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ON NATURAL OPERATIONS WITH LINEAR CONNECTIONS

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Many authors such as Dodson, Radivoiovici [2], Kolář [7, 9], Oproiu [16], Puşcaş [17], Rybnikov [20] and others have dealt with prolongations of connections. By these we understand the rule transforming a given connection on a manifold into a connection on some prolongation of this manifold. For instance, a connection on $J^rY \to X$ or $VY \to X$ is associated with a connection on a fibre manifold $Y \to X$.

In the present paper we shall deal with the "prolongation" of a linear connection on an arbitrary manifold in the following sense. Any linear connection on a manifold M may be identified with a principal connection on the first order frame bundle $H^1M(M, L^1_m) = \operatorname{inv} J^1_0(\boldsymbol{R}^m, M)$, $L^1_m = Gl(m, \boldsymbol{R})$; throughout the paper $m = \dim M$. We shall solve the problem how to construct a principal connection on the semi-holonomic (or holonomic) second order frame bundle $\bar{H}^2M$ (or H^2M) which depends only on finite order derivatives of a given linear connection. All our constructions will be natural in the sense of the theory of categories.

First we shall solve our problem using analytical methods. Then we shall show that there is a geometrical way of solution, which uses prolongations of a linear connection given by the geometrical constructions of Oproiu [16] and Kolář [7].

As a consequence we obtain natural prolongations of any linear connection. This means that there is a constructed connection on $\bar{H}^2M$ over a given connection on H^1M.

All connections on principal bundles will be principal. Our considerations are in the category C^∞.

1. Connections on principal fibre bundles. Let $P(M, G)$ be a principal fibre bundle. The r-th order (holonomic) prolongation of P is the principal fibre bundle $W^rP(M, G^r_m) = J^r_{0,e}(\boldsymbol{R}^m \times G, P)$ with the structure group $G^r_m = W^r_0(\boldsymbol{R}^m \times G)$ where e is the unit in G. It is known, [8], [10], that $W^rP = H^rM \oplus J^rP$ and $G^r_m = L^r_m \mathbin{\bar{\times}} T^r_mG$ where $\oplus$ is the fibre product and $\bar{\times}$ is the semidirect product of groups. Then W^r is a covariant functor from $\mathscr{PB}_m(G)$ (the category of all principal fibre bundles with m-dimensional bases and a structure group G and morphisms of such bundles over diffeomorphisms) into $\mathscr{PB}_m(G^r_m)$, [10]. W^r transforms any morphism $f \in \operatorname{Hom} \mathscr{PB}_m(G)$ over f_0 into $W^rf := (H^rf_0, J^rf)$ where H^rf_0 and J^rf are defined in the usual way, [5].

According to Ehresmann [1] an r-th order connection on any principal fibre bundle $P(M, G)$ can be interpreted as a section of the fibre manifold $Q^r P \to M$ of elements of connections. In the sense of the theory of categories, Q^r is a covariant functor from the category $\mathcal{PB}_m$ into the category $\mathcal{FM}_m$ (of all fibre manifolds with m-dimensional bases and fibre manifold morphisms over diffeomorphisms). Q^r transforms any principal fibre bundle $P(M, G)$ into $Q^r P \to M$ where $Q^r P$ is a fibre manifold associated with $W^r P(M, G_m^r)$. The standard fibre of $Q^r P$ is $T_{m,e}^r G = J_0^r(\boldsymbol{R}^m, G)_e$ and the action of the group G_m^r on $T_{m,e}^r G$ is given by, [19],

$$(1) \qquad (A, S)\, Y = (S Y j_0^r[(\beta S)^{-1}]) \circ A^{-1}\,,$$

where $(A, S) \in L_m^r \rtimes T_m^r G$, $Y \in T_{m,e}^r G$, $S Y j_0^r[(\beta S)^{-1}]$ is the product in $T_m^r G$ induced by the product in G and "$\circ$" is the composition of jets. β is the target projection $J_0^r(\boldsymbol{R}^m, G) \to G$ and $(\beta S)^{-1}$ means the inverse element of βS in G. $[(\beta S)^{-1}]$ means the constant map of $\boldsymbol{R}^m$ on $(\beta S)^{-1}$. Hence

$$Q^r P = W^r P \times T_{m,e}^r G / G_m^r\,.$$

Any morphism f of $P(M, G)$ into $\bar{P}(\bar{M}, \bar{G})$ given by a triplet (f, φ, f_0), $f\colon P \to \bar{P}$ over $f_0\colon M \to \bar{M}$, $\varphi\colon G \to \bar{G}$ is a homomorphism of groups such that $f(ug) = f(u)\,\varphi(g)$ for all $u \in P$ and $g \in G$, is transformed into the morphism $Q^r f\colon Q^r P \to Q^r \bar{P}$ by defining $Q^r f := (W^r f, T_m^r \varphi)$ where $T_m^r \varphi\colon T_m^r G \to T_m^r \bar{G}$ is given by $T_m^r \varphi(j_0^r \alpha) = j_0^r(\varphi \circ \alpha)$, $\alpha\colon \boldsymbol{R}^m \to G$.

