing set functors is [12]:

If F is a covariant set functor, then a cardinal $\alpha > 1$ is called an *unattainable cardinal* of F if for a set X with $\operatorname{card} X = \alpha$ we have $FX \neq \bigcup \operatorname{Im} Ff$ where the union is taken over all $f: Y \to X$ with $\operatorname{card} Y < \alpha$.

The class of all unattainable cardinals of F is denoted by $\mathscr{A}_F$.

The connection between unattainable cardinals of F and filters of points is shown in the following propositon:

Proposition 1,3 [12]. *Let F be a covariant set functor. Then a cardinal $\alpha > 1$ is an unattainable cardinal of F iff there exists a set X and $x \in FX$ with $\alpha = \|\mathscr{F}^X_F(x)\|$.*

In the sequel we restrict ourselves only to the covariant set functors, calling them just *set functors*. Here is a list of some basic ones:

- I – the identity set functor;
- C_M – the constant set functor onto M;
- Q_M – the covariant hom-functor of the set M;
- P_α – the covariant power set functor defined as $P_\alpha X = \{Z; Z \subset X, \operatorname{card} Z < \alpha\}$ and for $f: X \to Y$, $P_\alpha f(Z) = f(Z)$.

As follows from the definition of strongly ultimately $\mathscr{K}$-spanned category the basic role is played by functors of the following special kind:

Definition. Given a cardinal α, a set functor F is said to be *non-increasing from α upwards*, if $\operatorname{card} FX \leq \operatorname{card} X$ for every set X with $\operatorname{card} X \geq \alpha$.

Then we have

Theorem 1,4. *For a set functor F, the following are equivalent:*

a) *F is non-increasing from α upwards for a cardinal α;*
b) *if α is an unattainable cardinal of F, then α is finite;*
c) *F is a colimit of homfunctors of finite sets;*
d) *F is a factor-functor of a sum of homfunctors of finite sets.*

Proof. If we combine Theorem 3,4 in [13] and Lemma 2,3 in [12] we get that if F has an infinite unattainable cardinal α, then there exists a proper class A of cardinals such that for $\beta \in A$, $\operatorname{cf} \beta = \operatorname{cf} \alpha$ and $\operatorname{card} FX \geqq 2^{\beta}$ for any set with $\operatorname{card} X = \beta$. Hence a) $\Rightarrow$ b). Further, by an easy application of Theorem 1,3 in [12] we get b) $\Rightarrow$ a). For the implications b) $\Leftrightarrow$ c) $\Leftrightarrow$ d) see Theorem 3,4 in [12] or [2].

We next prove some auxiliary lemmas about set functors.

Convention. For a set functor F, $x \in FX$, F^x is a subfunctor of F defined as $F^x Z = \{z \in FZ;\ \exists f: X \to Z,\ Ff(x) = z\}$. Further, let us denote $F(x)\, Z = \{z \in FZ;\ \exists f: X \to Z,\ \exists V \in \mathscr{F}_F^X(x);\ f$ is one-to-one onto V and $Ff(x) = z\}$.

Lemma 1,5. *Let F: Set $\to$ Set be a set functor, $x \in FX$. Then for every infinite set Y with* $\operatorname{card} Y \geqq \operatorname{card} X$ *we have* $\operatorname{card} F^x Y = \operatorname{card} F(x)\, Y$.

Proof. For every mapping $h: X \to Y$ there exists a one-to-one mapping $f_h: X \to X \times Y$ with $h = \Pi_Y \circ f_h$ ($\Pi_Y: X \times Y \to Y$ is the projection). Hence for every $z \in F^x Y$ there exists a point $y_z \in F(x)(X \times Y)$ with $F\, \Pi_Y(y_z) = z$, i.e. $\operatorname{card} F^x Y \leqq \operatorname{card} F(x)(X \times Y)$. Since $\operatorname{card} Y = \operatorname{card} X \times Y$ and $F(x)\, Y \subseteq F^x Y$ we get $\operatorname{card} F(x)\, Y = \operatorname{card} F^x Y$.

Lemma 1,6. *Let F be a set functor, $x \in FX$ with $\|\mathscr{F}_F^X(x)\| \geqq \aleph_0$. Then for every finite set $A \subset X$ with* $\operatorname{card} A \leqq \|\bigcap(\mathscr{F}_F^X(x))\|$ *we have* $\operatorname{card} F(x)\, X = \operatorname{card} \{y \in F(x)\, X;\ A \subset \bigcap(\mathscr{F}_F^X(y))\}$.

Proof. Let $\{X_z;\ z \in X\}$ be a family of disjoint subsets of $X - A$ with $\operatorname{card} X_z = \operatorname{card} X$ for every $z \in X$ (since X is infinite, such a family exists). Choose a bijection $\varphi: \{B \subset X;\ \operatorname{card} B = \operatorname{card} A\} \to X$, and for every $B \subset X$ with $\operatorname{card} B = \operatorname{card} A$ choose an injection $\psi_B: X \to X$ with $\psi_B(X) = X_{\varphi(B)} \cup A$ and $\psi_B(B) = A$. Now for every $y \in F(x)\, X$ choose $B_y \subset \bigcap(\mathscr{F}_F^X(y))$ with $\operatorname{card} B_y = \operatorname{card} A$ and put $z_y = F\, \psi_{B_y}(y)$. Since ψ_B is an injection we get that $F\psi_B$ is an injection, which means that if $y_1 \neq y_2$ and $y_1, y_2 \in F(x)\, X$ and $B_{y_1} = B_{y_2}$, then $z_{y_1} \neq z_{y_2}$. By Proposition 1, 2, $\mathscr{F}_F^X(z_y) = \psi_{B_y}(\mathscr{F}_F^X(y))$, hence if $y_1, y_2 \in F(x)\, X$ and $B_{y_1} \neq B_{y_2}$, then $z_{y_1} \neq z_{y_2}$. Since $\{z_y;\ y \in F(x)\, X\} \subseteq \{y \in F(x)\, X;\ A \subset \bigcap(\mathscr{F}_F^X(y))\}$ (indeed, $\mathscr{F}_F^X(z_y) = \psi_{B_y}(\mathscr{F}_F^X(y))$ because ψ_{B_y} is an injection, thus $\bigcap(\mathscr{F}_F^X(z_y)) = \psi_{B_y}(\bigcap(\mathscr{F}_F^X(y)))$ and because $B_y \subset \bigcap(\mathscr{F}_F^X(y))$ and $\psi_{B_y}(B_y) = A$, necessarily $A \subset \bigcap(\mathscr{F}_F^X(z_y))$), we get the required equality.

Proposition 1,7. *Let F be a set functor, $x \in FX$ with $\|\mathscr{F}_F^X(x)\| \geqq \aleph_0$ and $F = F^x$. Then for every finite set $A \subset X$ with* card $A \leqq \|\bigcap(\mathscr{F}_F^X(x))\|$ *there exists a set $\mathfrak{A}$ of objects of $S(F)$ such that*

a) *if $(Y, S) \in \mathfrak{A}$ then $Y = X$, $\emptyset \neq S \subset F(x)X$, and $y \in S$ implies $A \subset \bigcap(\mathscr{F}_F^X(y))$;*
b) *if $(X, S_1), (X, S_2) \in \mathfrak{A}$ then $S_1 - S_2 \neq \emptyset \neq S_2 - S_1$;*
c) card $S(F)_{\mathrm{card} X}$ = card $\mathfrak{A}$.

Proof. Let $(Z, S) \in S(F)_{\mathrm{card}X}$, then card $Z \leqq$ card X and therefore we can choose a bijection $f: Z \to Y$ where $Y \subset X$; then (Z, S) is isomorphic with $(Y, Ff(S))$. Thus card $S(F)_{\mathrm{card}X} \leqq \max\{\mathrm{card}\ 2^X, \mathrm{card}\ 2^{FX}\}$. Since the converse equality is clear we get card $S(F)_{\mathrm{card}X} = \max\{\mathrm{card}\ 2^X, \mathrm{card}\ 2^{FX}\}$. Since $\|\mathscr{F}_F^X(x)\| \geqq \aleph_0$ we get by [12] that card $FX \geqq$ card X, which means that card $S(F)_{\mathrm{card}X}$ = card 2^{FX}. Choose two disjoint subsets X_1, X_2 of X with card X_1 = card X_2 = card X and $A \cap X_1 = \emptyset = A \cap X_2$. Let $\varphi_1, \varphi_2: X \to X$ be injections with $\varphi_1(A) = A = \varphi_2(A)$, $\varphi_1(X) = X_1 \cup A$, $\varphi_2(X) = X_2 \cup A$. Let $T \subset \{y \in F(x)X;\ A \subset \bigcap(\mathscr{F}_F^X(y))\}$, put $\mathscr{A}_T = (X, F\varphi_1(T) \cup F\varphi_2(\{y \in F(x)X - T;\ A \subset \bigcap(\mathscr{F}_F^X(y))\}))$. Put $\mathfrak{A} = \{\mathscr{A}_T;\ T \subset \{y \in F(x)X;\ A \subset \bigcap(\mathscr{F}_F^X(y))\}$. From the definition of $\mathscr{A}_T$ and by Proposition 1,2 we get that $\mathfrak{A}$ fulfils a), b), Lemmas 1,5 and 1,6 imply card FX = card $\{y \in F(x)X;\ A \subset \bigcap(\mathscr{F}_F^X(y))\}$ and hence $\mathfrak{A}$ fulfils c) as well.

To conclude this section we formulate two simple but very useful propositions:

Proposition 1,8. *Let $(\mathscr{K}, U)$, $(\mathscr{L}, V)$, $(\mathscr{M}, W)$ be concrete categories such that $(\mathscr{K}, U)$ is (strongly) $\mathscr{M}$-spanned from α upwards and there exists a (strong) full embedding Φ of $(\mathscr{K}, U)$ to $(\mathscr{L}, V)$ such that for a cardinal β we have* card $V\Phi A \leqq \max\{\mathrm{card}\ UA, \beta\}$ *for every object A of $\mathscr{K}$. Let γ be a cardinal such that* card $\mathscr{K}_\delta \geqq$ card $\mathscr{L}_\delta$ *for every $\delta \geqq \gamma$. Then $(\mathscr{L}, V)$ is (strongly) $\mathscr{M}$-spanned from* $\max\{\alpha, \beta, \gamma\}$ *upwards.*

Proposition 1,9. *Let $(\mathscr{K}, U)$, $(\mathscr{L}, V)$, $(\mathscr{M}, W)$ be concrete categories such that $(\mathscr{K}, U)$ is (strongly) $\mathscr{M}$-spanned from α upwards. If there is a (strong) full embedding Φ from $(\mathscr{L}, V)$ to $(\mathscr{M}, W)$ such that for a cardinal β we have* card $W\Phi A \leqq \max\{\mathrm{card}\ UA, \beta\}$, *then $(\mathscr{K}, U)$ is (strongly) $\mathscr{L}$-spanned from* $\max\{\alpha, \beta\}$ *upwards.*

Since both propositions have straightforward proofs we omit them.

