Czechoslovak Mathematical Journal

Štefan Schwarz

Common consequents in directed graphs

*Czechoslovak Mathematical Journal*, Vol. 35 (1985), No. 2, 212–247

Persistent URL: http://dml.cz/dmlcz/102012

# COMMON CONSEQUENTS IN DIRECTED GRAPHS

ŠTEFAN SCHWARZ, Bratislava

(Received June 1, 1983)

Let $V = \{a_1, a_2, \ldots, a_n\}$ be a finite set with $n \geqq 2$ different elements. By a binary relation on $V$ we mean a subset $\varrho$ of $V \times V$. Let $B_n(V)$ be the set of all binary relations on $V$ (including the empty relation). Then under the usual multiplication of binary relations $B_n(V)$ becomes a semigroup.

Let $M_n$ denote the set of all $n \times n$ matrices over the Boolean algebra $\{0, 1\}$. Then $M_n$ is a semigroup under the Boolean matrix multiplication. The map

$$\varrho \to M(\varrho) = (m_{ij}), \tag{1}$$

where $m_{ij} = 1$ if $(a_i, a_j) \in \varrho$ and $m_{ij} = 0$ otherwise, is an isomorphism of $B_n(V)$ onto $M_n$. In particular,

$$\varrho \,.\, \sigma \to M(\varrho) \,.\, M(\sigma) = M(\varrho \,.\, \sigma)\,.$$

$B_n(V)$ is closed under the set-theoretical union $\varrho \cup \sigma$ and $M_n$ admits an addition $M' + M''$ (the sum of Boolean matrices). The map (1) preserves these operations, i.e.

$$\varrho \cup \sigma \to M(\varrho) + M(\sigma) = M(\varrho \cup \sigma)\,.$$

Let $G_n(V)$ be the set of all directed graphs with $n$ vertices $\{a_1, a_2, \ldots, a_n\}$ with allowable loops and simple directed edges. Each matrix in $M_n$ can be considered as the adjacency matrix of a directed graph $H$ in $G_n(V)$ and determines $H$ uniquely up to an isomorphism. Conversely, each graph in $G_n(V)$ with labelled vertices determines a unique Boolean matrix in $M_n$ as its adjacency matrix.

We have a one-to-one correspondence between $B_n(V)$, $M_n$ and $G_n(V)$:

$$\varrho \leftrightarrow M(\varrho) \leftrightarrow G(\varrho)\,.$$

Here $G(\varrho)$ is the graph corresponding to the matrix $M(\varrho)$.

In the following we shall freely use these obvious correspondences.

If $\varrho$ is given and $a_i \in V$, we define $a_i\varrho = \{x \in V: (a_i, x) \in \varrho\}$. If $K$ is a non-empty subset of $V$, we put $K\varrho = \bigcup_{a_i \in K} a_i\varrho$.

The next notions concern the (finite directed) graph $G(\varrho)$. A sequence of vertices $\langle a_{i_0}, a_{i_1}, \ldots, a_{i_k} \rangle$ is a *path of length* $k$ in $G(\varrho)$ if every pair of adjacent vertices in this sequence is in $\varrho$. An *edge* is a path of length 1.

A vertex $a_j$ is a *consequent of length $k$ of a vertex $a_i$* if there is a (directed) path of length $k$ beginning with $a_i$ and ending with $a_j$. This is equivalent to the statement $a_j \in a_i\varrho^k$.

A path of the form $\langle a_{i_0}, a_{i_1}, \ldots, a_{i_0}\rangle$ is called a *cycle*. If all its elements with the exception of the last one are pairwise different, the cycle is called a *circuit of length $k$*. A circuit $\langle a_i, a_i\rangle$ is a *loop*.

Suppose there is a circuit going through a fixed chosen vertex $a_i \in V$, i.e. $a_i \in a_i\varrho^k$ for some integer $k > 0$. Denote by $h_i$ the least integer such that $a_i \in a_i\varrho^{h_i}$ (i.e. $a_i \notin a_i\varrho^h$ for $h < h_i$). Then any circuit of length $h_i$ going through $a_i$ will be called an *elementary circuit belonging to the vertex $a_i$* (or an elementary circuit going through $a_i$).

**Definition.** Let $\varrho$ be given. We shall say that a pair of vertices $(a_i, a_j)$, $a_i \neq a_j$, has a *common consequent* (*c.c.*) if there is an integer $l > 0$ such that

(2) $$a_i\varrho^l \cap a_j\varrho^l \neq \emptyset\,.$$

In terms of Boolean matrices this means: The rows corresponding to $a_i$ and $a_j$ in $M(\varrho^l)$ have a 1 in the same column.