In our case we consider only the first order connections and P is the first order frame bundle $H^1 M(M, L_m^1)$ (or the second order semiholonomic frame bundle $\bar{H}^2 M(M, \bar{L}_m^2)$). Then the fibre manifold of elements of connections $Q^1 H^1 M$ (or $Q^1 \bar{H}^2 M$) is associated with $W^1 H^1 M$ (or $W^1 \bar{H}^2 M$). $W^1 H^1 M$ (or $W^1 \bar{H}^2 M$) has the reduction $H^2 M$ (or $H^3 M$), [10], and hence $Q^1 H^1 M$ (or $Q^1 \bar{H}^2 M$) is a fibre manifold associated with $H^2 M$ (or $H^3 M$). $T_{m,e}^1 L_m^1$ (or $T_{m,e}^1 \bar{L}_m^2$) is the standard fibre of $Q^1 H^1 M$ (or $Q^1 \bar{H}^2 M$) and the action of the group L_m^2 (or L_m^3) on the standard fibre is given by the restriction of the action (1) to the subgroup $L_m^2 \subset (L_m^1)_m^1$ (or $L_m^3 \subset (\bar{L}_m^2)_m^1$). Then according to [5] $Q^1 H^1$ (or $Q^1 \bar{H}^2$) is a lifting functor of order two (or three) and a connection on $H^1 M$ (or $\bar{H}^2 M$) is a field of geometrical objects of order two (or three) in the sense of [15].

2. Natural operators. Let $F, G\colon \mathcal{M}_m \to \mathcal{FM}$ be two lifting functors of orders r and s. $\mathcal{M}_m$ is the category of all m-dimensional manifolds and their embeddings. If for any manifold $M \in \operatorname{Ob} \mathcal{M}_m$ a differential operator $A_M\colon FM \to GM$ is given then A is a differential operator of the functor F into the functor G.

If a differential operator A satisfies

$$(2) \qquad (A_M \sigma) \mid U = A_U(\sigma \mid U)$$

for all open subsets $U \subset M$ and all sections $\sigma\colon M \to FM$, then the operator A is called *inclusion-preserving*.

Let $f: M \to \bar{M}$ be any diffeomorphism. Then the map $Ff: FM \to F\bar{M}$ transforms any section $\sigma: M \to FM$ into the section $Ff \circ \sigma \circ f^{-1}: \bar{M} \to F\bar{M}$. An inclusion-preserving operator A is natural if for any section $\sigma: M \to FM$ and any diffeomorphism $f: M \to \bar{M}$,

$$A_{\bar{M}}(Ff \circ \sigma \circ f^{-1}) = Gf \circ A_M\sigma \circ f^{-1} \tag{3}$$

is fulfilled.

A natural operator $A: F \to G$ is of order k if A_M is of order k for all $M \in \mathrm{Ob}\, \mathcal{M}_m$. Hence for any $M \in \mathrm{Ob}\, \mathcal{M}_m$ we have the associated base-preserving morphism

$$\alpha_M: J^kFM \to GM\,, \quad \alpha_M(j^k\sigma) = A_M\sigma\,.$$

From (3) we obtain

$$(Gf \circ \alpha_M)(j^k_x\sigma) = (Gf \circ A_M\sigma \circ f^{-1})(\bar{x}) = (A_{\bar{M}}(Ff \circ \sigma \circ f^{-1}))(\bar{x}) = \\ = (\alpha_{\bar{M}} \circ J^kFf)(j^k_x\sigma)$$

where $\bar{x} = f(x)$ and this yields a commutative diagram

$$\begin{array}{ccc} J^kFM & \xrightarrow{\alpha_M} & GM \\ \downarrow{\scriptstyle J^kFf} & & \downarrow{\scriptstyle Gf} \\ J^kF\bar{M} & \xrightarrow{\alpha_{\bar{M}}} & G\bar{M} \end{array}$$

Hence $\alpha: J^kF \to G$ is a natural transformation of J^kF into G.

On the other hand, let $\alpha: J^kF \to G$ be a natural transformation. Then it is easy to prove that the rule A transforming any section $\sigma: M \to FM$ into $A_M\sigma = \alpha_M(j^k\sigma): M \to GM$ is a natural operator of order k.

So we have proved

Proposition 1. *There is a bijective correspondence between the set of k-th order natural operators of a lifting functor F into a lifting functor G and the set of natural transformations of J^kF into G.*

3. Natural operations with linear connection. Now we can solve our problem. A linear connection on M can be interpreted as a section $\Gamma: M \to Q^1H^1M$. We have to transform any section Γ into a section $p_M\Gamma: M \to Q^1\bar{H}^2M$ such that $p: Q^1H^1 \to Q^1\bar{H}^2$ is a finite order natural operator. This means that a connection $p_M\Gamma$ depends only on finite order derivatives of Γ.

