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ON ISOMORPHISMS OF GRAPHS OF LATTICES

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All lattices dealt with in this paper are assumed to be locally finite. Isomorphisms of graphs of distributive lattices, modular lattices and semimodular lattices were investigated in the fifties (cf. [10], [5], [6]); these investigations were inspired by a problem proposed by G. Birkhoff ([1], Problem 8). Some sharpenings of the results of [5] were obtained in [8]. In the recent paper [3] concerning applications of universal algebra G. Birkhoff proved that projectivity of prime intervals is in a certain sense invariant under graph isomorphisms of modular lattices and appplied this result when investigating simple subdirect factors of modular lattices L and M of finite lengths having isomorphic graphs.

Isomorphisms of graphs of some types of partially ordered sets (semilattices, multilattices) were studied by M. Kolibiar [11], M. Tomková [12] and the author [7]; in all these considerations certain covering conditions were assumed.

In this paper graph isomorphisms of lattices are studied (without the assumption of modularity). The properties of some types of cycles, transposed intervals and subdirect decompositions under graph isomorphisms of lattices are established.

1. PRELIMINARIES

A lattice $\mathscr{L} = (L; \leqq)$ is called *locally finite* if each bounded chain in $\mathscr{L}$ is finite In what follows, all lattices are supposed to be locally finite. Given $a, b \in L$ we write $a \prec b$ (or $b \succ a$) if $[a, b]$ is a prime interval, i.e., if $a < b$ and $a < c < b$ for no $c \in L$; in such a case we also say that b covers a.

By the graph $G(\mathscr{L})$ we mean the (undirected) graph whose set of vertices is L and whose edges are those pairs $\{a, b\}$, which satisfy either $a \prec b$ or $b \prec a$. Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be graphs whose sets of vertices are G_1 and G_2 and let $f: G_1 \to G_2$ be a bijection such that, for each $x, y \in G_1$, $\{x, y\}$ is an edge in $\mathscr{G}_1$ if and only if $\{f(x), f(y)\}$ is an edge in $\mathscr{G}_2$. Then f is said to be an isomorphism of $\mathscr{G}_1$ onto $\mathscr{G}_2$.

Let $\mathscr{L} = (L; \leqq)$ and $\mathscr{L}_1 = (L_1; \leqq_1)$ be lattices. If f is an isomorphism of $G(\mathscr{L})$ onto $G(\mathscr{L}_1)$, then f is called a *graph isomorphism of the lattice $\mathscr{L}$ onto $\mathscr{L}_1$*.

We denote by $\mathscr{L}^{\sim}$ the dual of $\mathscr{L}$. The symbol $\times$ is used for denoting the operation of direct product of algebras. The following result is valid:

(A) (Cf. [5].) *Let $\mathscr{L}$ and $\mathscr{L}_1$ be modular lattices. The graphs $G(\mathscr{L})$ and $G(\mathscr{L}_1)$ are isomorphic if and only if there are lattices $\mathscr{A}$ and $\mathscr{B}$ such that $\mathscr{L} \cong \mathscr{A} \times \mathscr{B}$ and $\mathscr{L}_1 \cong \mathscr{A} \times \mathscr{B}^{\sim}$.*

Let us remark that the assumption of modularity of $\mathscr{L}$ in (A) cannot be omitted (cf. [5], [6]); this concerns the assertion "only if" in (A). (The assertion "if" in (A) holds without assuming modularity (cf. [10], § 7.)

Without loss of generality we can assume that $L = L_1$ and that f is the identity on L. Let us apply the assumption that $L = L_1$ and let P and P_1 be the set of all prime intervals in $\mathscr{L}$ or in $\mathscr{L}_1$, respectively. We denote

$$Q = P \cap P_1\,, \quad Q' = P \setminus Q\,.$$

Let $u, v, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ be distinct elements of L such that (i) $u \prec x_1 \prec x_2 \prec \ldots \prec x_m \prec v, u \prec y_1 \prec y_2 \prec \ldots \prec y_n \prec v$, and (ii) either $x_1 \vee y_1 = v$, or $x_m \wedge y_n = u$. Then the set $C = \{u, v, x_1, \ldots, x_m, y_1, \ldots, y_n\}$ is said to be a *cell in $\mathscr{L}$*. A cell C is called *proper* if either $m \geqq 2$ or $n \geqq 2$. A cell C in $\mathscr{L}$ is called *regular* if either each prime interval of C belongs to Q, or each prime interval of C belongs to Q'. The notions of a cell in $\mathscr{L}_1$ and the regularity of cells in $\mathscr{L}_1$ are defined analogously.

2. CELLS AND GRAPH ISOMORPHISMS

Again, let $\mathscr{L} = (L; \leqq)$ and $\mathscr{L}_1 = (L; \leqq_1)$ be lattices. (The lattice operations in $\mathscr{L}$ and in $\mathscr{L}_1$ will be denoted by $\wedge$, $\vee$ or by $\wedge_1$, $\vee_1$, respectively. If $a, b \in L$ and $a \leqq b$, then $[a, b]$ is the corresponding interval in $\mathscr{L}$; if $c, d \in L$ and $c \leqq_1 d$, then the interval of $\mathscr{L}_1$ with the endpoints c and d will be denoted by $[c, d]_1$.) Consider the following conditions:

(a) There exist lattices $\mathscr{A} = (A; \leqq)$, $\mathscr{B} = (B; \leqq)$ and a bijection $g: L \to A \times B$ such that g is an isomorphism of $\mathscr{L}$ onto $\mathscr{A} \times \mathscr{B}$ and, at the same time, g is an isomorphism of $\mathscr{L}_1$ onto $\mathscr{A} \times \mathscr{B}^{\sim}$.

(b) The identity mapping is a graph isomorphism of $\mathscr{L}$ onto $\mathscr{L}_1$.

(c) All proper cells of $\mathscr{L}$ and all proper cells of $\mathscr{L}_1$ are regular.