II

This section is devoted to set functors F which have a finite unattainable cardinal α (i.e., there exists a set X and a point $x \in FX$ with $\|\bigcap(\mathscr{F}_F^X(x))\| = \alpha$ and $\mathrm{d}(\mathscr{F}_F^X(x)) = \exp X$). We shall show that $S(F)$ is strongly ultimately $\mathscr{D}$-spanned. This result follows by a simple compilation of results proved in other papers.

Definition. The category of symmetric connected graphs without loops and compatible mappings is denoted by $\mathscr{S}$.

We recall some propositions:

Proposition 2,1 [19]. *There exists a strong embedding of $\mathscr{D}$ to $\mathscr{S}$ such that the underlying functor is non-increasing from $\aleph_0$ upwards.*

Proposition 2,2 [13]. *For every set Z there exists a full subcategory $\mathscr{S}_Z$ of $\mathscr{S}$ and a strong embedding Π_Z from $\mathscr{S}$ to $\mathscr{S}_Z$ such that*

1) *for every symmetric graph $(X, R) \in \mathscr{S}_Z$ we have $Z \subset X$, and for every compatible mapping $f: (X, R) \to (Y, S)$ where $(X, R), (Y, S) \in \mathscr{S}_Z$ we have $f/Z = 1_Z$;*
2) *the underlying functor of the embedding Π_Z is non-increasing from* $\max \{\aleph_0, \text{card } Z\}$ *upwards.*

Proposition 2,3 [19]. *There exists a strong embedding from the following concrete category:*

a) *objects are (X, R, S) where (X, S) and (X, R) are symmetric graphs;*
b) *morphisms from (X_1, R_1, S_1) to (X_2, R_2, S_2) are mappings which are compatible mappings from (X_1, R_1) to (X_2, R_2) and at the same time compatible mappings from (X_1, S_1) to (X_2, S_2);*

to $\mathscr{S}$ such that the underlying functor is non-increasing from $\aleph_0$ upwards.

Note. We can get an independent proof of Proposition 2,3 if we use a technique from [14] and [15] and make two šíp-constructions where šíps are independent.

Proposition 2,4 [13]. *If a set functor F has a finite unattainable cardinal then there exists a strong embedding of $\mathscr{S}$ to $S(F)$ such that the underlying functor is non-increasing from $\aleph_0$ upwards.*

Theorem 2,5. *$\mathscr{S}$ is strongly $\mathscr{S}$-spanned from $\aleph_0$-upwards.*

Proof. Let α be an infinite cardinal, then by Proposition 2,2 there exists a strong embedding Π_α from $\mathscr{S}$ to $\mathscr{S}_\alpha$ (α is the set of ordinals smaller than α) with an underlying functor F_α non-increasing from α upwards. Let $(\mathscr{S}, \mathscr{S})$ be the category described in Proposition 2,3, then for every symmetric graph $\mathscr{G} = (\alpha, S)$ we can define $\Phi_{\mathscr{G}}$: $\mathscr{S} \to (\mathscr{S}, \mathscr{S})$, $\Phi_{\mathscr{G}}(X, R) = (Y, S_1, S_2)$ where $(Y, S_1) = \Pi_\alpha(X, R)$ and $S_2 = S$, $\Phi_{\mathscr{G}} f = \Pi_\alpha f$. Then $\Phi_{\mathscr{G}}$ evidently is a strong embedding of $\mathscr{S}$ to $(\mathscr{S}, \mathscr{S})$ with the underlying functor F_α. By Proposition 2,3 there exists a strong embedding Ψ from $(\mathscr{S}, \mathscr{S})$ to $\mathscr{S}$ with an underlying functor G non-increasing from $\aleph_0$ upwards. Hence $\Psi \circ \Phi_{\mathscr{G}}$ is a strong embedding from $\mathscr{S}$ to $\mathscr{S}$ with the underlying functor $G \circ F_\alpha$ non-increasing from α upwards and obviously $\Psi \circ \Phi_{\mathscr{G}_1}$ and $\Psi \circ \Phi_{\mathscr{G}_2}$ are independent where $\mathscr{G}_i = (\alpha, S_i)$, $i = 1, 2$ iff $S_1 - S_2 \neq \emptyset \neq S_2 - S_1$. Let $\varphi_1, \varphi_2: \alpha \to \alpha$ be two injections with $\varphi_1(\alpha) \cap \varphi_2(\alpha) = \emptyset$. For $x = \{0, 1\} \in P_2 2$ and $T \subset P_2(x)\,\alpha$ set $U_T = \varphi_1(T) \cup \cup\, \varphi_2(P_2(x)\,\alpha - T)$. For different $S, T \subset P_2(x)\,\alpha$ we have $U_S - U_T \neq \emptyset \neq U_T - U_S$ and thus the proof is complete.

Note. Since Ψ is a full embedding, G must be faithful and therefore we can consider I as a subfunctor of G, see [19].

To prove the basic theorem of this section we first prove an auxiliary lemma:

Lemma 2,6. *Let $x \in FX$ where F is a set functor. Let α be a cardinal with $\alpha \geqq \|\mathscr{F}_F^X(x)\|$. Let $S(F^x)$ be (strongly) $(\mathscr{K}, \alpha)$-spanned for a concrete category $(\mathscr{K}, U)$ such that the independent full embeddings $\{\Omega_a: \mathscr{K} \to S(F^x);\ a \in S(F^x)_\alpha\}$ fulfil:*

(*) *for every object A of $\mathscr{K}$ and $a \in S(F^x)_\alpha$, α is a subset of the underlying set of $\Omega_a A$, and for every morphism $f: A \to B$ in $\mathscr{K}$, $\Omega_a f/\alpha = 1_\alpha$.*

Then $S(F)$ is (strongly) $(\mathscr{K}, \alpha)$-spanned.

Proof. Clearly $\operatorname{card} S(F)_\alpha = \operatorname{card} \{(\alpha, R);\ R \subset F(\alpha)\}$. For every $(\alpha, R) \in S(F)$ we give a (strong) full embedding Σ_R such that:

a) the power of the underlying set of $\Sigma_R A$ is equal to the power of the underlying set of $\Omega_{(\alpha, R \cap F^x \alpha)} A$ for every object A of $\mathscr{K}$,

b) if there exists a morphism $f: \Sigma_{R_1} A \to \Sigma_{R_2} B$ of $S(F)$ then $R_1 \subset R_2$.

Define $\Sigma_R A = (Z, V)$ where $\Omega_{(\alpha, R \cap F^x \alpha)} A = (Z, W)$ and $V = W \cup Fi(R - F^x \alpha)$ where $i: \alpha \to Z$ is the inclusion. Since $\Omega_a f/\alpha = 1_\alpha$ we get that Σ_R is a functor. If $f: \Sigma_R A_1 = (Z_1, V_1) \to \Sigma_R A_2 = (Z_2, V_2)$ is a morphism of $S(F)$ and $\Omega_{(\alpha, R \cap F^x \alpha)} A_i = (Z_i, W_i)$, $i = 1, 2$, then for $y \in W_1$ we have $Ff(y) \in V_2$ and $Ff(y) \in F^x Z_2$, thus $Ff(y) \in W_2$. Hence $f: (Z_1, W_1) \to (Z_2, W_2)$ is a morphism of $S(F)$ and since $\Omega_{(\alpha, R \cap F^x \alpha)}$ is full we get that Σ_R is a full embedding fulfilling a) and b). Moreover, if $\Omega_{(\alpha, R \cap F^x \alpha)}$ is strong then Σ_R is strong. If we use the same technique as in Proposition 1,7 we get that there exists a set $\mathfrak{A}$ of subsets of F_α such that $\operatorname{card} \mathfrak{A} = \operatorname{card} 2^{F^\alpha}$ and $R_1, R_2 \in \mathfrak{A}$ implies $R_1 - R_2 \neq \emptyset \neq R_2 - R_1$, which completes the proof.

Theorem 2,7. *If F is a set functor with a finite unattainable cardinal, then $S(F)$ is strongly $\mathscr{D}$-spanned from $\aleph_0$ upwards.*

Proof. By Proposition 1,3 there exists $x \in FX$ with $\mathrm{d}(\mathscr{F}_F^X(x)) = \exp X$, $\aleph_0 > \|\bigcap(\mathscr{F}_F^X(x))\| > 1$ and hence by Lemma 1,5, Proposition 2,4 and Theorem 2,5 we get that the assumptions of Proposition 1,8 are fulfilled for $\mathscr{S}, \mathscr{S}$ and $S(F^x)$. Thus $S(F^x)$ is strongly $\mathscr{S}$-spanned from $\aleph_0$ upwards. Proposition 1,9 and Lemma 2,6 conclude the proof.

III

Let $\Psi: \mathscr{K} \to S(F)$ be a full embedding, let (Z, V) be an object of $S(F)$ such that

a) Z is a subset of the underlying set of ΨA for every object $A \in \mathscr{K}$;
b) $\Psi f / Z = 1_Z$ for every morphism $f: A \to B$ of $\mathscr{K}$;
c) for every pair of objects A, B of $\mathscr{K}$, if $\Psi A = (X, R)$, $\Psi B = (Y, S)$ then every morphism $f: (X, R) \to (Y, S \cup Fi(V))$ of $S(F)$ where $i: Z \to Y$ is the inclusion, satisfies $F f(R) \subset S$ and $S \cap Fi(V) = \emptyset$.

Then (Z, V) is called a *graft* of Ψ.

Now we give the basic lemma of this paper:

Lemma 3,1. *Let $(\mathscr{K}, U)$ be a concrete category. Let β be such a cardinal that for every cardinal $\gamma \geqq \beta$ there exist a set T with* card $T = \gamma$, *a (strong) full embedding $\Psi_T: \mathscr{K} \to S(F)$ such that the power of the underlying set $\Psi_T A$ is smaller than* max $\{\gamma$, card $UA\}$ *for any object A of $\mathscr{K}$, and a class of objects $\mathfrak{A}_T$ with the underlying set T such that*

a) *if $(T, V) \in \mathfrak{A}_T$ then (T, V) is a graft of Ψ_T;*
b) *if $(T, V_1), (T, V_2) \in \mathfrak{A}_T$ then $V_1 - V_2 \neq \emptyset \neq V_2 - V_1$;*
c) card $\mathfrak{A}_T =$ card $S(F)_\gamma$.

Then $S(F)$ is (strongly) $\mathscr{K}$-spanned from β upwards.