Assume first that p is a first order operator, so that we have the associated map

$$\pi_M: J^1Q^1H^1M \to Q^1\bar{H}^2M\,.$$

The standard fibre $S_1 = T^1_{m,e}L^1_m$ of Q^1H^1M is an L^2_m-space. We denote the canonical coordinates on S_1 by (Γ^i_{jk}) (Christoffel's). Then from (1) the action of L^2_m on S_1 is

$$(a^i_j, a^i_{jk})(\Gamma^i_{jk}) = (a^i_{lp}\tilde{a}^p_k\tilde{a}^l_j + a^i_l\,\Gamma^l_{mp}\tilde{a}^p_k\tilde{a}^m_j)\,, \tag{4}$$

where $(a^i_j, a^i_{jk}) \in L^2_m$ and $\tilde{A} = (\tilde{a}^i_j, \tilde{a}^i_{jk})$ is the inverse element of A.

The r-th jet prolongation of Q^1H^1M is a fibre manifold $J^rQ^1H^1M$ associated with $H^{r+2}M$, [13]. Recall that according to [5] $J^rQ^1H^1$ is a lifting functor of order$(r+2)$. Denote by S_1^r the standard fibre of $J^rQ^1H^1M$ and consider the canonical coordinates $(\Gamma^i_{jk}, \Gamma^i_{jk,l}, \ldots, \Gamma^i_{jk,l_1,\ldots,l_r})$ (Christoffel's and their partial derivatives up to order r). Then the action of the group L_m^{r+2} on S_1^r is given by gradual formal differentiation up to order r of the action (4), [11]. For $r = 1$ we obtain the action of L_m^3 on S_1^1:

(5) $$(a^i_j, a^i_{jk}, a^i_{jkl})(\Gamma^i_{jk}, \Gamma^i_{jk,l}) = (a^i_{pm}\tilde{a}^m_k\tilde{a}^p_j + a^i_p\Gamma^p_{lm}\tilde{a}^m_k\tilde{a}^l_j\,,$$
$$a^i_{pmq}\tilde{a}^q_l\tilde{a}^m_k\tilde{a}^p_j + a^i_{pm}\tilde{a}^m_{kl}\tilde{a}^p_j + a^i_{pm}\tilde{a}^m_k\tilde{a}^p_{jl} + a^i_{pq}\tilde{a}^q_l\Gamma^p_{rm}\tilde{a}^m_k\tilde{a}^r_j +$$
$$+ a^i_p\Gamma^p_{rm,q}\tilde{a}^q_l\tilde{a}^m_k\tilde{a}^r_j + a^i_p\Gamma^p_{rm}\tilde{a}^m_{kl}\tilde{a}^r_j + a^i_p\Gamma^p_{rm}\tilde{a}^m_k\tilde{a}^r_{jl})\,.$$

Further, $Q^1\bar{H}^2M$ is a fibre manifold associated with H^3M with the standard fibre $R_2 = T^1_{m,e}\bar{L}^2_m$. Denote the canonical coordinates on R_2 by $(\Gamma^i_{jk}, \Gamma^i_{jkl})$ (there is no symmetry in subscripts). Then from (1) we have the coordinate expression of the action of L^3_m on R_2:

(6) $$(a^i_j, a^i_{jk}, a^i_{jkl})(\Gamma^i_{jk}, \Gamma^i_{jkl}) = (a^i_{lp}\tilde{a}^p_k\tilde{a}^l_j + a^i_l\Gamma^l_{mp}\tilde{a}^p_k\tilde{a}^m_j\,,$$
$$a^i_{qmp}\tilde{a}^p_l\tilde{a}^m_k\tilde{a}^q_j + a^i_{qm}\Gamma^q_{pr}\tilde{a}^r_l\tilde{a}^p_j\tilde{a}^m_k + a^i_{pm}\Gamma^m_{qr}\tilde{a}^r_l\tilde{a}^p_j\tilde{a}^q_k +$$
$$+ a^i_m\Gamma^m_{pqr}\tilde{a}^r_l\tilde{a}^q_k\tilde{a}^p_j + a^i_{mp}\tilde{a}^p_l\tilde{a}^m_{jk} + a^i_m\Gamma^m_{pr}\tilde{a}^r_l\tilde{a}^p_{jk})\,.$$

Now, according to [18] our problem is equivalent to finding all L^3_m-equivariant maps of S^1_1 into R_2. The coordinate expression for such an L^3_m-equivariant map is

(7) $$\Gamma^i_{jk} = F^i_{jk}(\Gamma^i_{jk}, \Gamma^i_{jk,l})\,, \quad \Gamma^i_{jkl} = F^i_{jkl}(\Gamma^i_{jk}, \Gamma^i_{jk,l})\,.$$