2.1. Lemma. *Let* (a) *be valid. Then* (b) *and* (c) *hold.*

Proof. The relation (a) $\Rightarrow$ (b) was proved in [10]. Assume that (a) is valid and let C be a proper cell in $\mathscr{L}$ (under the same notation as in § 1). Suppose that $x_1 \vee y_1 = v$ (if $x_m \wedge y_n = u$, we can apply a dual method). If $x \in L$ and $g(x) = (a, b)$, then we denote $a = x(A)$, $b = x(B)$. Without loss of generality we can assume that $m > 1$.

Since $u \prec x_1$, we have either

(i) $u(A) \prec x_1(A)$ and $u(B) = x_1(B)$,

or

(ii) $u(A) = x_1(A)$ and $u(B) \prec x_1(B)$.

Similar relations hold for u and y_1; let us denote them by (i_1) and (ii_1). Consider the case when (i) is valid.

If (ii_1) held, then we should have

$$v = x_1 \vee y_1 = g^{-1}(x_1(A), x_1(B)) \vee g^{-1}(y_1(A), y_1(B)) =$$
$$= g^{-1}(x_1(A), u(B)) \vee g^{-1}(u(A), y_1(B)) = g^{-1}(((x_1(A), u(B)) \vee$$
$$\vee (u(A), y_1(B))) = g^{-1}((x_1(A), y_1(B)) .$$

Because $(x_1(A), y_1(B)) \succ (x_1(A), u(B))$, we obtain $v \succ x_1$ which is a contradiction. Hence (i_1) must hold and thus

$$g(v) = g(x_1 \vee y_1) = g(x_1) \vee g(x_2) = (x_1(A), u(B)) \vee (y_1(A), u(B)) =$$
$$= (x_1(A) \vee y_1(A), u(B)) ,$$

implying $v(B) = u(B)$. For each x_i and each y_j we have $u \leqq x_i \leqq v$, $u \leqq y_j \leqq v$, whence $x_i(B) = u(B) = y_j(B)$, and therefore (since $g: \mathscr{L}_1 \to \mathscr{A} \times \mathscr{B}^\sim$ is an isomorphism) we get $u \prec_1 x_1 \prec_1 x_2 \prec_1 \dots \prec_1 x_m \prec_1 v$, $u \prec_1 y_1 \prec_1 y_2 \prec_1 \dots$ $\dots \prec_1 y_m \prec_1 v$. Therefore C is regular.

The proof for the case (ii) is analogous. We have verified that all proper cells of $\mathscr{L}$ are regular. Similarly we can verify that all proper cells of $\mathscr{L}_1$ are regular. Hence (c) is valid.

2.2. Lemma. *Let $C = \{u, v, x_1, y_1\}$ be a cell in $\mathscr{L}$ (under the notation from § 1). Let* (b) *hold. Then one of the following conditions is valid*: (i) *C is regular*; (ii) *$[u, x_1], [y_1, v] \in Q$ and $[u, y_1], [x_1, v] \in Q'$*; (iii) *$[u, x_1], [y_1, v] \in Q'$ and $[u, y_1]$, $[x_1, v] \in Q$.*

Proof. Cf. [5], Lemma 5.

In the following lemmas 2.3–2.9 we assume that the conditions (b) and (c) are fulfilled.

2.3. Lemma. *Let $u, v, x_1, \dots, x_m, y_1, \dots, y_n$ be distinct elements of L such that* (i) *$u \prec x_1 \prec x_2 \prec \dots \prec x_m \prec v$, $u \prec_1 x_1 \prec_1 x_2 \prec_1 \dots \prec_1 x_m \prec_1 v$,* (ii) *$u \prec y_1 \prec$ $\prec y_2 \prec \dots \prec y_n \prec v$. Then $u \prec_1 y_1 \prec {}_1 y_2 \prec_1 y_3 \prec_1 \dots \prec_1 y_n \prec_1 v$.*

Proof. (Cf. Fig. 2.3a and 2.3b.) By way of contradiction, assume that the as-

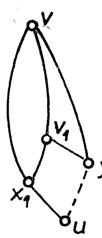

Fig. 2.3a.

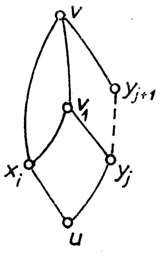

Fig. 2.3b.

sertion of the lemma does not hold. Since (b) is valid, the following condition (iii) is fulfilled:

(iii) there exists $j \in \{0, 1, 2, \ldots, n\}$ such that $y_j \succ_1 y_{j+1}$.

(Here we denote $u = y_0$, $v = y_{n+1}$.) Suppose that $[u, v]$ is a minimal element (with respect to the partial order defined by inclusion) of the system of all intervals of $\mathscr{L}$ which contain the elements $x_1, \ldots, x_m, y_1, \ldots, y_m$ fulfilling the conditions (i), (ii) and (iii).

If $m = n = 1$, then we have a contradiction with 2.2. Hence either $m > 1$ or $n > 1$. First assume that $j = 0$, i.e. $y_1 \prec_1 u$. Suppose that $x_1 \vee y_1 = v_1 < v$. In view of the minimality of $[u, v]$, all prime intervals of $[x_1, v_1]$ belong to Q. Hence it follows that all prime intervals of $[u, v_1]$ belong to Q, which is a contradiction. Therefore $x_1 \vee y_1 = v$. Thus $C = \{u, v, x_1, \ldots, x_m, y_1, \ldots, y_n\}$ is a proper cell in $\mathscr{L}$. According to (c), C is regular. In view of (ii), $u \succ_1 y_1$ cannot hold. By a dual argument we can verify that $v \prec_1 y_n$ cannot hold. Hence we have $u < y_j$, $y_{j+1} < v$.

Suppose that $x_1 \vee y_1 = v_1 < v$. Then all prime intervals of the interval $[x_1, v]$ belong to Q, hence the same is valid for the intervals $[u, v_1]$ and $[y_1, v]$, which is a contradiction. Therefore $x_1 \vee y_1 = v$. Hence, again, C is a proper cell in L. From (c) and (ii) we infer that (iii) cannot be valid, completing the proof.