Proof. If $\mathscr{T} = (T, V)$ is a graft of Ψ_T then define $\Sigma_{\mathscr{T}}: \mathscr{K} \to S(F)$ as follows: for an object A of $\mathscr{K}$, $\Sigma_{\mathscr{T}} A = (X, R \cup Fi(V))$ where $\Psi_T A = (X, R)$ and $i: T \to X$ is the inclusion, for a morphism $f: A \to B$ define $\Sigma_{\mathscr{T}} f = \Psi_T f$. Since (T, V) is a graft of Ψ_T, we have $T \subset X$ for every object A of $\mathscr{K}$, $\Psi_T A = (X, R)$. Thus the definition $\Sigma_{\mathscr{T}} A$ is correct and since for every morphism $f: A \to B$, $\Psi_T f / T = 1_T$ we get that $\Sigma_{\mathscr{T}}$ is a functor. If $\mathscr{T}_1 = (T, V_1)$, $\mathscr{T}_2 = (T, V_2)$ are grafts of Ψ_T and A, B are objects of $\mathscr{K}$ then the existence of a morphism $f: \Sigma_{\mathscr{T}_1} A = (X, R \cup Fi(V_1)) \to \Sigma_{\mathscr{T}_2} B = (Y, S \cup Fj(V_2))$ where $i: T \to X$, $j: T \to Y$ are the inclusions implies that $V_1 \subset V_2$ and $f: \Psi_T A \to \Psi_T B$. Indeed, since (T, V_2) is a graft of Ψ_T we get $Ff(R) \subset S$, i.e. $f: \Psi_T A \to \Psi_T B$ is a morphism of $S(F)$, which means that $f / T = 1_T$ and thus $V_1 \subset V_2$. If $V_1 = V_2$ then since Ψ_T is full we obtain that $\Sigma_{\mathscr{T}_1}$ is full, moreover, if Ψ_T is strong then $\Sigma_{\mathscr{T}_1}$ is strong with the same underlying functor. Now the existence of a class $\mathfrak{A}_T$ concludes the proof.

Note 3,2. Let F be a set functor such that, for a point $x \in FX$, $\bigcap(\mathscr{F}_F^X(x)) \neq \emptyset$, $\|\mathscr{F}_F^X(x)\| \geqq \aleph_0$ and there exists a strong embedding Ψ from $\mathscr{S}$ (or $\mathscr{D}$) to $S(F)$ with the underlying functor non-increasing from α upwards. If we prove that for each infinite set T there exists a finite set $A \subset T$ such that each object (T, V) is a graft of $\Psi \circ \Pi_T$ whenever $V \subset F(x)\, T$ and $A \subset \bigcap(\mathscr{F}_F^T(y)) \neq \emptyset$ for any $y \in V$ then combining Lemma 3,1 and Proposition 1,7 we get that $S(F)$ is strongly $\mathscr{S}$ (or $\mathscr{D}$)-spanned from α upwards.

Construction 3,3. Let such a set functor F be given that for a point $x \in FX$, $\|\mathscr{F}_F^X(x)\| \geqq \aleph_0 > n = \|\bigcap(\mathscr{F}_F^X(x))\| > 0$. If for a concrete category $(\mathscr{K}, U)$ there exist a set Z with card $Z = \|\mathscr{F}_F^X(x)\|$ and a full embedding $\Psi: \mathscr{K} \to S(P_{n+1})$ such that

a) for every object A of $\mathscr{K}$, Z is a subset of the underlying set of ΨA;
b) for every morphism $f: A \to B$ of $\mathscr{K}$, $\Psi f / Z = 1_Z$;
c) for every object A of $\mathscr{K}$ with $\Psi A = (Y, S)$ we have $S \subset P_{n+1}(n)(Y)$ (where $n = \{0, 1, \dots, n-1\}$, thus $P_{n+1}(n)(Y)$ is the set of all n-point subsets of Y), then we can define $\bar{\Psi}: \mathscr{K} \to S(F)$ as follows:

for an object A of $\mathscr{K}$ define $\bar{\Psi}A = (Y, R)$ where $\Psi A = (Y, S)$ and $R = \{y \in F(x)\ Y; \bigcap(\mathscr{F}_F^Y(y)) \in S,\ Z \cup (\bigcap(\mathscr{F}_F^Y(y))) \in \mathscr{F}_F^Y(y)\}$, for a morphism $f: A \to B$ define $\bar{\Psi}f = \Psi f$.

By a) $\bar{\Psi}A$ is correctly defined. We have to prove that $\bar{\Psi}f$ is a morphism of $S(F)$, then clearly $\bar{\Psi}$ is a functor. Let us denote $\bar{\Psi}A = (Y, R)$, $\Psi A = (Y, S)$, $\bar{\Psi}B = (Y_1, R_1)$, $\Psi B = (Y_1, S_1)$. Choose $y \in R$. Then $Z \cup (\bigcap(\mathscr{F}_F^Y(y))) \in \mathscr{F}_F^Y(y)$ and since by b) $\bar{\Psi}f / Z = 1_Z$ we get that $\bigcap \bar{\Psi}f(\mathscr{F}_F^Y(y)) = \bar{\Psi}f(\bigcap(\mathscr{F}_F^Y(y)))$. Since $\bigcap(\mathscr{F}_F^Y(y)) \in S$ and $\bar{\Psi}f = \Psi f$ we have $\bar{\Psi}f(\bigcap(\mathscr{F}_F^Y(y))) \in S_1$, thus by c) card $\bar{\Psi}f(\bigcap(\mathscr{F}_F^Y(y))) = \|\bigcap\mathscr{F}_F^Y(y)\|$ and so $\bar{\Psi}f / \bigcap(\mathscr{F}_F^Y(y))$ is one-to-one. Therefore there exists $U \in \mathscr{F}_F^Y(y)$ such that $\bar{\Psi}f / U$ is one-to-one and by Proposition 1.2 $\bar{\Psi}f(\mathscr{F}_F^Y(y)) = \mathscr{F}_F^{Y_1}(F(\bar{\Psi} f(y)))$ and so $F(\bar{\Psi} f(y)) \in R_1$. Hence $\bar{\Psi}f$ is a morphism of $S(F)$. We prove that $\bar{\Psi}$ is full. If $\bar{\Psi}A_i = (Y_i, R_i)$, $\Psi A_i = (Y_i, S_i)$ for $i = 1, 2$ and $f: (Y_1, R_1) \to (Y_2, R_2)$ is a morphism of $S(F)$ then for every $y \in R_1$ and for every $Z' \in \mathscr{F}_F^{Y_2}(F f(y))$, card $Z' \geqq \|\mathscr{F}_F^X(x)\|$ and hence card $f(Z) = \|\mathscr{F}_F^X(x)\|$. We choose $Z_1 \subset Z$ such that card $Z_1 =$ card Z, and f / Z_1 is one-to-one. Then for every $y \in R_1$ there exists $y' \in R_1$ with $\bigcap(\mathscr{F}_F^{Y_1}(y)) = \bigcap(\mathscr{F}_F^{Y_1}(y'))$ and $Z_1 \cup (\bigcap(\mathscr{F}_F^{Y_1}(y))) \in \mathscr{F}_F^{Y_1}(y')$. By Proposition 1.2, $\mathscr{F}_F^{Y_2}(F f(y')) \supset f(\mathscr{F}_F^{Y_1}(y'))$ and since $f(\bigcap\mathscr{F}_F^{Y_1}(y')) = f(\bigcap(\mathscr{F}_F^{Y_1}(y')))$ we get that $\bigcap(\mathscr{F}_F^{Y_2}(F f(y'))) \subset f(\bigcap(\mathscr{F}_F^{Y_1}(y')))$. However, both sets have the same finite cardinality and hence coincide, which means that $f(\bigcap(\mathscr{F}_F^{Y_1}(y))) \in S_2$ for any $y \in R_1$ and thus $f: (Y_1, S_1) \to (Y_2, S_2)$ is a morphism of $S(P_{n+1})$. Since Ψ is full we get that $\bar{\Psi}$ is full. Moreover, if Ψ is strong then $\bar{\Psi}$ is strong with the same underlying functor.

Lemma 3,4. *Let F be a set functor such that $1 < \|\bigcap(\mathscr{F}_F^X(x))\| < \aleph_0 \leqq \|\mathscr{F}_F^X(x)\|$ for some $x \in FX$. Then for each infinite set T with $T \cap Z = \emptyset$ and each finite subset A of T with* card $A = n$ *we have. if (T, V) is an object of $S(F)$ such that $V \subset F(x)\ T$ and $y \in V$ implies $\bigcap(\mathscr{F}_F^T(y)) = A$, then (T, V) is a graft of $\bar{\Psi} \circ \Pi_T$ for each full embedding $\Psi: \mathscr{S} \to S(P_n)$ which extends the underlying sets (i.e. if $(Y, S) \in \mathscr{S}$, $\Psi(Y, S) = (W, Q)$ then $Y \subset W$ and if $f: (Y, S) \to (Y_1, S_1)$ then $\Psi f / Y = f$).*

Proof. Let $\Psi: \mathscr{S} \to S(P_n)$ be a full embedding which extends the underlying sets. For a set Z we construct $\bar{\Psi}$ by Construction 3,3. Let T be an infinite set with $T \cap Z = \emptyset$. Let (Y_j, S_j) be graphs, $\Psi \circ \Pi_T(Y_j, S_j) = (W_j, Q_j)$, $\bar{\Psi} \circ \Pi_T(Y_j, S_j) = (W_j, R_j)$, $j = 1, 2$. If $f: (W_1, R_1) \to (W_2, R_2 \cup Fi(V))$ is a morphism of $S(F)$ where $i: T \to W_2$ is the inclusion then we have to prove $F f(R_1) \subset R_2$. Assume that $F f(y) \notin R_2$ for some $y \in R_1$. Then $T \in \mathscr{F}_F^{Y_2}(F f(y))$ and so there exists $Z' \subset Z$ such that card $Z' =$

$= \operatorname{card} Z$, f/Z is one-to-one and $f(Z') \subset T$. For every $y \in R_1$ choose $y' \in R_1$ with

$$\bigcap(\mathscr{F}_F^{Y_1}(y)) = \bigcap(\mathscr{F}_F^{Y_1}(y')), \quad Z' \cup (\bigcap(\mathscr{F}_F^{Y_1}(y))) \in \mathscr{F}_F^{Y_1}(y').$$

Then card $(f(Z' \cup (\bigcap(\mathscr{F}_F^{Y_1}(y')))) - T)$ is finite and $f(Z' \cup (\bigcap \mathscr{F}_F^{Y_1}(y')))$ is an element of $\mathscr{F}_F^{Y_2}(F f(y'))$, thus $Ff(y') \in R_2 \cup Fi(V)$. Since f/Z' is one-to-one and $\bigcap(\mathscr{F}_F^{Y_1}(y'))$ is finite we get that $\bigcap f(\mathscr{F}_F^{Y_1}(y')) = f(\bigcap(\mathscr{F}_F^{Y_1}(y')))$ and by Proposition 1,2 we conclude that $\bigcap(\mathscr{F}_F^{Y_2}(F f(y'))) \subseteq \bigcap f(\mathscr{F}_F^{Y_1}(y')) = f(\bigcap(\mathscr{F}_F^{Y_1}(y')))$. Now if we use the fact that $\bigcap(\mathscr{F}_F^{Y_1}(y'))$ and $\bigcap(\mathscr{F}_F^{Y_2}(F f(y')))$ are finite and have the same cardinality we get the equality. By the property of V we obtain that $f(\bigcap(\mathscr{F}_F^{Y_1}(y))) = A$ for each $y \in R_1$. Thus there exists a morphism from $\Psi \circ \Pi_T(Y_1, S_1)$ to $(n, \{n\})$. Hence if $Q_1 \neq \emptyset$ then there exists a morphism $h: (W_1, Q_1) \to (W_1, Q_1)$ of $S(F)$ which factorizes by $(n, \{n\})$. Then card Im $h \leqq n$ and this contradicts $h/T = 1_T$. If $Q_1 = \emptyset$ then we get a contradiction by the same argument.