Let $(A^i_j, A^i_{jk}, A^i_{jkl})$ be the canonical coordinates on the Lie algebra of L^3_m. Then according to [10, 12] all L^3_m-equivariant maps $F\colon S^1_1 \to R_2$ are just all global solutions of the following systems of partial differential equations:

(8) $$\frac{\partial F^i_{jk}}{\partial \Gamma^m_{np}}(A^m_{np} + A^m_q\Gamma^q_{np} - \Gamma^m_{qp}A^q_n - \Gamma^m_{nq}A^q_p) +$$
$$+ \frac{\partial F^i_{jk}}{\partial \Gamma^m_{np,q}}(A^m_{npq} + A^m_{rq}\Gamma^r_{np} + A^m_r\Gamma^r_{np,q} - \Gamma^m_{rp,q}A^r_n - \Gamma^m_{nr,q}A^r_p -$$
$$- \Gamma^m_{np,r}A^r_q - \Gamma^m_{rp}A^r_{nq} - \Gamma^m_{nr}A^r_{pq}) = A^i_mF^m_{jk} - F^i_{mk}A^m_j - F^i_{jm}A^m_k + A^i_{jk}\,,$$

(9) $$\frac{\partial F^i_{jkl}}{\partial \Gamma^m_{np}}(A^m_{np} + A^m_q\Gamma^q_{np} - \Gamma^m_{qp}A^q_n - \Gamma^m_{nq}A^q_p) +$$
$$+ \frac{\partial F^i_{jkl}}{\partial \Gamma^m_{np,q}}(A^m_{npq} + A^m_{rq}\Gamma^r_{np} + A^m_r\Gamma^r_{np,q} - \Gamma^m_{rp,q}A^r_n - \Gamma^m_{nr,q}A^r_p -$$
$$- \Gamma^m_{np,r}A^r_q - \Gamma^m_{rp}A^r_{nq} - \Gamma^m_{nr}A^r_{pq}) = A^i_{jkl} + A^i_{mk}F^m_{jl} + A^i_{jm}F^m_{kl} +$$
$$+ A^i_mF^m_{jkl} - F^i_{mkl}A^m_j - F^i_{jml}A^m_k - F^i_{jkm}A^m_l - F^i_{ml}A^m_{jk}\,.$$

First we shall solve the system (8). Assume that $A^i_j = \delta^i_j$, $A^i_{jk} = A^i_{jkl} = \emptyset$, then the system (8) is reduced to

$$\frac{\partial F^i_{jk}}{\partial \Gamma^m_{np}} \Gamma^m_{np} + 2 \frac{\partial F^i_{jk}}{\partial \Gamma^m_{np,q}} \Gamma^m_{np,q} = F^i_{jk} . \tag{10}$$

(10) is the system of type

$$a \frac{\partial f}{\partial x^i} x^i + b \frac{\partial f}{\partial y^p} y^p = kf .$$

According to [4] all global solutions are homogeneous polynomials of degree r in the variables x^i and of degree s in the variables y^p such that $ar + bs = k$. In our case $a = 1$, $b = 2$, $k = 1$ and hence $r = 1$, $s = \emptyset$. Hence the solution of (10) is a linear function dependent only on Christoffel's Γ^i_{jk}. Then we have

$$F^i_{jk} = a^{iqr}_{jkp} \Gamma^p_{qr}$$

and if we assume $A^i_j = \emptyset$ we obtain from (8)

$$a^{ijk}_{jki} + a^{ikj}_{jki} = 1$$

and $a^{iqr}_{jkp} = \emptyset$ if $i \neq p$ or $\{j, k\} \neq \{q, r\}$. If we denote $a^{ijk}_{jki} = \bar{c}$ we can express the solution of (8) in the form

$$F^i_{jk} = \bar{c}\Gamma^i_{jk} + (1 - \bar{c}) \Gamma^i_{kj} . \tag{11}$$

Now, we shall solve the system (9). Assume again $A^i_j = \delta^i_j$, $A^i_{jk} = A^i_{jkl} = \emptyset$. Then the system (9) is reduced to

$$\frac{\partial F^i_{jkl}}{\partial \Gamma^m_{np}} \Gamma^m_{np} + 2 \frac{\partial F^i_{jkl}}{\partial \Gamma^m_{np,q}} \Gamma^m_{np,q} = 2F^i_{jkl} \tag{12}$$

and according to [4] the solutions of (12) are homogeneous polynomials of degree r in Γ^m_{np} and of degree s in $\Gamma^m_{np,q}$ such that $r + 2s = 2$. Then $r = 2$, $s = \emptyset$ or $r = \emptyset$, $s = 1$. Hence the solutions are sums of linear functions in $\Gamma^i_{jk,l}$ and quadratic functions in Γ^i_{jk}. Express these solutions in the form

$$F^i_{jkl} = a^{iqrs}_{jklp} \Gamma^p_{qr,s} + b^{irsnp}_{jklqm} \Gamma^m_{np} \Gamma^q_{rs}$$