Analogously we can prove

2.3.1. Lemma. *Let $u, v, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ be distinct elements of L such that* (i) $u \prec x_1 \prec x_2 \prec \ldots \prec x_m \prec v$, $u \succ_1 x_1 \succ_1 x_2 \succ_1 \ldots \succ_1 x_m \succ_1 v$, (ii) $u \prec y_1 \prec y_2 \prec \ldots \prec y_n \prec v$. *Then* $u \succ_1 y_1 \succ_1 y_2 \succ_1 \ldots \succ_1 y_n \succ_1 v$.

2.4. Lemma. *Let $u, v, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ be distinct elements of L such that* (i) $u \prec x_1 \prec x_2 \prec \ldots \prec x_m \prec v$, $u \prec y_1 \prec y_2 \prec \ldots \prec y_n \prec v$, (ii) *there are* $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$ *such that* $x_i \wedge y_j = u$, $x_i \vee y_j = v$, $u \prec_1 x_1 \prec_1 \ldots \ldots \prec_1 x_i$, $u \prec_1 y_1 \prec_1 \ldots \prec_1 y_j$. *Then we have* $x_i \prec_1 x_{i+1} \prec_1 \ldots \prec_1 x_m \prec_1 v$ *and* $y_j \prec_1 y_{j+1} \prec_1 \ldots \prec_1 y_n \prec_1 v$.

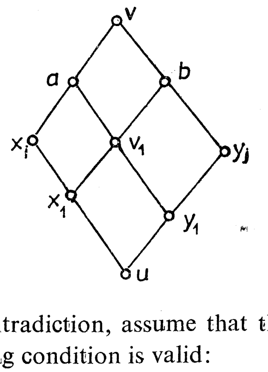

Fig. 2.4.

Proof. (Cf. Fig. 2.4.) By contradiction, assume that the assertion of the lemma does not hold. Then the following condition is valid:

(iii) either there is $k \in \{i, i + 1, \ldots, m\}$ with $x_k \succ_1 x_{k+1}$, or there is

$k \in \{j, j+1, \ldots, n\}$ with $y_k \succ_1 y_{k+1}$. (As above, we put $x_{m+1} = v = y_{n+1}$.) Suppose that $[u, v]$ is a minimal element (with respect to the partial order defined by inclusion) of the system of all intervals of $\mathscr{L}$ which contain elements $x_1, \ldots, x_m, y_1, \ldots, y_n$ fulfilling the conditions (i), (ii) and (iii). From 2.2 it follows that $m > 1$ or $n > 1$. If $x_1 \vee y_1 = v$, then $C = \{u, v, x_1, \ldots, x_m, y_1, \ldots, y_n\}$ is a proper cell, thus C is regular; now (i), (ii) and (iii) give a contradiction. Thus $x_1 \vee y_1 = v_1 < v$. Denote $a = x_i \vee \vee v_1$, $b = y_j \vee v_1$.

Now by using repeatedly the minimality of $[u, v]$ and 2.3, we obtain that all prime intervals of the following intervals belong to Q:

$$[u, v_1], \quad [x_1, v_1], \quad [y_1, v_1],$$
$$[x_i, a], \quad [v_1, a], \quad [y_j, b], \quad [v_1, b],$$
$$[a, v], \quad [b, v].$$

Hence in view of 2.2 and 2.3, the condition (iii) cannot hold.

2.5. Lemma. *Let $u, v, x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n$ be distinct elements of L such that the condition* (i) *from* 2.4 *is valid. Assume that there are $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, n\}$ such that $x_i \wedge y_j = u$, $x_i \vee y_j = v$, $u \succ_1 x_1 \succ_1 \ldots \succ_1 x_i$, $u \succ_1 \succ_1 y_1 \succ_1 \ldots \succ_1 y_j$. Then all prime intervals of $[x_i, v]$ and of $[y_j, v]$ belong to Q'.*

Proof can be performed by using a method analogous to that of 2.4 and by applying 2.3.

Also, the assertions dual to 2.4 and to 2.5 are valid.

2.6. Lemma. *Let $u, v, x, y \in L$, $x \wedge y = u$, $x \vee y = v$. Assume that* (i) *all prime intervals of the interval $[u, x]$ belong to Q and* (ii) *all prime intervals of $[u, y]$ belong to Q'. Then all prime intervals of the interval $[x, v]$ belong to Q' and all prime intervals of $[y, v]$ belong to Q.*

Proof. By way of contradiction, assume that the assertion of the lemma fails to hold and that $[u, v]$ is a minimal interval containing elements x and y with $x \wedge y = u$, $x \vee y = v$, such that either some prime interval of $[x, v]$ belongs to Q or some prime interval of $[y, v]$ belongs to Q'. Then we obviously have $x \neq u \neq v$.

Choose $x_1 \in [u, x]$ and $y_1 \in [u, y]$ such that $u \prec x_1$ and $u \prec y_1$. Hence $[u, x_1] \in Q$ and $[u, y_1] \in Q'$. Therefore in view of (c), the relation $x_1 \vee y_1 = v$ cannot hold. Thus we have $x_1 \vee y_1 = v_1 < v$. Put $a = x \vee v_1$, $b = y \vee v_1$.

According to the minimality of $[u, v]$ we infer that all prime intervals of $[x_1, v_1]$ belong to Q'. Hence we must have $x \wedge v_1 = x_1$. Again, by the minimality of $[u, v]$ we get (by considering the interval $[x_1, a]$) that all prime intervals of $[x, a]$ belong to Q' and all prime intervals $[v_1, a]$ belong to Q.

By a similar reasoning we infer that all prime intervals of $[y, b]$ belong to Q and all prime intervals of $[v_1, b]$ belong to Q'. Hence we must have $a \wedge b = v_1$.

If we consider the interval $[v_1, v]$, then the minimality of $[u, v]$ implies that all prime intervals of $[a, v]$ belong to Q' and all prime intervals of $[b, v]$ belong to Q.

Hence according to 2.3 and 2.3.1 all prime intervals of $[x, v]$ belong to Q' and all prime intervals of $[y, v]$ belong to Q.