Theorem 3,5. *Let F be such a set functor that for some $x \in FX$, $1 < \|\bigcap(\mathscr{F}_F^X(x))\| < \aleph_0$. Then $S(F)$ is strongly $\mathscr{D}$-spanned from $\|\mathscr{F}_F^X(x)\|$ upwards.*

Proof. By Lemmas 2,6, 3,1 and 3,4, Note 3,2 and Theorems 2,7 and 1,9 it suffices to find a strong embedding $\Psi: \mathscr{S} \to S(P_n)$ with an underlying faithful functor non-increasing from $\aleph_0$ upwards. Such a functor was constructed in [13].

IV

The aim of this section is to prove that if there is a point $x \in FX$ for a set functor F with $\|\bigcap(\mathscr{F}_F^X(x))\| > 1$ then $S(F)$ is strongly $\mathscr{D}$-spanned from $\max\{\|\mathscr{F}_F^X(x)\|, \aleph_0\}$ upwards. We recall the construction of a strong embedding from $\mathscr{S}$ to $S(F)$ for $\bigcap(\mathscr{F}_F^X(x))$ infinite which was given in [13].

Construction 4,1. Let F be a set functor such that there is a point $x \in FX$ with $\|\bigcap(\mathscr{F}_F^X(x))\| \geqq \aleph_0$ and $d(\mathscr{F}_F^X(x)) = \exp X$. Put $\alpha = \|\bigcap(\mathscr{F}_F^X(x))\|$. We shall assume that (Y, R) is a symmetric connected graph without loops such that $Y \cap \alpha = \emptyset$ and neither Y nor α contain points a, b. Put $Z = Y \cup \alpha \cup \{a, b\}$. Let S be the set of all points $z \in F(x) Z$ such that $\bigcap(\mathscr{F}_F^Z(z))$ coincides with some of the sets described in a)–h):

a) $E = \{x + 2n$; x is a limit ordinal smaller than α, n is a natural number$\}$;

b) $0 = \{x + 2n + 1$; x is a limit ordinal smaller than α, n is a natural number$\}$;

c) $D = \{x + 3n$; x is a limit ordinal smaller than α, n is a natural number$\}$;

d) $P_x = \{y \in E; y > x\} \cup \{x\}$ for some $x \in 0$;

e) $P_x = \{y \in 0; y > x + 2\} \cup \{x\}$ for some $x \in E$;

f) $V \cup \{a, x, y\}$ for some $x, y \in 0$ $(x \neq y)$ and some $V \subset E$ with card $V = \operatorname{card}(E - V) = \alpha$;

g) $V \cup \{b, x, y\}$ for some $x, y \in E$ $(x \neq y)$ and some $V \subset 0$ with card $V =$ $=$ card $(0 - V) = \alpha$;

h) $D \cap E \cup \{x, y\}$ for some $(x, y) \in R$ (i.e. $x, y \in Y$).

Put $\Lambda(Y, R) = (Z, S) \in S(F)$ and for a compatible mapping $f: (Y_1, R_1) \to (Y_2, R_2)$ define $\Lambda f: \Lambda(Y_1, R_1) \to \Lambda(Y_2, R_2)$ such that $\Lambda f / Y_1 = f$ and $\Lambda f / \alpha = 1_\alpha$, $\Lambda f(a) = a$, $\Lambda f(b) = b$. Then Λ is a strong embedding from $\mathscr{S}$ to $S(F)$ with an underliyng functor $I \vee C_{\alpha \cap \{a,b\}}$ (see [13]), non-increasing from α upwards.

Lemma 4,2. *Let T be an infinite set, $t \in T$. Then for a set functor F with a point $x \in FX$ such that $\alpha = \|\bigcap(\mathscr{F}_F^X(x))\| \geqq \aleph_0$ and $\mathrm{d}(\mathscr{F}_F^X(x)) = \exp X$ an object (T, V) is a graft of $\Lambda \circ \Pi_T$ (for Λ see Construction 4,1) whenever $V \subset F(x)\, T$ and $y \in V$ implies $t \in \bigcap(\mathscr{F}_F^T(y))$.*

Proof. Let (Y_j, S_j) be graphs, $\Lambda \circ \Pi_T(Y_j, S_j) = (W_j, Q_j)$, $j = 1, 2$. Then for a morphism $f: (W_1, Q_1) \to (W_2, Q_2 \cup Fi(V))$ where $i: T \to W_2$ is the inclusion we have to prove that $Ff(Q_1) \subset Q_2$. Assume the contrary, i.e. $Ff(q) \in Fi(V)$ for some $q \in Q_1$. By Construction 4,1, card $(\bigcap(\mathscr{F}_F^{W_1}(q)) \cap \alpha) = \alpha$ and by Proposition 1,2, $\bigcap(\mathscr{F}_F^{W_2}(Ff(q))) \subset f((\bigcap(\mathscr{F}_F^{W_1}(q)))$. Since card $\bigcap(\mathscr{F}_F^{W_2}(Ff(q))) = \alpha$ and $\bigcap(\mathscr{F}_F^{W_2}(Ff(q))) \subset T$, we have card $(f(\alpha) \cap T) = \alpha$. Therefore there exists $A \subset \alpha$, card $A = \alpha$, such that $f(A) \subset T$ and f/A is one-to-one. Clearly we can assume that $A \subset O$ (we have card $(A \cap O) = \alpha$ or card $(A \cap E) = \alpha$, hence in the first case put $A = A \cap O$, in the second case put $A = A \cap E$ and substitute E instead of O). Since $E = \bigcap(\mathscr{F}_F^{W_1}(q_1))$ for some $q_1 \in Q_1$, necessarily card $f(E) = \alpha$. Now we choose $A' \subset A$, $i_1, i_2 \in E$ such that card $A' =$ card $(O - A') = \alpha$ and f is one-to-one onto $A' \cup \{a, i_1, i_2\}$. Since there exists $y \in Q_1$ with $\bigcap(\mathscr{F}_F^{W_1}(y)) = A' \cup$ $\cup \{a, i_1, i_2\}$ we get by Proposition 1,2 that $\bigcap(\mathscr{F}_F^{W_2}(Ff(y))) = f(\bigcap(\mathscr{F}_F^{W_1}(y)))$, hence card $(\bigcap(\mathscr{F}_F^{W_2}(Ff(y))) \cap T) = \alpha$ and so $Ff(y) \in Fi(V)$, thus $f(A' \cup \{a, i_i, i_2\}) \subset T$. Since i_1, i_2 were arbitrary points of E we get $f(E) \subset T$. Now, if we exchange E and O we get $f(O) \subset T$. Further, we prove that for every $A \subset O$, card $A = \alpha$, we have card $f(A) = \alpha$. Otherwise we can choose $A' \subset A$ with card $A' =$ card $(O - A') = \alpha$ and then there exists $y \in Q_1$ with $\bigcap(\mathscr{F}_F^{W_1}(y)) = A' \cup \{a, i_1, i_2\}$ for some $i_1, i_2 \in E$, $i_1 \neq i_2$. Then by Proposition 1,2, $\bigcap(\mathscr{F}_F^{W_2}(Ff(y))) \subset f(\bigcap(\mathscr{F}_F^{W_1}(y))) = f(A' \cup$ $\cup \{a, i_1, i_2\})$ but card $f(A' \cup \{a, i_1, i_2\}) < \alpha$, which contradicts $Ff(y) \in Q_2 \cup$ $\cup Fi(V) \subset F(x)\, W_2$. Analogously we can prove that for every $A \subset E$, card $A = \alpha$ we have card $f(A) = \alpha$. Thus there exists an ordinal $k < \alpha$ such that for all ordinals j, $k \leqq j < \alpha$, $f(j) \neq t$. Assume that $k \in 0$, then there exists $y \in Q_1$ with $\bigcap(\mathscr{F}_F^{W_1}(y)) =$ $= \{k\} \cup \{j \in E;\ j > k\}$. Now by Proposition 1,2 we get that $\bigcap(\mathscr{F}_F^{W_2}(Ff(y))) \subset$ $\subset f(\bigcap(\mathscr{F}_F^{W_1}(y))) = \{f(k)\} \cup \{f(j);\ j \in E,\ j > k\}$; this implies $t \notin \bigcap(\mathscr{F}_F^{W_2}(Ff(y)))$, therefore $Ff(y) \notin Fi(V)$. On the other hand, $f(\bigcap(\mathscr{F}_F^{W_1}(y))) \subset T$ implies $Ff(y) \notin$ $\notin Q_2$ – a contradiction. Hence $Ff(Q_1) \subset Q_2$.

Construction 4,3. Let F be such a set functor that for some $x \in FX$, $\mathrm{d}(\mathscr{F}_F^X(x)) \neq$ $\neq \exp X$ and $\alpha = \|\bigcap(\mathscr{F}_F^X(x))\| \geqq \aleph_0$. Then for $\Lambda: \mathscr{S} \to S(P_\alpha)$ defined in Construction 4,1 we can define $\Sigma: \mathscr{S} \to S(F)$ as follows: if $(Y, S) \in \mathscr{S}$ and $A \circ \Pi_Z(Y, S) = (W, Q)$ where Z is a set with card $Z = \|d(\mathscr{F}_F^X(x))\|$ then put $\Sigma(Y, S) = (W, R)$ where $R = \{y \in F(x)W;\ \bigcap(\mathscr{F}_F^W(y)) \in Q,\ Z \cup (\bigcap(\mathscr{F}_F^W(y))) \in \mathscr{F}_F^W(y)\}$; if $f: (Y_1, S_1) \to (Y_2, S_2)$ is a compatible mapping of $\mathscr{S}$ then define $\Sigma f = A \circ \Pi_Z f$. Then $\Sigma: \mathscr{S} \to S(F)$ is a strong embedding with an underlying functor non-increasing from $\|\mathscr{F}_F^X(x)\|$ upwards as was proved in [13].