where $b^{irsnp}_{jklqm} = b^{inprs}_{jklmq}$. Using a simple algebraic evaluation we can easily find the relations between the coefficients. If we denote $a^{ijkl}_{jkli} = a$, $a^{ijlk}_{jkli} = b$, $a^{ikjl}_{jkli} = c$, $a^{iklj}_{jkli} = d$, $a^{iljk}_{jkli} = e$, $a^{ilkj}_{jkli} = f$ we obtain the solutions of (9) in the form

$$\begin{aligned} F^i_{jkl} = {} & a\Gamma^i_{jk,l} + b\Gamma^i_{jl,k} + c\Gamma^i_{kj,l} + d\Gamma^i_{kl,j} + e\Gamma^i_{lj,k} + f\Gamma^i_{lk,j} + \\ & + (\bar{c} - c - e - d + \alpha) \Gamma^i_{jm}\Gamma^m_{kl} + (1 - f - \bar{c} - \alpha) \Gamma^i_{jm}\Gamma^m_{lk} + \\ & + (c + e - \alpha) \Gamma^i_{mj}\Gamma^m_{kl} + \alpha\Gamma^i_{mj}\Gamma^m_{lk} + (c + d + e - 1 + \bar{c} + \beta) \Gamma^i_{km}\Gamma^m_{jl} + \\ & + (1 - e - \bar{c} - \beta) \Gamma^i_{km}\Gamma^m_{lj} + (a + f - \beta) \Gamma^i_{mk}\Gamma^m_{jl} + \beta\Gamma^i_{mk}\Gamma^m_{lj} + \end{aligned} \tag{13}$$

$$+ (\bar{c} - a - b - d + \gamma) \Gamma^i_{lm}\Gamma^m_{jk} + (-c - \gamma) \Gamma^i_{lm}\Gamma^m_{kj} +$$
$$+ (b + d - \bar{c} - \gamma) \Gamma^i_{ml}\Gamma^m_{jk} + \gamma\Gamma^i_{ml}\Gamma^m_{kj} ,$$

where $a, b, c, d, e, f, \bar{c}, \alpha, \beta, \gamma \in \boldsymbol{R}$ and $a + b + c + d + e + f = 1$.

We have proved

Theorem 1. *There is a 9-parameter family of first order natural operators of Q^1H^1 into $Q^1\bar{H}^2$ and any such natural operator transforms a connection Γ $(=\Gamma^i_{jk})$ on H^1M for any $M \in \mathrm{Ob}\,\mathcal{M}_m$ into connections $p_M\Gamma$ on $\bar{H}^2M$ with components given by (11) and (13) for some real coefficients $a, b, c, d, e, f, \bar{c}, \alpha, \beta, \gamma$, where $a + b + c + d + e + f = 1$.*

Remark. According to (11) the connections $p_M\Gamma$ are over connections $\bar{c}\Gamma + (1 - \bar{c})\,\tilde{\Gamma}$ where $\tilde{\Gamma}$ is the conjugate connection to Γ. If $\bar{c} = 1$ then $p_M\Gamma$ are over Γ and we obtain prolongation natural operators. Hence we have

Corollary. *There is an 8-parameter family of first order prolongation operators of Q^1H^1 into $Q^1\bar{H}^2$.*

4. Geometrical construction of natural operators. In this section we shall show that any first order natural operator of Q^1H^1 into $Q^1\bar{H}^2$ may be constructed in the geometrical way. There are two basic prolongation natural operators of order one of Q^1H^1 into $Q^1\bar{H}^2$. The first one is given by the geometrical construction of Oproiu [16]. This construction can be generalized in the following form. Let $P(M, G, \pi)$ be a principal bundle. Then $W^1P \subset T^1_mP$ is the set of such 1-jets of $\varphi: \boldsymbol{R}^m \to P$ that $\bar{\varphi} = \pi \circ \varphi$ is a local diffeomorphism of $\boldsymbol{R}^m$ on M, [14]. Consider a connection Γ on P given by $\Gamma(p) = j^1_x\,\gamma(t)$, where $\gamma: M \to P$ is a section such that $\gamma(x) = p$, and a connection Λ on H^1M given by $\Lambda(u) = j^1_x\,\lambda(t)$, where $\lambda: M \to H^1M$ is a section such that $\lambda(x) = u$, [9]. Let $x \in M$ be an arbitrary point and $\xi \in T_xM$ an arbitrary vector. Then $u^{-1}(\xi)$, $u \in H^1_xM$, is an element of $\boldsymbol{R}^m$. Then $\lambda(\bar{\varphi})\,(u^{-1}(\xi)): \boldsymbol{R}^m \to TM$ and $\Gamma(\lambda(\bar{\varphi})\,(u^{-1}(\xi))): \boldsymbol{R}^m \to TP$. The 1-jet of this map is an element of T^1_mTP. Using the diffeomorphism $T^1_mTP \approx TT^1_mP$ we obtain a lifting which defines a connection $\bar{p}_M(\Gamma, \Lambda)$ on W^1P.