The assertion dual to 2.6 can be proved analogously.

2.7. Lemma. *Let u, a, v be distinct elements of L, $u < a < v$. Assume that all prime intervals of $[u, a]$ belong to Q' and all prime intervals of $[a, v]$ belong to Q. Then there is $b \in L$ such that $a \wedge b = u$, $a \vee b = v$, all prime intervals of $[u, b]$ belong to Q and all prime intervals of $[b, v]$ belong to Q'.*

Proof. According to the assumption, we have $a <_1 v$ and $a <_1 u$. Moreover, from the dual of 2.3 (if the roles of $\mathscr{L}$ and $\mathscr{L}_1$ are interchanged) we infer that all prime intervals of $[a, u]_1$ belong to Q_1' and all prime intervals of $[a, v]_1$ belong to Q_1. Hence we must have $u \wedge_1 v = a$. Put $b = u \vee_1 v$. According to 2.6, all prime intervals of $[v, b]_1$ belong to Q_1' and all prime intervals of $[u, b]_1$ belong to Q_1. Thus in view of 2.3, we have $u < b < v$, all prime intervals of $[u, b]$ belong to Q and all prime intervals of $[b, v]$ belong to Q'. Therefore $a \wedge b = u$, $a \vee b = v$.

2.8. Lemma. *Let $a, b \in L$, $u = a \wedge b$, $v = a \vee b$. Assume that all prime intervals of $[u, a]$ belong to Q. Then all prime intervals of $[b, v]$ belong to Q.*

Proof. If $u = b$, then the assertion is trivial. Let $u < b$. There exist elements $y_0, \ldots, y_n \in L$ with $u = y_0 \prec y_1 \prec y_2 \prec \ldots \prec y_n = b$. If $n = 1$, then the assertion is valid according to 2.4 and 2.6. Suppose that the assertion holds for $n - 1$. Put $v_1 = a \vee y_{n-1}$. Then all prime intervals of $[y_{n-1}, v_1]$ belong to Q. We have either $v_1 = v$, or $v_1 \wedge b = y_{n-1}$ and $v_1 \vee b = v$. Hence in view of 2.4 and 2.6, all prime intervals of $[b, v]$ belong to Q.

2.9. Lemma. *Let a, b, u, v be as in 2.8. Assume that all prime intervals of $[u, a]$ belong to Q'. Then all prime intervals of $[b, v]$ belong to Q'.*

The proof is the same as in 2.8. The assertions dual to 2.7, 2.8 and 2.9 are also valid.

3. DIRECT DECOMPOSITIONS

Let us assume that the conditions (b) and (c) from § 2 are fulfilled. Let $a, b \in L$. We put $a R b$, if all prime intervals of $[a \wedge b \ \ a \vee b]$ belong to Q. Analogously, we put $a R' b$, if all prime intervals of $[a \wedge b, a \vee b]$ belong to Q'.

From 2.2, 2.4 and the dual of 2.4 we infer that $a R b$ is equivalent with each of the following two conditions:

(α_1) All prime intervals of $[a \wedge b, a]$ and of $[a \wedge b, b]$ belong to Q.

(α_2) All prime intervals of $[a, a \vee b]$ and of $[b, a \vee b]$ belong to Q.

A similar equivalence (with Q replaced by Q') is valid for the relation R'. In an analogous way we can define the relations R_1 and R_1' on L by taking the operations $\wedge_1$ and $\vee_1$ instead of $\wedge$ and $\vee$. It is easy to verify (by using 2.3) that R_1 coincides with R and R_1' coincides with R'.

3.1. Lemma. *R and R' are equivalence relations on L.*

Proof. R is obviously reflexive and symmetric. Let $a\ R\ b$, $b\ R\ c$. Put $a \wedge b = u_1$, $b \wedge c = u_2$, $u_1 \wedge u_2 = u$. In view of (α_1), all prime intervals of $[u_1, a]$, $[u_1, b]$, $[u_2, b]$, $[u_2, c]$ belong to Q. Hence all prime intervals of $[u_1, u_1 \vee u_2]$ and $[u_2, u_1 \vee u_2]$ belong to Q. According to the dual of 2.4, all prime intervals of $[u, u_1]$ and $[u, u_2]$ belong to Q. In view of 2.3, all prime intervals of $[u, a]$ and $[u, c]$ belong to Q as well, hence the same holds for $[a \wedge c, a]$ and $[a \wedge c, c]$. Therefore $a\ R\ c$ and so R is an equivalence relation. The proof for R' is analogous.

3.2. Lemma. *R and R' are congruence relations on $\mathscr{L}$ and on $\mathscr{L}_1$.*

Proof. Let $a, b, c \in L$, $a\ R\ b$. In view of 3.1 it suffices to verify that $a \vee c\ R\ b \vee c$ is valid (the remaining cases are analogous). Put $u = a \wedge b$, $v = a \vee b$. According to the definition of R we have $u\ R\ v$. Denote $c \vee u = u_1$, $c \vee v = v_1$, $v \wedge u_1 = u_2$. All prime intervals of $[u_2, v]$ belong to Q, hence according to 2.8, all prime intervals of $[u_1, v_1]$ belong to Q. Since $a \vee c$, $b \vee c \in [u_1, v_1]$, we infer that $a \vee c\ R\ b \vee c$.

Let us denote by 0 and I the least or the largest equivalence relation on $\mathscr{L}$, respectively.

3.3. Lemma. *$R \wedge R' = 0$ and $R \vee R' = I$.*

Proof. The first identity follows from $Q \cap Q' = \emptyset$. If $[a, b]$ is a prime interval in $\mathscr{L}$, then we have eher $a\ R\ b$ or $a\ R'\ b$; since $\mathscr{L}$ is locally finite, $R \vee R' = I$ is valid.

3.4. Lemma. *R and R' are permutable.*

Proof. If we apply 2.7, 2.8 and their duals, then by the same method as in [5], Lemma 7 we obtain the permutability of R and R' (Alternatively, we can use also [2], p. 163, Ex. 10.)