Lemma 4,4. *Let T be an infinite set with $T \cap Z = \emptyset$. Choose three distinct points t_1, t_2, t_3 of T. Then for a set functor F with a point $x \in FX$ such that* card $Z = \beta = \|d(\mathscr{F}_F^X(x))\| \geqq \aleph_0$, $\alpha = \|\bigcap \mathscr{F}_F^X(x)\| \geqq \aleph_0$, *an object (T, V) is a graft of $\Sigma \circ \Pi_T$ (for Σ see Construction 4,3) whenever $V \subset F(x)\,T$ and $y \in V$ implies $t_1, t_2, t_3 \in \bigcap(\mathscr{F}_F^T(y))$. (Since $T \cap Z = \emptyset$ we can assume that for any graph (Y, S), $T \subset W$ where $\Sigma \circ \Pi_T(Y, S) = (W, Q)$ and for any morphism f of $\mathscr{S}$, $\Sigma \circ \Pi_T f/T = 1_T$.)*

Proof. Let (Y_j, S_j) be graphs, $\Sigma \circ \Pi_T(Y_j, S_j) = (W_j, Q_j)$ for $j = 1, 2$. Let $f: (W_1, Q_1) \to (W_2, Q_2 \cup Fi(V)$ be a morphism of $S(F)$ where $i: T \to W_2$ is the inclusion. We have to prove that $Ff(Q_1) \subset Q_2$. Assume that $Ff(q) \in Fi(V)$ for a point $q \in Q_1$. Then there exists $A \in \mathscr{F}_F^{W_2}(Ff(q))$ with $A \subset T$ and card $A = \|\mathscr{F}_F^X(x)\|$. Hence either card $(f(Z) \cap T) = \|\mathscr{F}_F^X(x)\|$ or card $(f(\alpha) \cap T) = \alpha$ where $\alpha = \|\bigcap(\mathscr{F}_F^X(x))\|$.

1) Assume card $(f(Z) \cap T) =$ card Z. Then there exists a set $Z' \subset Z$ such that card $Z' =$ card Z and f/Z' is one-to-one. By Construction 4,3, for every $y \in Q_1$ there is $y' \in Q_1$ with $\bigcap(\mathscr{F}_F^{W_1}(y)) = \bigcap(\mathscr{F}_F^{W_1}(y'))$, $Z' \cup (\bigcap(\mathscr{F}_F^{W_1}(y))) \in \mathscr{F}_F^{W_1}(y')$. Then Proposition 1,2 yields $\bigcap(\mathscr{F}_F^{W_2}(F f(y'))) \subseteq \bigcap f(\mathscr{F}_F^{W_1}(y')) = f(\bigcap(\mathscr{F}_F^{W_1}(y')))$ and hence card $f(\bigcap(\mathscr{F}_F^{W_1}(y))) = \alpha$ for each $y \in Q_1$. Choose $A \subset O$, $i_1, i_2 \in E$ such that card $A =$ card $(O - A) = \alpha$, f is one-to-one onto $A \cup \{a, i_1, i_2\}$. There exists $y \in Q_1$ with $\bigcap(\mathscr{F}_F^{W_1}(y)) = A \cup \{i_1, i_2, a\}$ and $Z' \cup A \cup \{a, i_1, i_2\} \in \mathscr{F}_F^{W_1}(y)$, hence by Proposition 1,2, $\mathscr{F}_F^{W_2}(Ff(y)) = f(\mathscr{F}_F^{W_1}(y))$ and since $f(Z') \subset T$ we get that necessarily $F f(y) \in F\,i(V)$ and so $f(A \cup \{i_1, i_2, a\}) \in T$. Since A and i_1, i_2 are arbitrary we have $f(\alpha) \subset T$. Since for each $A \subset \alpha$, card $A = \alpha$, there exist two points $i_1, i_2 \in \alpha$ such that $A \cup \{a, b, i_1, i_2\} \supset \bigcap(\mathscr{F}_F^{W_1}(y))$ for some $y \in Q_1$, we conclude that for each $A \subset \alpha$, card $A = \alpha$, we have card $f(A) = \alpha$. Hence there exists $k < \alpha$ such that for each j, $k \leqq j < \alpha$, we have $f(j) \neq t_1, t_2, t_3$. Then there exists $y \in Q_1$ with $\{2k + 1\} \cup \{j > 2k + 1;\ j \in E\} \cup Z' \in \mathscr{F}_F^{W_1}(y)$ and $\bigcap(\mathscr{F}_F^{W_1}(y)) = \{2k + 1\} \cup \{j > 2k + 1;\ j \in E\}$. But then $t_1, t_2\ t_3 \notin f(\bigcap(\mathscr{F}_F^{W_1}(y))) \supset \bigcap(\mathscr{F}_F^{W_2}(F f(y)))$ – a contradiction with $F f(y) \in F\,i(V)$.

2) Assume card $(f(\alpha) \cap T) = \alpha$. Then there exists $A \subset \alpha$, moreover, we can assume that $A \subset O$ since otherwise we can substitute E instead of O, such that card $A =$ card $(O - A) = \alpha$, f/A is one-to-one and $f(A) \subset T - \{t_1, t_2, t_3\}$. If card $f(Z) =$ card Z, then we choose $Z' \subset Z$ such that card $Z' =$ card Z and f/Z' is one-to-one. There is $y \in Q_1$ with $Z' \cup A \cup \{a, i_i, i_2\} \in \mathscr{F}_F^{W_1}(y)$, $\bigcap(\mathscr{F}_F^{W_1}(y)) = A \cup \{a, i_1, i_2\}$ for some $i_1, i_2 \in E$, $i_1 \neq i_2$. Then $\bigcap(\mathscr{F}_F^{W_2}(Ff(y))) \subset f(\bigcap(\mathscr{F}_F^{W_1}(y)))$ and so $Ff(y) \in Fi(V)$ but we can assume that $f(i_1), f(i_2) \notin \{t_1, t_2, t_3\}$ and hence

$\{t_1, t_2, t_3\} \not\subset \bigcap(\mathscr{F}_F^{W_2}(F f(y)))$ — a contradiction. Therefore $\operatorname{card} f(Z) < \operatorname{card} Z$. For arbitrary $i_1, i_2 \in E$ choose $y \in Q_1$ with $\bigcap(\mathscr{F}_F^{W_1}(y)) = A \cup \{a, i_1, i_2\}$ and $Z \cup \cup A \cup \{a, i_1, i_2\} \in \mathscr{F}_F^{W_1}(y)$. Since $\operatorname{card}(f(Z \cup A \cup \{a, i_1, i_2\}) - T) < \|\mathscr{F}_F^X(x)\|$ and $F f(y) \in F(x) W_2$ we get that $F f(y) \in F\, i(V)$. Since i_1, i_2 are arbitrary we can assume that $f(i_1), f(i_2) \notin \{t_1, t_2, t_3\}$ and so for arbitrary $B \in \mathrm{d}(\mathscr{F}_F^{W_1}(y))$ we get that $t_1, t_2, t_3 \in \in f(B) \cup \{f(a)\}$. Assume that $f(a) \neq t_1, t_2$. Then either $\operatorname{card}(Z - f^{-1}(t_1)) = = \operatorname{card} Z$ or $\operatorname{card}(Z - f^{-1}(t_2)) = \operatorname{card} Z$. Assume the first case occurs then there is $y_1 \in Q_1$ with $\bigcap(\mathscr{F}_F^{W_1}(y)) = \bigcap(\mathscr{F}_F^{W_1}(y_1))$ hence $F f(y_1) \in F\, i(V)$ and $(\bigcap(\mathscr{F}_F^{W_1}(y_1))) \cup \cup (Z - f^{-1}(t_1)) \in \mathscr{F}_F^{W_1}(y_1)$. Then $t_1 \notin f((\bigcap(\mathscr{F}_F^{W_1}(y_1))) \cup (Z - f^{-1}(t_1)))$ and Proposition 1,2 yields $t_1 \notin \bigcap(\mathscr{F}_F^{W_2}(F f(y_1)))$ — a contradiction since $F f(y_1) \notin F\, i(V)$.

Theorem 4,5. *Let F be a set functor with a point $x \in FX$ $\|\bigcap(\mathscr{F}_F^X(x))\| > 1$. Then $S(F)$ is strongly $\mathscr{D}$-spanned from $\|\mathscr{F}_F^X(x)\|$ upwards.*

Proof. By Lemmas 3,1, 4,2 and 4,4 and by Note 3,2 and Theorems 3,5 and 1,9 and Constructions 4,1 and 4,3, $S(F^x)$ is strongly $\mathscr{D}$-spanned from $\|\mathscr{F}_F^X(x)\|$ upwards. Now Lemma 2,6 concludes the proof.

V

In this section we deal with the categories $S(F)$ such that for some $x \in FX$, $\|\bigcap(\mathscr{F}_F^X(x))\| = 1$ and $\|\mathrm{d}(\mathscr{F}_F^X(x))\| \geqq \aleph_0$ (hence $\mathscr{F}_F^X(x)$ is not an ultrafilter). We prove that $S(F)$ is strongly ultimately $\mathscr{S}$-spanned. We first recall a construction in [13].