Now, we consider $P = H^1M$. Then $W^1H^1M \approx \tilde{H}^2M$ and two connections Γ, Λ on H^1M determine a connection $\bar{p}_M(\Gamma, \Lambda)$ on $\tilde{H}^2M$. The connection $\bar{p}_M(\Gamma, \Lambda)$ determines the connection on $\bar{H}^2M \subset \tilde{H}^2M$ if and only if $\Lambda \equiv \tilde{\Gamma}$ (the conjugate connection to Γ). Then $\bar{p}$ is a first order natural prolongation operator and its associated map has the coordinate expression

$$(14) \qquad F^i_{jk} = \Gamma^i_{jk} \quad \text{(the prolongation condition)},$$
$$F^i_{jkl} = \Gamma^i_{jl,k} + \Gamma^i_{jm}\Gamma^m_{kl} .$$

It is easy to see that Oproiu's prolongation operator is obtained from (11) and (13) if $\bar{c} = 1$, $b = 1$ and the other coefficients vanish.

The second basic prolongation operator has been constructed by Kolář [7] who developed the original idea of Gollek [3]. Let Γ be a connection on P and Λ a connection on H^1M given as above. Then $(\lambda(t), \Gamma(\gamma(t)))$ is a section of $W^1P = H^1M \oplus J^1P$ and $j^1_x(\lambda(t), \Gamma(\gamma(t)))$ determines a connection on W^1P.

In our case $P = H^1M$ and $W^1P = \tilde{H}^2M$. Then two connections Γ and Λ on H^1M determine the connection $p_M(\Gamma, \Lambda)$ on $\tilde{H}^2M$. This connection determines the connection on $\bar{H}^2M$ if and only if $\Lambda \equiv \Gamma$ and then p is a first order prolongation natural operator with the associated map

$$F^i_{jk} = \Gamma^i_{jk}, \tag{15}$$

$$F^i_{jkl} = \Gamma^i_{jk,l} + \Gamma^i_{mk}\Gamma^m_{jl} + \Gamma^i_{jm}\Gamma^m_{kl} - \Gamma^i_{ml}\Gamma^m_{jk}.$$

It is easy to see that Kolář's prolongation operator is obtained from (11) and (13) if $\bar{c} = 1$, $a = 1$ and the other coefficients vanish.

According to (6) $Q^1\bar{H}^2M$ is an affine fibre manifold. Hence if $\Sigma_1, \ldots, \Sigma_p$ are connections on $\bar{H}^2M$, i.e. sections of $Q^1\bar{H}^2M$, then $k_1\Sigma_1 + \ldots + k_p\Sigma_p$, $k_i \in \boldsymbol{R}$, $k_1 + \ldots + k_p = 1$, is also a connection. Denote by $E(Q^1\bar{H}^2M)$ the vector bundle associated with $Q^1\bar{H}^2M$. Then $E(Q^1\bar{H}^2M)$ is associated with H^2M. For an arbitrary connection $\Gamma\colon M \to Q^1\bar{H}^2M$ and an arbitrary section $\tau\colon M \to E(Q^1\bar{H}^2M)$ the sum $\Gamma + k\tau$, $k \in \boldsymbol{R}$, is also a connection on $\bar{H}^2M$.

Consider the canonical involutive automorphism $i_M\colon \bar{H}^2M \to \bar{H}^2M$ which maps the point (x^i, x^i_j, x^i_{jk}) onto the point (x^i, x^i_j, x^i_{kj}). This automorphism transforms any connection Σ on $\bar{H}^2M$ given by $\Sigma(u) = j^1_x\,\sigma(t)$ into the connection $i_M\Sigma(u) = j^1_x i_M\,\sigma(t)$. If $(\Sigma^i_{jk}, \Sigma^i_{jkl})$ are components of Σ, then $i_M\Sigma$ has components $(\Sigma^i_{jk}, \Sigma^i_{kjl})$. Then i is a natural operator of order zero of $Q^1\bar{H}^2$ into $Q^1\bar{H}^2$.