From 3.2, 3.3, 3.4 and Thm. 5, Chap. VII, [2] it follows that the natural bijections

(1) $\mathscr{L} \to (\mathscr{L}/R) \times (\mathscr{L}/R')$,

(2) $\mathscr{L}_1 \to (\mathscr{L}_1/R) \times (\mathscr{L}_1/R')$

are isomorphisms.

Let x_0 be a fixed dement of L. Put

$$A = \{x \in L : x\ R\ x_0\}\ , \quad B = \{y \in L : y \in R'\ x_0\}\ .$$

Then $\mathscr{A} = (A; \leqq)$ and $\mathscr{B} = (B; \leqq)$ are convex sublattices of $\mathscr{L}$, and $\mathscr{A}_1 = (A; \leqq_1)$, $\mathscr{B}_1 = (B; \leqq_1)$ are convex sublattices of $\mathscr{L}_1$. In view of the definition of the relations R and R', $\mathscr{A}_1 = \mathscr{A}$ and $\mathscr{B}_1 = \mathscr{B}^\sim$. For $z \in L$ there exist uniquely determined elements $x \in A$ and $y \in B$ such that $z\ R\ y$ and $z\ R'\ x$. Put $f(z) = (x, y)$. From (1) and (2) we obtain:

3.5. Lemma. *The mapping f is an isomorphism of the lattice $\mathscr{L}$ onto $\mathscr{A} \times \mathscr{B}$; at the same time, f is an iomorphism of $\mathscr{L}_1$ onto $\mathscr{A}_1 \times \mathscr{B}_1$.*

3.6. Corollary. *Let $\mathscr{L} = (L; \leqq)$ and $\mathscr{L}_1 = (L; \leqq_1)$ be lattices fulfilling the conditions* (b) *and* (c). *Then $\mathscr{L}$ and $\mathscr{L}_1$ satisfy the condition* (a).

Now let us omit the assumption that the lattices $\mathscr{L}$ and $\mathscr{L}_1$ are defined on the same set L; let $\mathscr{L} = (L; \leqq)$ and $\mathscr{L}_1 = (L_1; \leqq_1)$. Suppose that $h: L \to L_1$ is a bijection. A proper cell C of $\mathscr{L}$ is said to be regular under h, if either $f(x_1) \prec_1 f(x_2)$ for each prime interval $[x_1, x_2]$ of C, or $f(x_1) \succ_1 f(x_2)$ for each prime interval $[x_1, x_2]$ of C. Analogously we define regularity of proper cells in $\mathscr{L}_1$ (with h replaced by h^{-1}).

From 2.1 and 3.5 we immediately obtain:

3.7. Theorem. *Let $\mathscr{L} = (L; \leqq)$ and $\mathscr{L}_1 = (L_1; \leqq_1)$ be lattices. Let $h: L \to L_1$ be a bijection. Then the following conditions are equivalent:*

(α) *h is a graph isomorphism of the lattice $\mathscr{L}$ onto $\mathscr{L}_1$, all proper cells of $\mathscr{L}$ are regular under h and all proper cells of $\mathscr{L}_1$ are regular under h^{-1}.*

(β) *There exist lattices $\mathscr{A} = (A; \leqq)$ and $\mathscr{B} = (B; \leqq)$ and direct representations $f: \mathscr{L} \to \mathscr{A} \times \mathscr{B}$, $g: \mathscr{L}_1 \to \mathscr{A} \times \mathscr{B}^{\sim}$ such that $h = g^{-1}f$.*

From the well-known covering conditions for modular lattices we infer:

3.8. Lemma. *If $\mathscr{L} = (L; \leqq)$ is a modular lattice, then there are no proper cells in $\mathscr{L}$.*

From 3.8 it follows that Theorem (A) of § 1 is a particular case of Theorem 3.7.

In [8] Theorem (A) was sharpened as follows:

(A_1) (Cf. [8].) Let $\mathscr{L}$ be a modular lattice and let $\mathscr{L}_1$ be a lattice. The graphs $G(\mathscr{L})$ and $G(\mathscr{L}_1)$ are isomorphic if and only if there are lattices $\mathscr{A}$ and $\mathscr{B}$ such that $\mathscr{L} \cong \mathscr{A} \times \mathscr{B}$ and $\mathscr{L}_1 \cong \mathscr{A} \times \mathscr{B}.^{\sim}$

Hence if $\mathscr{L}$ and $\mathscr{L}_1$ are lattices with isomorphic graphs and if $\mathscr{L}$ is modular, then $\mathscr{L}_1$ is modular as well. The possibility of setting conditions only on $\mathscr{L}$ (when studying graph isomorphisms of $\mathscr{L}$ and $\mathscr{L}_1$) suggests to formulate the open question whether the condition (α) of Thm. 3.7 is equivalent to the condition

(α') h is a graph isomorphism of the lattice $\mathscr{L}$ onto $\mathscr{L}_1$ and all proper cells of $\mathscr{L}$ are regular under h.

For the notion of the weak product of abstract algebras cf. Grätzer [4]. If $\{A_i\}$ $(i \in I)$ is a system of algebras of the same type and if and algebra A is isomorphic to the weak product of this system, then we shall write

$$A \to (w) \prod_{i \in I} A_i\,; \tag{3.1}$$

we also say that (3.1) is a weak product decomposition of A. If the set I is finite, then (3.1) implies that $A \cong \prod_{i \in I} A_i$.