Construction 5,1. Let F be a set functor such that for some $x \in FX$ $\|\bigcap(\mathscr{F}_F^X(x))\| = = 1$ and $\alpha = \|\mathscr{F}_F^X(x)\| > 1$. Then there exist objects (Z, U_1), (Z, U_2) of $S(F)$ and four distinct points $a\ b\ c\ d \in Z$ such that

a) $U_1 \subset U_2 \subset F(x) Z$, $U_1 \neq U_2$;
b) if $y \in U_i$ $(i = 1, 2)$ and $\mathscr{F}_F^Z(z) = \mathscr{F}_F^Z(y)$ for some $z \in F(x) Z$ then $z \in U_i$;
c) put $\beta_0 = 0$, $\beta_{i+1} = 2^{\beta_i}$ and for a limit ordinal γ, $\beta_\gamma = \sup_{i<\gamma} \beta_i$; then $\operatorname{card} Z = \beta_\alpha$;
d) for $f: Z \to Z$ such that $\operatorname{card} f(\{z; f(z) \neq z\}) \geqq \alpha$ there exist $y \in U_1$ and $Z_1 \subset Z$ such that $\bigcap(\mathscr{F}_F^Z(y)) = \{b\}$, $Z_1 \cup \{b\} \in \mathscr{F}_F^Z(y)$, f/Z_1 is one-to-one, $f(Z_1) \cap Z_1 = = \emptyset$, $\operatorname{card} Z_1 = \alpha$ and for every $v \in U_2$, $\mathrm{d}(\mathscr{F}_F^Z(v)) \neq f(\mathrm{d}(\mathscr{F}_F^Z(y)))$;
e) for $f: Z \to Z$ with $\operatorname{card}\{z; f(z) \neq z\} \geqq \alpha$ and $\operatorname{card} f(\{z; f(z) \neq z\}) < \alpha$ there exists $y \in U_1$ such that $\bigcap(\mathscr{F}_F^Z(y)) = \{b\}$ and $\{z; f(z) \neq z\} \cup \{b\} \in \mathscr{F}_F^Z(y)$;
f) for every $Z_1 \subset Z$ with $\operatorname{card} Z_1 = \alpha$ there is $y \in U_1$ with $\bigcap(\mathscr{F}_F^Z(y)) = \{a\}$ and $Z_1 \cup \{a\} \in \mathscr{F}_F^Z(y)$;
g) for $f: Z \to Z$ such that $\operatorname{card}\{z; f(z) \neq z\} < \alpha$ and for some $z \in Z - \{a, b, c, d\}$, $f(z) \neq z$, there is $y \in U_1$ with $\bigcap(\mathscr{F}_F^Z(y)) = \{z\}$ and $\{z\} \cup \{v \in Z; f(v) = v\} \in \in \mathscr{F}_F^Z(y)$ such that for each $w \in U_2$, $\mathscr{F}_F^Z(w) \neq f(\mathscr{F}_F^Z(y))$;
h) there are $y_1, y_2 \in U_1$ with $\mathscr{F}_F^Z(y_1) = \{c\}$, $\mathscr{F}_F^Z(y_2) = \{d\}$, $\mathrm{d}(\mathscr{F}_F^Z(y_1)) \neq \neq \mathrm{d}(\mathscr{F}_F^Z(y_2))$, and for each $z \in U_2$ either $\mathscr{F}_F^Z(z) = \mathscr{F}_F^Z(y_1)$ or $\mathscr{F}_F^Z(z) = \mathscr{F}_F^Z(y_2)$ or $\mathrm{d}(\mathscr{F}_F^Z(y_1)) \neq \mathrm{d}(\mathscr{F}_F^Z(z)) \neq \mathrm{d}(\mathscr{F}_F^Z(y_2))$;

i) for a graph (Y, S) put $W = Y \cup (Y \times Y \times (Z - \{c, d\}))$ (we assume $Y \cap \cap (Y \times Y \times (Z - \{c, d\})) = \emptyset$). For $(y_1, y_2) \in Y$ define $\varphi_{(y_1,y_2)}: Z \to W$ as follows: $\varphi_{(y_1,y_2)}(z) = (y_1, y_2\ z)$ for $z \in Z - \{c, d\}$ $\varphi_{(y_1,y_2)}(c) = y_1$, $\varphi_{(y_1,y_2)}(d) = y_2$ and put $Q = \bigcup F\varphi_{(y_1,y_2)}(U_1) \cup \bigcup F\varphi_{(y_1,y_2)}(U_2)$ where the first union is taken over all $(y_1, y_2) \in Y$ and the second union is taken over all $(y_1, y_2) \in S$. Define $\Sigma(Y, S) = (W, Q)$ and for a compatible mapping $f: (Y_1, S_1) \to (Y_2, S_2)$ define $\Sigma f/Y = f$, $\Sigma f/(Y \times Y \times (Z - \{c, d\})) = f \times f \times 1$.

Then Σ is a strong embedding of $\mathscr{D}$ to $S(F)$ with the underlying functor non-increasing from β_α upwards.

Lemma 5,2. *Let (T, V) be an object of $S(F)$ and let $t \in T$ be such that $V \subset F(x)\, T$ and $y \in V$ implies $\bigcap(\mathscr{F}_F^T(y)) = \{t\}$. For every graph (Y, S) with $\Sigma \circ \Pi_T(Y, S) = (W, Q)$ and for every morphism $f: (Z, U_j) \to (W, Q \cup Fi(V))$ of $S(F)$ where $i: T \to W_2$ is the inclusion and $j = 1, 2$ we have either $Ff(U_j) \subset Q$ or* card $(Z - f^{-1}(T)) < \alpha$ *and for any $W' \subset W$ with* card $W' < \alpha$ *we have* card $f^{-1}(W') < \alpha$.

Proof. Put $(\bar{Y}, \bar{S}) = \Pi_T(Y, S)$. If card $f^{-1}(W') \geqq \alpha$ for some $W' \subset W$ with card $W' < \alpha$ then by f) there is $y \in U_j$ with $\{a\} \cup f^{-1}(W') \in \mathscr{F}_F^Z(y)$. Since card $(f(a) \cup W') < \alpha$ we get by Proposition 1,2 that $Ff(y) \notin F(x)W$, a contradiction. To prove the first statement define $\bar{f}: Z \to Z/\sim$ where $\sim$ is the equivalence with the only non-trivial class $\{c, d\}$. Define $\bar{f}(z) = z'$ if $f(z) = (y_1, y_2, z')$ for some $y_1, y_2 \in Y$, $\bar{f}(z) = \{c, d\}$ if $f(z) \in \bar{Y}$. If card $\{z; \bar{f}(z) \neq z\} \geqq \alpha$ then by d) and e) there exist $y \in U_j$ and $Z' \subset \{z; \bar{f}(z) \neq z\}$ such that $Z' \cup \{b\} \in \mathscr{F}_F^Z(y)$, card $Z' = \alpha$ and either f/Z' is one-to-one and $\bar{f}(\mathrm{d}(\mathscr{F}_F^Z(y))) \neq \mathrm{d}(\mathscr{F}_F^Z(w))$ for all $w \in U_2$ or card $\bar{f}(Z) < \alpha$. By the definition of Σ and by Proposition 1,2 we get that in both these cases $\varphi_{(y_1,y_2)}(\bar{f}(\mathrm{d}(\mathscr{F}_F^Z(y)))) \neq \varphi_{(y_1',y_2')}(\mathscr{F}_F^Z(w)) = \mathscr{F}_F^W(F\varphi_{(y_1',y_2')}(w))$ for all $y_1, y_2, y_1', y_2' \in \bar{Y}$, $w \in U_2$, and hence $Ff(y) \notin Q$. Thus card $\{z; \bar{f}(z) \neq z, \bar{f}(z) \neq \{c, d\}\} < \alpha$. On the other hand if card $\bar{f}^{-1}(\{c, d\}) \geqq \alpha$ then there is $A \subset \bar{f}^{-1}(\{c, d\})$ such that $f/A \cup \{a\}$ is one-to-one and card $A = \alpha$, and then by f) there is $y \in U_j$ with $A \cup \{a\} \in \mathscr{F}_F^Z(y)$ and $\{a\} = \bigcap(\mathscr{F}_F^Z(y))$. Since $f(A \cup \{a\}) \subset \bar{Y}$, Proposition 1,2 yields $f(a) = t$. Now for every $B \subset Z$ such that $\bar{b} \in B$ satisfies either $\bar{f}(\bar{b}) \neq \bar{b}$ or $f(\bar{b}) \notin \bar{Y} - T$ we have card $B < \alpha$. Otherwise either card $f(B) = \alpha$ or card $f(B) < \alpha$. In the second case, by f) there is $y \in U_1$ with $B \cup \{a\} \in \mathscr{F}_F^Z(y)$ but then $f(B) \cup \{t\} \in \mathscr{F}_F^W(F f(y))$, hence $\|\mathscr{F}_F^W(F f(y))\| < \alpha$ and thus $F f(y) \notin F(x)\, W$, a contradiction. In the first case we can assume that $f/B \cup \{a\}$ is one-to-one and by f) there is $y \in U_j$ with $B \cup \{a\} \in \mathscr{F}_F^Z(y)$ and $\{a\} = \bigcap(\mathscr{F}_F^Z(y))$. By Proposition 1,2, $\{t\} = \bigcap(\mathscr{F}_F^W(F f(y)))$ and $f(B) \cup \{t\} \in \mathscr{F}_F^W(F f(y))$. By d), $F f(y) \notin Q$ and since $f(B) \cap T = \emptyset$ we obtain $F f(y) \notin F\, i(V)$. Thus if card $\{z; \bar{f}(z) \neq z\} \geqq \alpha$ then card $\{z; f(z) \notin T\} < \alpha$. On the other hand, if $F f(y) \notin Q$ then $F f(y) \in F\, i(V)$ and so $T \in \mathscr{F}_F^W(F f(y))$. By Proposition 1,2, card $(T \cap \operatorname{Im} f) \geqq \alpha$ and so card $(z; \bar{f}(z) \neq z\} \geqq \alpha$. The proof is complete.

Lemma 5,3. *Let F be such a set functor that for some $x \in FX$, $\|\bigcap(\mathscr{F}_F^X(x))\| = 1$, $\alpha = \|\mathscr{F}_F^X(x)\| > 1$ and whenever, for a set $X' \subset X$, $X' \cap A \neq \emptyset$ for all $A \in \mathrm{d}(\mathscr{F}_F^X(x))$ then* card $X' \geqq \alpha$. *Then for any infinite set T and $t \in T$, an object (T, V) is a graft of $\Sigma \circ \Pi_T$ whenever $V \subset F(x)\,T$ and $y \in V$ implies $\{t\} = \bigcap(\mathscr{F}_F^T(y))$.*

Proof. Let (Y_j, S_j), $j = 1, 2$ be graphs, $\Pi_T(Y_j, S_j) = (\bar{Y}_j, \bar{S}_j)$, $\Sigma \circ \Pi_T(Y_j, S_j) = (W_j, Q_j)$. Let $f: (W_1, Q_1) \to (W_2, Q_2 \cup Fi(V))$ be a morphism of $S(F)$ where $i: T \to W_2$ is the inclusion. If $Ff(y) \notin Q_2$ for some $y \in Q_1$, then by Construction 5,1 i) there exist $\bar{y}_1, \bar{y}_2 \in \bar{Y}_1$ and $v \in U_2$ such that $F\varphi_{(\bar{y}_1,\bar{y}_2)}(v) = y$ and by Lemma 5,2 we get that card $\{z \in Z;\ f \circ \varphi_{(\bar{y}_1,\bar{y}_2)}(z) \notin T\} < \alpha$. Choose $z \in Z - \{a, b, c, d\}$ with $f \circ \varphi_{(\bar{y}_1,\bar{y}_2)}(z) \neq t$. Then by g) there exists $y \in U_1$ with $\{z\} = \bigcap(\mathscr{F}_F^Z(y))$. Since $Ff \circ \varphi_{(\bar{y}_1,\bar{y}_2)}(y) \notin Q$ we obtain $\bigcap(\mathscr{F}_F^W(Ff \circ \varphi_{(\bar{y}_1,\bar{y}_2)}(y))) = \{t\}$. Since $f \circ \varphi_{(\bar{y}_1,\bar{y}_2)}(z) \neq t$ we get $A \cap (f \circ \varphi_{(\bar{y}_1,\bar{y}_2)})^{-1}(t) \neq \emptyset$ for each $A \in \mathrm{d}(\mathscr{F}_Y^Z(y))$. By the assumptions of our lemma, card $(f \circ \varphi_{(\bar{y}_1,\bar{y}_2)})^{-1}(t) \geqq \alpha$ which contradicts Lemma 5,2.