Using the natural operator i and the conjugate connection $\tilde{\Gamma}(=\Gamma^i_{kj})$ we can construct for any connection $\Gamma(=\Gamma^i_{jk})$ the following connections on $\bar{H}^2M$ over Γ:

$$p_M(\Gamma, \Gamma),\ \bar{p}_M(\Gamma, \tilde{\Gamma}),\ (ip)_M(\Gamma, \Gamma),\ (i\bar{p})_M(\Gamma, \tilde{\Gamma}) \tag{16}$$

and the following connections on $\bar{H}^2M$ over $\tilde{\Gamma}$:

$$p_M(\tilde{\Gamma}, \tilde{\Gamma}),\ \bar{p}_M(\tilde{\Gamma}, \Gamma),\ (ip)_M(\tilde{\Gamma}, \tilde{\Gamma}),\ (i\bar{p})_M(\tilde{\Gamma}, \Gamma). \tag{17}$$

Denote $\Sigma = \frac{1}{2}(\Gamma + \tilde{\Gamma})$. Then with Kolář's operator p we can associate the section of $E(Q^1\bar{H}^2M)$ given by

$$P = 4\{p_M(\Sigma, \Sigma) - \tfrac{1}{2}p_M(\Gamma, \Gamma) - \tfrac{1}{2}p_M(\tilde{\Gamma}, \tilde{\Gamma})\} \tag{18}$$

and with Oproiu's operator $\bar{p}$ we can associate two sections of $E(Q^1\bar{H}^2M)$

$$Q = 4\{\bar{p}_M(\Sigma, \Sigma) - \tfrac{1}{2}\bar{p}_M(\Gamma, \tilde{\Gamma}) - \tfrac{1}{2}\bar{p}_M(\tilde{\Gamma}, \Gamma)\}, \tag{19}$$

$$R = 4\{(i\bar{p})_M(\Sigma, \Sigma) - \tfrac{1}{2}(i\bar{p})_M(\Gamma, \tilde{\Gamma}) - \tfrac{1}{2}(i\bar{p})_M(\tilde{\Gamma}, \Gamma)\}. \tag{20}$$

Then from (16)–(20), using the affine structure of $Q^1\bar{H}^2M$, we can construct a family of connections on $\bar{H}^2M$ which is equivalent to the family given by Theorem 1.

Thus we have proved

Theorem 2. *Every first order natural operator of Q^1H^1 into $Q^1\bar{H}^2$ can be deduced from Oproiu's and Kolář's operators.*

5. Natural operators of $\mathbf{Q^1H^1}$ into $\mathbf{Q^1H^2}$. Up to now we have considered the second order semiholonomic frame bundle. Now, we shall consider the second order holonomic frame bundle H^2M. For the components of connections on H^2M we have the additional condition $\Gamma^i_{jkl} = \Gamma^i_{kjl}$. From the coordinate expression (15) of Kolář's operator we immediately obtain

Proposition 2. *If $\Gamma: M \to Q^1H^1M$ is a connection without torsion then $p_M(\Gamma, \Gamma)$ is the connection on H^2M.*

If Γ is a linear connection in our sense, then the classical connection, [6], is $-\tilde{\Gamma}$. From this and the coordinate expression (14) of Oproiu's operator we immediately obtain

Proposition 3. *$\bar{p}_M(\Gamma, \tilde{\Gamma})$ is a connection on H^2M if and only if Γ is a connection on H^1M such that $\tilde{\Gamma}$ is the connection without curvature.*

In general, the additional condition $\Gamma^i_{jkl} = \Gamma^i_{kjl}$ leads to a four-parameter family of natural operators of order one. This family is obtained from (11) and (13) by putting $a = c$, $b = d$, $e = f$, $\alpha = \beta$, $\gamma = (2b - \bar{c})/2$.

Theorem 3. *There is a four-parameter family of first order natural operators of Q^1H^1 into Q^1H^2 and any such natural operator transforms any connection Γ $(=\Gamma^i_{jk})$ on H^1M into connections on H^2M with components*

$$\Gamma^i_{jk} = \bar{c}\Gamma^i_{jk} + (1 - \bar{c})\,\Gamma^i_{kj} \tag{21}$$

$$\begin{aligned}\Gamma^i_{jkl} = {} & a(\Gamma^i_{jk,l} + \Gamma^i_{kj,l}) + b(\Gamma^i_{jl,k} + \Gamma^i_{kl,j}) + e(\Gamma^i_{lj,k} + \Gamma^i_{lk,j}) + \\ & + (\bar{c} + \alpha - \tfrac{1}{2})(\Gamma^i_{jp}\Gamma^p_{kl} + \Gamma^i_{kp}\Gamma^p_{jl}) + (1 - e - \bar{c} - \alpha)(\Gamma^i_{jp}\Gamma^p_{lk} + \Gamma^i_{kp}\Gamma^p_{lj}) + \\ & + (a + e - \alpha)(\Gamma^i_{pj}\Gamma^p_{kl} + \Gamma^i_{pk}\Gamma^p_{jl}) + \alpha(\Gamma^i_{pj}\Gamma^p_{lk} + \Gamma^i_{pk}\Gamma^p_{lj}) + \\ & + \left(\frac{\bar{c}}{2} - a - b\right)(\Gamma^i_{lp}\Gamma^p_{kj} + \Gamma^i_{lp}\Gamma^p_{jk}) + \left(b - \frac{\bar{c}}{2}\right)(\Gamma^i_{pl}\Gamma^p_{jk} + \Gamma^i_{pl}\Gamma^p_{kj})\end{aligned}$$

where $2a + 2b + 2e = 1$.