The following result is easy to verify:

3.9. Lemma. *Let $\mathscr{L}$ and $\mathscr{L}_i$ $(i \in I)$ be lattices with* card $\mathscr{L}_i > 1$ *for each $i \in I$ and suppose that $L \to (w) \prod_{i \in I} L_i$. Assume that $\mathscr{L}$ is of finite length. Then all $\mathscr{L}_i$ are of finite length and the set I is finite.*

3.10. Theorem. (Cf [9].) *Let $\mathscr{L}$ be a (locally finite) lattice. Then there are lattices $\mathscr{L}_\iota$ $(\iota \in I)$ such that* (i) *all $\mathscr{L}_\iota$ are directly indecomposable, and* (ii) *$\mathscr{L}$ has a weak product decomposition $\mathscr{L} \to (w) \prod_{i \in I} \mathscr{L}_i$.*

From 3.7 and 3.10 we obtain:

3.11. Theorem. *Let $\mathscr{L} = (L; \leqq)$ and $\mathscr{L}_1 = (L_1; \leqq_1)$ be lattices. Let $h: L \to L_1$ be a bijection. Suppose that the condition* (α) *from* 3.7 *is valid. Then there exist weak product decompositions $\mathscr{L}_1 \to (w) \prod_{i \in I} \mathscr{L}_i$ and $L_1 \to (w) \prod_{i \in I} \mathscr{L}_i^1$, such that* (i) *all $\mathscr{L}_i$ and all $\mathscr{L}_i^1$ are directly indecomposable, and* (ii) *for each $\iota \in I$, either $\mathscr{L}_i \cong \mathscr{L}_i^1$, or $\mathscr{L}_i \cong (\mathscr{L}_i^1)^\sim$.*

4. TRANSPOSED INTERVALS AND GRAPH ISOMORPHISMS

Let us recall the following notions (cf. [2]). Let $\mathscr{L} = (L; \leqq)$ be a lattice, $a, b, u, v \in \in L$, $a \wedge b = u$, $a \vee b = v$. Then the intervals $[u, a]$ and $[b, v]$ are said to be (*mutually*) *transposed*; in such a case we write $[u, a] \sim_t [b, v]$ or $[b, v] \sim_t [u, a]$. Intervals $[a_1, b_1]$ and $[a_2, b_2]$ of $\mathscr{L}$ are called *projective* if there exist intervals $[x_1, y_1], [x_2, y_2], \ldots, [x_n, y_n]$ in $\mathscr{L}$ such that $[x_1, y_1] = [a_1, b_1]$, $[x_n, y_n] = [a_2, b_2]$ and $[x_i, y_i] \sim_t [x_{i+1}, y_{i+1}]$ for $i = 1, 2, \ldots, n - 1$. The projectivity of $[a_1, b_1]$ and $[a_2, b_2]$ will be denoted by $[a_1, b_1] \sim [a_2, b_2]$.

Similarly as in § 2 let us assume that $\mathscr{L} = (L; \leqq)$ and $\mathscr{L}_1 = (L; \leqq_1)$ are lattices with the same underlying set. Let Q, Q', Q_1 and Q_1' be defined as in § 2. Let us consider following condition:

(d($\mathscr{L}$)) If $[a, b]$ is an interval in $\mathscr{L}$ which is transposed to a prime interval $[p_1, p_2]$ of $\mathscr{L}$ and if $q_1, q_2 \in [a, b]$, $q_1 \prec q_2$, then (i) $[p_1, p_2] \in Q \Rightarrow [q_1, q_2] \in Q$, and (ii) $[p_1, p_2] \in Q' \Rightarrow [q_1, q_2] \in Q'$.

Let (d($\mathscr{L}_1$)) be defined analogously and let us denote by (d) the condition saying that both (d($\mathscr{L}$)) and (d($\mathscr{L}_1$)) are valid. Let (a), (b) and (c) be as in § 2.

4.1. Lemma. *Let* (b) *and* (d) *be valid. Then* (c) *holds.*

Proof. (Cf. Fig. 4.1.) Assume that the conditions (b) and (d) are fulfilled. Let C be a proper cell in $\mathscr{L}$. We use the analogous notation for C as in § 2 and § 3. Without loss of generality we can suppose that $x_1 \vee y_1 = v$. (In the case $x_m \wedge y_n = u$ the dual argument can be used.) Also, we can assume that $m > 1$. Assume that $[u, y_1]$ belongs to Q. Since $[u, y_1] \sim_t [x_1, v]$, according to (d) we infer that all prime intervals of $[x_1, v]$ belong to Q; in particular, we have $x_{m-1} \prec_1 x_m$. In order to prove that C is regular it suffices to verify that $u \prec_1 x_1$ (since we can then imply that $y_1 \prec_1 y_2 \prec_1 \ldots \prec_1 y_m \prec_1 v$ is valid).

By way of contradiction, assume that $u \succ_1 x_1$ holds. Then in view of (d), all prime intervals of $[y_1, v]$ belong to Q'. If $x_m \wedge y_n = u$ then $[y_n, v] \sim_t [u, x_m]$, hence (d) implies that $x_{m-1} \succ_1 x_m$, which is a contradiction. Thus $x_m \wedge y_n = u_1 > u$. There exists $u_2 \in [u, u_1]$ with $u \prec u_2$. Since $x_1 \nleqq y_n$ and $y_1 \nleqq x_m$, we must have $x_1 \neq u_2 \neq$

$\neq y_1$. Hence $x_1 \wedge u_2 = u = y_1 \wedge u_2$. Put $z_1 = x_1 \vee u_2$, $z_2 = y_1 \vee u_2$. Then we have

(α_1) $x_1 < z_1$ and all prime intervals of $[x_1, z_1]$ belong to Q,

(α_2) $y_1 < z_2$ and all prime intervals of $[y_1, z_2]$ belong to Q'.

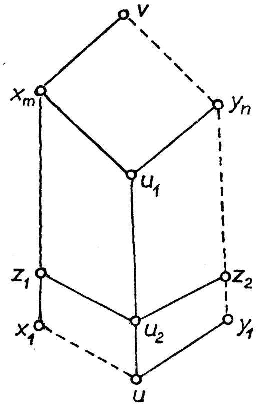

Fig. 4.1.

But in view of (d) this is impossible, since $[u, u_2] \sim_t [x_1, z_1]$ and, at the same time, $[u, u_2] \sim_t [y_1, z_2]$.

From 4.1 and 3.6 we infer

4.2. Lemma. *Let* (b) *and* (d) *be valid. Then* (a) *holds.*

4.3. Lemma. *Let* (a) *be valid. Let* $a, b, u, v \in L$, $a \wedge b = u$, $a \vee b = v$, $u < a$. *If* $u <_1 a$, *then* $b <_1 v$ *and all prime intervals of* $[b, v]$ *belong to* Q.