Corollary 5,4. *Let F be a set functor fulfilling the assumptions of Lemma 5,3. Then $S(F)$ is strongly $\mathscr{D}$-spanned from β_α upwards.*

Proof follows from Lemmas 2,5, 3,1 and 5,3 and from Note 3,2.

Lemma 5,5. *Let F be such a set functor that for some $x \in FX$, $\|\bigcap(\mathscr{F}_F^X(x))\| = 1$ and $\mathrm{d}(\mathscr{F}_F^X(x))$ is not an ultrafilter. Let T be an infinite set with $t \in T$ and let the subset $T_1 \subset T - \{t\}$ fulfil* card T_1 = card $(T - T_1)$. *Then every object (T, V) of $S(F)$ is a graft of $\Sigma \circ \Pi_T$ whenever $V \subset F(x)\,T$, and for $y \in V$ we have: $\bigcap(\mathscr{F}_F^T(y)) = \{t\}$ and $T_1 \cap A \neq \emptyset \neq [T - T_1] \cap A$ for each $A \in \mathrm{d}(\mathscr{F}_F^Y(y))$.*

Proof. Let (Y_j, S_j) be graphs, $j = 1, 2$. Put $\Pi_T(Y_j, S_j) = (\bar{Y}_j, \bar{S}_j)$, $\Sigma \circ \Pi_T(Y_j, S_j) = (W_j, Q_j)$, $j = 1, 2$. Let $f: (W_1, Q_1) \to (W_2, Q_2 \cup Fi(V))$ be a morphism of $S(F)$ where $i: T \to W_2$ is the inclusion. If $Ff(y) \in Fi(V)$, for some $y \in Q_1$, then there exist $\bar{y}_1, \bar{y}_2 \in Y$, $v \in U_2$ with $F\varphi_{(\bar{y}_1,\bar{y}_2)}(v) = y$. If we apply Lemma 5,2 to $f \circ \varphi_{(\bar{y}_1,\bar{y}_2)}$ we get that card $\{z;\ f \circ \varphi_{(\bar{y}_1,\bar{y}_2)}(z) \notin T\} < \alpha$. Choose $A \in \mathscr{F}_F^{W_2}(Ff(y))$, $A \subset \mathrm{Im}\, f \circ \varphi_{(\bar{y}_1,\bar{y}_2)}$. Then $A \cap T_1 \neq \emptyset$, $A \cap (T - (T_1 \cup \{t\})) \neq \emptyset$ and either card $(A \cap T_1) \geqq \alpha$ or card $(A \cap (T - T_1)) \geqq \alpha$. Hence we can assume that card $(\mathrm{Im}\, f \circ \varphi_{(\bar{y}_1,\bar{y}_2)} \cap T_1) \geqq \alpha$ (otherwise card $(\mathrm{Im}\, f \circ \varphi_{(\bar{y}_1,\bar{y}_2)} \cap (T - (T_1 \cup \{t\}))) \geqq \alpha$ and we can take $T - (T_1 \cup \{t\})$ instead of T_1). Then there is $Z_1 \subset Z$ with card $Z_1 = \alpha$ such that $f \circ \varphi_{(\bar{y}_1,\bar{y}_2)}$ is one-to-one onto Z_1 and $f \circ \varphi_{(\bar{y}_1,\bar{y}_2)}(Z_1) \subset T_1 \cup \{t\}$. By f) of Construction 5,1 there is $q \in U_1$ with $Z_1 \cup \{a\} \in \mathscr{F}_F^Z(q)$, $\{a\} = \bigcap(\mathscr{F}_F^Z(q))$. Now $F(f \circ \varphi_{(\bar{y}_1,\bar{y}_2)})(z) \notin Q_2$ since $f \circ \varphi_{(\bar{y}_1,\bar{y}_2)}(Z_1 \cup \{a\}) \subset T_1$ and $F(f \circ \varphi_{(\bar{y}_1,\bar{y}_2)})(z) \notin Fi(V)$ since $[f \circ \varphi_{(\bar{y}_1,\bar{y}_2)}(Z_1 \cup \{a\})] \cap (T - [T_1 \cup \{t\}]) = \emptyset$ – a contradiction.

Lemma 5,6. *Let F be a set functor with a point $x \in FX$ such that $\mathrm{d}(\mathscr{F}_F^X(x))$ is not an ultrafilter and $\|\bigcap(\mathscr{F}_F^X(x))\| = 1$. Let T be a set with* card $T \geqq \alpha = \|\mathscr{F}_F^X(x)\|$, *choose $t \in T$, $T_1 \subset T$ with* card T_1 = card T = card $(T - T_1)$. *Then there is a set $\mathfrak{A}$ of objects of $S(F)$ on T such that*

a) *if* $(T, V) \in \mathfrak{A}$ *then* $V \subset F(x)\,T$, *and* $y \in V$ *implies* $\bigcap(\mathscr{F}_F^T(y)) = \{t\}$, *and for each* $A \in \mathrm{d}(\mathscr{F}_F^T(y))$ *we have* $A \cap T_1 \neq \emptyset \neq A \cap (T - T_1)$;

b) *if* $(T, V_1), (T, V_2) \in \mathfrak{A}$ *then* $V_1 - V_2 \neq \emptyset \neq V_2 - V_1$;

c) $\operatorname{card} \mathfrak{A} = \operatorname{card} S(F^x)_{\operatorname{card} T}$.

Proof. Choose $A \notin \mathrm{d}(\mathscr{F}_F^X(x))$ such that $A \cap B \neq \emptyset$ for each $B \in \mathrm{d}(\mathscr{F}_F^X(x))$. Denote $\mathscr{F} = \{C; \exists B \in \mathrm{d}(\mathscr{F}_F^X(x)), C \supset B \cap A\}$, $\mathscr{G} = \{C; \exists B \in \mathrm{d}(\mathscr{F}_F^X(x)), C \supset B \cap (X - A)\}$. Then $\mathscr{F} \neq \exp X \neq \mathscr{G}$ are filters. Put $\beta = \operatorname{card} \{\mathscr{H}; \exists f: X \to T, \exists B \in \mathscr{F}, f/B$ is one-to-one, $\mathscr{H} = f(\mathscr{F})\}$, $\gamma = \operatorname{card} \{\mathscr{H}; \exists f: X \to T, \exists B \in \mathscr{G}, f/B$ is one-to-one, $\mathscr{H} = f(\mathscr{G})\}$, $\delta = \operatorname{card} \{y \in F(x)\,X; \mathscr{F}_F^X(y) = \mathscr{F}_F^X(x)\}$. Then $\operatorname{card} \{y \in F(x)\,T; \bigcap(\mathscr{F}_F^T(y)) = \{t\}\} = \beta \times \gamma \times \delta$. Indeed, if $\beta' = \operatorname{card} \{\mathscr{H}; \exists f: X \to T, \exists B \in \mathrm{d}(\mathscr{F}_F^X(x)), f/B$ is one-to-one, $\mathscr{H} = f(\mathrm{d}(\mathscr{F}_F^X(x)))\}$ then clearly $\operatorname{card} \{y \in F(x)\,T; \bigcap(\mathscr{F}_F^T(y)) = \{t\}\} = \beta' \times \delta$. On the other hand, obviously $\beta' = \beta \times \gamma$. Hence $\beta \times \gamma \times \delta = \operatorname{card} F(x)T = \operatorname{card} \{y \in F(x)T; \bigcap(\mathscr{F}_F^X(y)) = \{t\}\} \geqq \operatorname{card} \{y \in F(x)T; \bigcap(\mathscr{F}_F^X(y)) = \{t\}, \forall A \in \mathrm{d}(\mathscr{F}_F^T(y)), A \cap T_1 \neq \emptyset \neq A \cap (T - T_1)\} \geqq \beta \times \gamma \times \delta$. If we now use the same technique as in Proposition 1,7 we get Lemma 5,6.

Theorem 5,7. *Let F be a set functor such that for some* $x \in FX$, $\alpha = \|\mathscr{F}_F^X(x)\| > \|\bigcap(\mathscr{F}_F^X(x))\| = 1$. *Then* $S(F)$ *is strongly* $\mathscr{D}$*-spanned from* β_α *upwards.*

Proof. Clearly $\mathrm{d}(\mathscr{F}_F^X(x)) \neq \exp X$. If $\mathrm{d}(\mathscr{F}_F^X(x))$ is an ultrafilter then for every $A \subset X$ with $\operatorname{card} A < \alpha$ we have $X - A \in \mathrm{d}(\mathscr{F}_F^X(x))$ (otherwise $\|\mathscr{F}_F^X(x)\| = \|\mathrm{d}(\mathscr{F}_F^X(x))\| < \alpha$). Thus $\mathscr{F}_F^X(x)$ fulfils the assumption of Lemma 5,3 and by Corollary 5,4 we get the required assertion. If $\mathrm{d}(\mathscr{F}_F^X(x))$ is not an ultrafilter then Lemmas 2,6, 3,1, 5,5 and 5,6 conclude the proof.

VI

In this section we summarize all the preceding results.

Lemma 6,1. *Let F be such a functor that for every* $x \in FX$ *either* $\mathscr{F}_F^X(x)$ *is an ultrafilter or* $\bigcap(\mathscr{F}_F^X(x)) = \emptyset$. *Then for any object* (T, V) *of* $S(F)$, *if* $f: (T, V) \to (T, V)$ *is a morphism in* $S(F)$ *and* $f(t_1) = t_2 \neq t_1$, *then* $g: (T, V) \to (T, V)$ *with* $g(t) = t$ *for all* $t \neq t_1$, $g(t_1) = t_2$, *is also a morphism.*

Proof. Let $y \in V$. Then either $T - \{t_1\} \in \mathscr{F}_F^T(y)$ or $\bigcap(\mathscr{F}_F^T(y)) = \{t_1\}$ and $\mathscr{F}_F^T(y)$ is an ultrafilter. In the first case, Proposition 1,2 yields $Fg(y) = y$, in the second case $\{t_1\} \in \mathscr{F}_F^T(y)$ and hence by Proposition 1,2, $Ff(y) = Fg(y) \in V$.