Corollary. For $\bar{c} = 1$ we have prolongation natural operators and hence there is a three-parameter family of first order prolongation natural operators of Q^1H^1 into Q^1H^2.

Rybnikov [20] constructed geometrically a prolongation natural operator of Q^1H^1 into Q^1H^2 for connections without torsion. In coordinates

$$\Gamma^i_{jkl} = \tfrac{1}{6}\Gamma^i_{(jk,l)} + \tfrac{2}{3}\Gamma^i_{p(j}\Gamma^p_{k)l} - \tfrac{1}{3}\Gamma^i_{lp}\Gamma^p_{jk}\,,$$

where $(\ldots)$ denote symmetrisation. It is easy to see that this operator is obtained from (21) if $\bar{c} = 1$, $\alpha = \emptyset$, $a = b = e = \frac{1}{6}$.

6. The order of natural operators of Q^1H^1 into $Q^1\bar{H}^2$. Up to now we have assumed only first order natural operators. In what follows we shal prove that natural operators of finite order $r > 1$ do not exist.

We have denoted by $S_1 = \mathbf{R}^m \otimes \otimes^2 \mathbf{R}^{m*}$ the standard fibre of the lifting functor Q^1H^1 and by (Γ^i_{jk}) coordinates on S_1. Then (4) is the action of the group L^2_m on S_1. The standard fibre of the lifting functor $J^rQ^1H^1$ is $S^r_1 = T^r_m S_1 = S_1 \oplus S_1 \otimes \otimes \mathbf{R}^{m*} \oplus \ldots \oplus S_1 \otimes \bigcirc^r \mathbf{R}^{m*}$. Let i_r denote the canonical monomorphism of groups $i_r\colon L^1_m \to L^r_m$, $(a^i_j) \mapsto (a^i_j, \emptyset, \ldots, \emptyset)$. Then from the coordinate expression of the action of L^{r+2}_m on S^r_1 we immediately prove the following

Lemma. *The action of the structure group L^{r+2}_m on the standard fibre S^r_1 of the lifting functor $J^rQ^1H^1$ is such that its restriction to the subgroup $i_{r+2}(L^1_m)$ is the tensor action of type $(1, s + 2)$ on any component $S_1 \otimes \bigcirc^s \mathbf{R}^{m*}$, $0 \leqq s \leqq r$, of S^r_1.*

Consider an L^1_m-equivalent map f of the tensor space of type (r, s), $r \neq s$, into the tensor space of type (p, q). Then using the standard methods of the theory of differential invariants, [10], [12], we see that f is the solution of the equation

$$(r - s) \frac{\partial f}{\partial x^i} x^i = (p - q) f .$$

According to [4] f is a homogeneous polynomial of degree m where $(r - s)\, m = (p - q)$. Hence f must be a homogeneous polynomial of degree $(p - q)/(r - s)$.

Theorem 4. *All natural operators of Q^1H^1 into Q^1H^1 of finite orders are zero order natural operators with the associated maps given by* (11).

Proof. Let $f\colon Q^1H^1 \to Q^1H^1$ be an r-th order natural operator, $r \geqq \emptyset$. Its associated map F is an L^{r+2}_m-equivariant map of $T^r_m S_1$ into S_1, [18]. F must be also an L^1_m-equivariant map if we identify L^1_m with $i_{r+2}(L^1_m)$. If we restrict F to the component $S_1 \otimes \bigcirc^s \mathbf{R}^{m*}$, $\emptyset \leqq s \leqq r$, we have an L^1_m-equivariant map of the tensor space of type $(1, s + 2)$ into the tensor space of type $(1, 2)$. Hence F must be a homogeneous polynomial of degree $1/(1 + s)$ which yields $s = \emptyset$. In Section 3 we have proved that all such maps are (11), QED.

Theorem 5. *If the order of a natural operator $f\colon Q^1H^1 \to Q^1\bar{H}^2$ is finite then its order is less than or equal to one.*

Proof. The standard fibre of the lifting functor $Q^1\bar{H}^2$ is $R_2 = S_1 \oplus S_1 \otimes \mathbf{R}^{m*}$ and the action of L^3_m is given by (6). It is easy to see that the restriction of the action (6) to the subgroup $i_3(L^1_m)$ is, on the first component, the tensor action of type $(1, 2)$ and, on the second component, the tensor action of type $(1, 3)$. Let the natural operator $f\colon Q^1H^1 \to Q^1\bar{H}^2$ be of finite order r. Its associated map F is an L^{r+2}_m-equivariant map of S_1^r into R_2. F has to be an L_m^1-equivariant map and resticting F to the component $S_1 \otimes \bigcirc^s R^{m*}$ we obtain for the first component of R_2 the map described in Theorem 4 and for the second component of R_2 a polynomial map of degree $2/(1 + s)$. Hence $s \leqq 1$, QED.

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