Proof. We have $b < v$. Let A and B be as in § 3. According to the definition of B we have $u(B) = a(B)$, hence $v(B) = (a \vee b)(B) = a(B) \vee b(B) = u(B) \vee b(B) = b(B)$ and $v(A) \geqq b(A)$. Thus $v >_1 b$. If $[x, y]$ is a prime interval of $[b, v]$, then $b(B) \leqq x(B) \leqq y(B) \leqq v(B) = b(B)$ and $b(A) \leqq x(A) \leqq y(A) \leqq v(A)$, hence $x \prec_1 y$ and $x, y \in [b, v]_1$.

4.4. Lemma. *Let* (a) *be valid. Let* a, b, u, v *be as in* 4.3. *If* $u >_1 a$, *then* $b >_1 v$ *and all prime intervals of* $[b, v]$ *belong to* Q'.

The proof is analogous to that of 4.3. The duals of 4.3 and 4.4 are also valid. Hence we obtain:

4.5. Corollary. (a) $\Rightarrow$ (d).

As a consequence of 3.6, 4.1 and 4.5 we get

4.6. Corollary. *Let* (b) *be valid. Then* (a) $\Leftrightarrow$ (d).

Now let $\mathscr{L} = (L; \leqq)$ and $\mathscr{L}_1 = (L_1; \leqq_1)$ be lattices and let $h: L \to L_1$ be a bijection. Consider the following condition:

($\mathrm{d}_h(\mathscr{L})$) If $[a, b]$ is an interval in $\mathscr{L}$ which is transposed to a prime interval $[p_1, p_2]$ of $\mathscr{L}$ and if $[q_1, q_2]$ is a prime interval of $[a, b]$, then (i) $h(p_1) \prec_1 h(p_2) \Rightarrow h(q_1) \prec_1 h(q_2)$, and (ii) $h(p_1) \succ_1 h(p_2) \Rightarrow h(q_1) \succ_1 h(q_2)$.

Let ($\mathrm{d}_h(\mathscr{L}_1)$) and ($\mathrm{d}_h$) be defined analogously (cf. the definitions of ($\mathrm{d}(\mathscr{L}_1)$) and (d)).

From 4.6 we obtain:

4.7. Theorem. *Let $\mathscr{L} = (L; \leqq)$ and $\mathscr{L}_1 = (L_1; \leqq_1)$ be lattices. Let $h: L \to L_1$ be a graph isomorphism. Then the condition (β) from 3.7 is equivalent to the condition (d_h).*

4.8. Lemma. *Let $\mathscr{L} = (L; \leqq)$ and $\mathscr{L}_1 = (L_1; \leqq_1)$ be modular lattices and let $h: L \to L_1$ be a graph isomorphism of $\mathscr{L}$ onto $\mathscr{L}_1$. Then the condition (d_h) is fulfilled.*

Proof. Since $\mathscr{L}$ and $\mathscr{L}_1$ are modular, they have no proper cells, hence the condition (α) from 3.7 is valid. Thus (β) of 3.7 is valid as well. In view of 4.5 we infer that (d_h) holds.

The assertion "only if" of Theorem (A) (cf. § 1) is a corollary of 4.7 and 4.8.

5. SUBDIRECT DECOMPOSITIONS

The following result is due to G. Birkhoff ([3], § 5):

(B) *Let L and M be any two modular lattices of finite lengths and let their graphs G(L) and G(M) be isomorphic. Let $L_1, \ldots, L_r$ and $M_1, \ldots, M_s$ be the "simple" subdirect factors of L and M, respectively. Then $r = s$, and we can so order the M_j's that $M_j \cong L_j$ or $M_j \cong L_j^{\sim}$.*

The proof of (B) was established by investigating the properties of pairs of projective prime intervals under graph isomorphisms ([3], Lemma 12).

Let us now consider the subdirect product decompositions with subdirectly indecomposable factors in the case when neither the condition concerning finite length nor modularity is assumed.

Let $\mathscr{L} = (L; \leqq)$ be a lattice, card $L > 1$. The system of all congruence relations of $\mathscr{L}$ is denoted by Con $\mathscr{L}$. Let x and y be distinct elements of L and let $K(x, y)$ be the system of all $\theta \in$ Con $\mathscr{L}$ such that $x \,\theta\, y$ fails to hold. Let $K_m(x, y)$ be the set of all maximal elements of $K(x, y)$. Further let $\theta_0(x, y)$ be a fixed element of $K_m(x, y)$ and let $\{\theta_i\}$ $(i \in I)$ be the set of all such $\theta_0(x, y)$ (for all pairs of distinct elements x and y of L). Then all lattices $\mathscr{L}/\theta_i$ are subdirectly indecomposable. Let I_1 be a subset of I such that $\bigwedge_{i \in I_1} \theta_i = 0$. Then $\mathscr{L}$ is a subdirect product of the lattices $\mathscr{L}/\theta_i (i \in I_1)$ (cf., e.g., [1], Chap. VII, 6, or [2], Chap. VII, 8).

We obviously have $K(x, y) = K(x \wedge y, x \vee y)$. If the intervals $[x_1, y_1]$ and $[x_2, y_2]$ are projective, then $K(x_1, y_1) = K(x_2, y_2)$. Hence we may choose $\theta_0(x, y)$

in such a way that $\theta_0(x_1, y_1) = \theta_0(x_2, y_2)$ is valid whenever $[x_1, y_1]$ is projective to $[x_2, y_2]$. Let this condition be fulfilled.

Now we apply the assumption that $\mathscr{L}$ is locally finite. Put $I_0 = \{i \in I : \theta_i = \theta_0(x_1, y_1)$ for some prime interval $[x_1, y_1]$ of $\mathscr{L}\}$. Then we clearly have $\bigwedge_{i \in I_0} \theta_i = 0$. Hence $\mathscr{L}$ is a subdirect product of lattices $\mathscr{L}/\theta$ $(i \in I_0)$. This subdirect decomposition of $\mathscr{L}$ will be called *canonical*; we also say that the system $\{\theta_i\}$ $(i \in I_0)$ is canonical for L.