Construction 6,2. Let F be such a set functor that for every $x \in FX$ either $\bigcap(\mathscr{F}_F^X(x)) = \emptyset$ or $\mathscr{F}_F^X(x)$ is an ultrafilter. Let $1 = \{0\}$, $2 = \{0, 1\}$. For $x \in X$ denote by $p_x^X: 1 \to X$ a mapping with $p_x^X(0) = x$. Set $D = \{x \in F1; Fp_0^2(x) \neq Fp_1^2(x)\}$. Now for every object $\mathscr{A} = (X, V)$ of $S(F)$ define a quasi-order $\leqq_{\mathscr{A}}$ on X as follows: $x \leqq_{\mathscr{A}} y$ iff $(Fp_x^X)^{-1}(V) \cap D \subset (Fp_y^X)^{-1}(V) \cap D$.

Lemma 6,3. *Let F be such a set functor that for each $x \in FX$ either $\mathscr{F}_F^X(x)$ is an ultrafilter or $\bigcap(\mathscr{F}_F^X(x)) = \emptyset$. Then for every object (X, V) of $S(F)$ and $x \in X$, a mapping $g: X \to X$ such that all points except x are fixed points of g is an endomorphism of $\mathscr{A}$ iff $x \leqq_{\mathscr{A}} g(x)$.*

Proof. If g is an endomorphism then put $\mathscr{B} = (1, B) = (Fp_x^X)^{-1}(V) \cap D$. Clearly $V \cap Fp_x^X(D) = Fp_x^X(B)$ and so $p_x^X: \mathscr{B} \to \mathscr{A}$ is a morphism. Then $g \circ p_x^X$ is a morphism and so $V \cap Fp_y^X(D) = V \cap F(g \circ p_x^X)(D) \supseteq Fg(V) \cap F(g \circ p_x^X)(D) = = Fg(V \cap Fp_x^X(D)) = Fg(Fp_x^X(B)) = Fp_y^X(B)$, thus $y \ {}_{\mathscr{A}}\!\geqq x$. If $x \leqq_{\mathscr{A}} y$ then for $v \in V$ either $X - \{x\} \in \mathscr{F}_F^X(v)$ and then $Fg(v) = v$ by Proposition 1,2, or $\{x\} = = \bigcap(\mathscr{F}_F^X(v))$, then $\mathscr{F}_F^X(v)$ is an ultrafilter and so $v \in Fp_x^X(D)$ – thus $Fg(v) \in V \cap \cap Fp_y^X(D) \subset V$.

Lemma 6,4. *Let F be such a set functor that for every $x \in X$ either $\bigcap(\mathscr{F}_F^X(x)) = \emptyset$ or $\mathscr{F}_F^X(x)$ is an ultrafilter. Then for any object $\mathscr{A} = (X, V)$, $g: X \to X$ is an endomorphism of $\mathscr{A}$ whenever $x \leqq_{\mathscr{A}} g(x)$ for all $x \in X$, and $g(x) = x$ for all $x \in X$ but a finite set.*

Proof. If $v \in (X, V)$ then either $X - \{x; g(x) \neq x\} \in \mathscr{F}_F^X(v)$ and so $Fg(v) = v$ or there exists $x \in X$ with $g(x) \neq x$ such that $\{x\} = \bigcap(\mathscr{F}_F^X(v))$ and $\bigcap(\mathscr{F}_F^X(x))$ is an ultrafilter. Since $x \leqq_{\mathscr{A}} g(x)$ we get $Fg(v) \in V$ and the proof is complete.

Now we introduce some definitions which will be used in the characterization theorem.

Definition. A monoid $\mathscr{M}$ is said to be *given by a quasiordering*, if there exists a quasi-ordered set $(X, \leqq)$ and $\mathscr{M}$ is isomorphic with the monoid $(\{f: X \to X; \forall x \in X,\ x \leqq f(x)\}, \cdot)$.

Definition. An object $\mathscr{A}$ of a category $\mathscr{K}$ with the trivial monoid of endomorphisms is called *rigid*. A concrete category $(\mathscr{K}, U)$ is binding if every category of algebras has a full embedding into $(\mathscr{K}, U)$ (or equivalently, the category $\mathscr{D}$ has a full embedding into $(\mathscr{K}, U)$).

A set functor F *preserves the union of sets* A, B if $F(A \cup B) = \operatorname{Im} Fi \cup \operatorname{Im} Fj$ where $i: A \to A \cup B$, $j: B \to A \cup B$ are the inclusions. A set functor F *preserves unions with a finite set* if for every pair A, B of sets where A is finite, F preserves the union of A and B.

(f, r) is a *transposition pair* on a set X if $r, f: X \to X$ are such mappings that $r(x) = x$ iff $f(x) \neq x$ for all $x \in X$, $r^2 = 1_X$ and card $\{x \in X; r(x) \neq x\} = 2$.

For a cardinal α, α^+ denotes the cardinal successor of α.

Now we have:

Characterization theorem 6,5. *Let F be a covariant set functor.*

Then the following are equivalent:

1) *$S(F)$ is binding;*
2) *there is a strong embedding from $\mathscr{D}$ to $S(F)$;*

3) *$S(F)$ contains more than* card $2^{F(2^{F1})}$ *non-isomorphic rigid objects;*
4) *$S(F)$ contains a rigid object on a set with power* $>$ card 2^{F1};
5) *there is an object of $S(F)$ such that the monoid of endomorphisms is isomorphic with a non-trivial group;*
6) *there is an object of $S(F)$ such that the monoid of endomorphisms is isomorphic with a finite monoid which is not given by a quasiordering;*
7) *$S(F)$ is strongly ultimately $\mathcal{D}$-spanned;*
8) *$S(F)$ is strongly ultimately V-spanned for a variety V of finitary algebras;*
9) *$S(F)$ is strongly ultimately R-spanned for a category R of an n-ary relation (n is finite);*
10) *$S(F)$ is ultimately $\mathcal{D}$-spanned;*
11) *$S(F)$ is ultimately $\mathcal{M}$-spanned for a finite monoid $\mathcal{M}$;*
12) *$S(F)$ is ultimately discretely spanned;*
13) *$S(F)$ has an $(\mathcal{M}, \alpha)$-span for a finite monoid which is not given by a quasi-ordering and for an infinite cardinal α;*
14) *$S(F)$ has an $(\mathcal{M}, \alpha)$-span for a finite monoid and $\alpha >$* card 2^{F1};
15) *$S(F)$ has an $(\mathcal{M}, \alpha)$-span for a non-trivial group and for an infinite cardinal α;*
16) *F does not preserve unions with a finite set;*
17) *F does not preserve unions of a set with a onepoint set;*
18) *for some $x \in FX$, $\bigcap(\mathcal{F}_F^X(x)) \neq \emptyset$ and $\mathcal{F}_F^X(x)$ is not an ultrafilter;*
19) *there exists a transposition pair (f, r) on a set X such that for some $x \in FX$ both $Fr(x) \neq x$ and $Ff(x) \neq x$;*
20) *there exists a cardinal α such that for each transposition pair (f, r) on a set X with power at least α, there is $x \in FX$ with both $Fr(x) \neq x$ and $Ff(x) \neq x$.*

Proof. The equivalence of 1)$-$4), 16)$-$20) is proved in [13]. From Theorems 3,5, 4,5 and 5,7 we get that 18) implies 7). 7) $\Rightarrow$ 8) follows from Proposition 1,9, 7) $\Rightarrow$ 9) follows from [8] and Proposition 1,9. 7) $\Rightarrow$ 10) is trivial. On the other hand, 8) $\Rightarrow$ 1), 9) $\Rightarrow$ 1) and 10) $\Rightarrow$ 1) are evident. By [8] we get 7) $\Rightarrow$ 11) & 12) & 13) & 14) & 15). Clearly 12) $\Rightarrow$ 3), 13) $\Rightarrow$ 6), 14) $\Rightarrow$ 3) and 15) $\Rightarrow$ 5). Further, 1) $\Rightarrow$ 5) and 1) $\Rightarrow$ 6) and by Lemmas 6,1, 6,3 and 6,4, 5) $\Rightarrow$ 18) and 6) $\Rightarrow$ 18). Analogously, 11) implies that for every cardinal α there exist a cardinal $\beta > \alpha$ and an object (β, V) of $S(F)$ such that monoid of endomorphisms is isomorphic with $\mathcal{M}$, but then Lemmas 6,1 and 6,3 imply 18). The proof is complete.

As a consequence we get

Main Theorem. *For a covariant set functor F the following are equivalent:*

$S(F)$ is binding;
$S(F)$ is strongly ultimately $\mathcal{D}$-spanned;
$S(F)$ is ultimately discretely spanned;
F does not preserve the union with finite sets.

One may ask whether 12) could be strengthened so as to read "$S(F)$ has a discrete span". The following example shows that it is impossible.

Exmaple 6,6. Let $F = I \times C_N$ where N is the set of all natural numbers. Then card $S(F)_{\aleph_0} = 2^{\aleph_0}$. On the other hand, there is $\mathscr{Z} \subset \exp N$ such that card $\mathscr{Z} = 2^{\aleph_0}$ and for $Z_1, Z_2 \in \mathscr{Z}$, $Z_1 - Z_2 \neq \emptyset \neq Z_2 - Z_1$. Let $\{\mathscr{Z}_i;\ i \in 2^{\aleph_0}\}$ be a decomposition of Z such that every Z_i is infinite. Choose a countable set X and for every i let $\varphi_i: X \to \mathscr{Z}_i$ be a bijection. For $i \in 2^{\aleph_0}$ let (X, S_i) be an object of $S(F)$ where $S_i = \{(x, n);\ x \in X,\ n \in \varphi_i(x)\}$. Then by Lemmas 6,1 and 6,3, (X, S_i) are rigid for every $i \in 2^{\aleph_0}$ and it is easy to verify that there is no morphism from (X, S_i) to (X, S_j) if $i \neq j$. Thus $S(F)$ has a discrete $\aleph_0$-span. In the following papers we will show that the estimate in Condition 14) can not be strengthened, either.

Note 6,7. Concrete binding categories are most often ultimately $\mathscr{D}$-spanned. One possibility to continue the hierarchy is to ask, whether a binding ultimately $\mathscr{D}$-spanned category $\mathscr{K}$ is ultimately $\mathscr{L}$-spanned by some other categories $\mathscr{L}$ of interest, e.g. by the category $S(Q_{\omega_0})$ of infinitary relations or by $S(P^+)$ for P^+ – the covariant power-set functor. This direction of investigation is still open.

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Author's address: 118 00 Praha 1, Malostranské n. 25, ČSSR (Matematicko-fyzikální fakulta UK).