Let $\mathscr{L} = (L; \leqq)$ and $\mathscr{L}_1 = (L_1; \leqq_1)$ be lattices with isomorphic graphs. As above, without loss of generality we can suppose that $L_1 = L$ and that the identity mapping is a graph isomorphism of L onto L_1. In what follows we shall assume that the condition (c) is valid; hence (a) holds as well.

5.1. Lemma. *Let $x, y \in L$. Assume that $x \prec y$ and $x \prec_1 y$. Then there is a prime interval $[x_1, y_1]$ of $\mathscr{A}$ which is projective to $[x, y]$.*

Proof. Let x_0 be as in the definition of A and B (cf. § 3, Lemma 3.5). Put $x_1 = x(A)$, $y_1 = y(A)$, $u = x \wedge x_1$, $v = y \wedge y_1$. Elementary calculations show that $[x, y] \sim_t [u, v]$ and $[u, v] \sim_t [x_1, y_1]$. Hence $[x_1, y_1]$ is projective to $[x, y]$.

Analogously we can verify

5.2. Lemma. *Let $x, y \in L$. Assume that $x \prec y$ and $x \succ_1 y$. Then there is a prime interval $[x_1, y_1]$ of $\mathscr{B}$ which is projective to $[x, y]$.*

5.3. Corollary. *Let the system $\{\theta_i\}$ $(i \in I_0)$ be canonical for $\mathscr{L}$. Then for each $i \in I_0$ there exists a prime interval $[x_1, y_1]$ of $\mathscr{L}$ such that* (i) $\theta_i = \theta_0(x_1, y_1)$, *and* (ii) *either $x_1, y_1 \in A$ or $x_1, y_1 \in B$.*

For each $\theta \in \operatorname{Con} \mathscr{L}$ we define the relation $\theta|_A$ by putting, for $a_1, a_2 \in A$, $a_1\, \theta|_A\, a_2$ if $a_1\, \theta\, a_2$; let $\theta|_B$ be defined analogously. Conversely, let $\theta_1 \in \operatorname{Con} \mathscr{A}$, $\theta_2 \in \operatorname{Con} \mathscr{B}$. For $z_1, z_2 \in L$ we put $z_1\, \theta_1^*\, z_2$ if $z_1(A)\, \theta_1\, z_2(A)$; similarly we set $z_1\, \theta_2^*\, z_2$ if $z_1(B)\, \theta_2\, z_2(B)$. Then $\theta|_A \in \operatorname{Con} \mathscr{A}$, $\theta|_B \in \operatorname{Con} \mathscr{B}$; θ_1^* and θ_2^* belong to $\operatorname{Con} \mathscr{L}$.

If $\{\theta_i\}_{i \in I_0}$ is a canonical system for $\mathscr{L}$, then we denote $I_1 = \{i \in I_0 : \theta_i = \theta_0(x_1, y_1)$ for some prime interval $[x_1, y_1] \subseteqq A\}$, $I_2 = \{i \in I_0 : \theta_i = \theta_0(x_1, y_1)$ for some prime interval $[x_1, y_1] \subseteqq B\}$. In view of 5.1 and 5.2 we have $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = I_0$.

The following lemma is an immediate consequence of (a):

5.4. Lemma. (i) *Let $\{\theta_i\}_{i \in I_0}$ be a anonical system for $\mathscr{L}$. Then $\{\theta_i|_A\}_{i \in I_1}$ is a canonical system for $\mathscr{A}$ and $\{\theta_i|_B\}_{i \in I_2}$ is a canonical system for $\mathscr{B}$.*

(ii) Let $\{\theta_i\}_{i \in I_1^0}$ be a canonical system for $\mathscr{A}$ and let $\{\theta_i\}_{i \in I_2^0}$ be a canonical system for $\mathscr{B}$. Then $\{\theta_i^*\}_{i \in I_1^0 \cup I_2^0}$ is a canonical system for $\mathscr{L}$.

The following assertion is obvious:

5.5. Lemma. *If $\theta \in \operatorname{Con} \mathscr{B}$, then $\theta \in \operatorname{Con} \mathscr{B}^{\sim}$ and $\mathscr{B}/\theta$ is dually isomorphic to $\mathscr{B}^{\sim}/\theta$.*

If $[x_1, y_1]$ is a prime interval in $\mathscr{A}$ and $\theta = \theta_0(x_1, y_1)$, then for each pair of elements $x_2, y_2 \in B$ we have $x_2 \, \theta \, y_2$; hence $\mathscr{L}/\theta \cong \mathscr{A}/(\theta/_A)$. Similarly, if $[x_2, y_2]$ is a prime interval in $\mathscr{B}$ and $\theta = \theta_0(x_2, y_2)$, then $\mathscr{L}/\theta \cong \mathscr{B}/(\theta/_B)$. Thus 5.3, 5.4 and 5.5 yield:

5.6. Theorem. *Let $\mathscr{L} = (L; \leqq)$ and $\mathscr{L}_1 = (L; \leqq_1)$ be lattices fulfilling* (b) *and* (c). *Let a canonical subdirect representation of $\mathscr{L}$ with subdirect factors $\mathscr{L}/\theta_i$ $(i \in I)$ be given. Then $\{\theta_i\}$ $(i \in I)$ is a canonical system for $\mathscr{L}_1$ and for each $i \in I$, either $\mathscr{L}/\theta_i$ is isomorphic to $\mathscr{L}_1/\theta_i$ or $\mathscr{L}/\theta_i$ is dually isomorphic to $\mathscr{L}_1/\theta_i$.*

If the lattices $\mathscr{L} = (L; \leqq)$ and $\mathscr{L}_1 = (L_1; \leqq_1)$ and the bijection $h: L \to L_1$ fulfil (α), then an assertion analogous to 5.6 holds.

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