Example 0,1. Let $V = \{a_1, a_2, a_3\}$ and let $\varrho_1$ be defined by

$$M(\varrho_1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

We have

$$M(\varrho_1^2) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \qquad M(\varrho_1^3) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

The pair $(a_1, a_2)$ has a common consequent of length 3 since

$$a_1\varrho_1 \cap a_2\varrho_1 = \emptyset\,, \quad a_1\varrho_1^2 \cap a_2\varrho_1^2 = \emptyset\,, \quad a_1\varrho_1^3 \cap a_2\varrho_1^3 = \{a_1\} \neq \emptyset\,.$$

We shall often use diagrams of the form

$$a_1 \to a_3 \to a_2 \to a_1\,,$$

$$a_2 \to a_1 \to a_3 \to a_1\,,$$

to specify the paths by which the *c.c.* can be reached.

Let again $V = \{a_1, a_2, a_3\}$ and let $\varrho_2$ be defined by

$$M(\varrho_2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is easy to see that none of the pairs $(a_1, a_2), (a_2, a_3), (a_1, a_3)$, have a common consequent.

Example 0,2. This example shows that some care is necessary. Let $V = \{a_1, a_2, a_3\}$ and let $\varrho$ be defined by

$$M(\varrho) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then

$$M(\varrho^2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad M(\varrho^3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence $a_2\varrho \cap a_3\varrho = \{a_1\} \neq \emptyset$, while $a_2\varrho^2 \cap a_3\varrho^2 = \emptyset$.

We show that this situation cannot occur if $M(\varrho)$ has no zero row.

By the first projection $\Pi(\varrho)$ of $\varrho$ we mean the subset of $V$ consisting of all $a_i \in V$ for which $a_i\varrho \neq \emptyset$.

If $\Pi(\varrho) = V$ and $a_i\varrho = \{a_k, a_l, \ldots\}$, then $a_i\varrho^2 = a_i\varrho \,.\, \varrho = \{a_k, a_l, \ldots\}\, \varrho \neq \emptyset$ and, repeating this argument, we obtain $\Pi(\varrho^l) = V$ for any $l > 0$.

If $\Pi(\varrho) = V$, then $a_i\varrho^l \cap a_j\varrho^l \neq \emptyset$ implies the existence of an $a_k \in V$ such that $a_k \in a_i\varrho^l$, $a_k \in a_j\varrho^l$, hence $a_k\varrho^t \subset a_i\varrho^{l+t}$, $a_k\varrho^t \subset a_j\varrho^{l+t}$, whence $a_i\varrho^{l+t} \cap a_j\varrho^{l+t} \neq \emptyset$ for any integer $t > 0$. We state this fact explicitly:

**Lemma 0,1.** *If $\Pi(\varrho) = V$, then $a_i\varrho^l \cap a_j\varrho^l \neq \emptyset$, $\{a_i, a_j\} \in V$, implies $a_i\varrho^{l+t} \cap$ $\cap\, a_j\varrho^{l+t} \neq \emptyset$ for any integer $t > 0$.*

**Definition.** Let $\varrho \in B_n(V)$, $V = \{a_1, \ldots, a_n\}$. If $a_i, a_j$ have a common consequent, then the least integer $l \geqq 1$ for which $a_i\varrho^l \cap a_j\varrho^l \neq \emptyset$ will be denoted by $L(a_i, a_j)$.

To shorten the terminology, if $a_i, a_j$ have a common consequent, we shall say that $L(a_i, a_j)$ exists.

If there is at least one couple $(a_i, a_j)$ for which $L(a_i, a_j)$ exists, we define $L(\varrho) =$ $= \max L(a_i, a_j)$, where $(a_i, a_j)$ runs through all couples for which $L(a_i, a_j)$ exists. If $M = M(\varrho)$ is the Boolean matrix corresponding to $\varrho$ in the correspondence (1), we shall write $L(M) = L(\varrho)$. If, finally, there is no couple $(a_i, a_j)$ for which $L(a_i, a_j)$ exists, we define $L(\varrho) = L(M) = 0$. [Note that in contradistinction to this definition $L(a_i, a_j)$ is either $\geqq 1$ or does not exist.]

The cardinality of a set $Q$ will be denoted by $|Q|$.

The following lemma is known. (See A. Paz [1], p. 89.)

**Lemma 0,2.** *Let $\varrho \in B_n(V)$, $V = \{a_1, \ldots, a_n\}$, $\{a_i, a_j\} \in V$. If $L(a_i, a_j)$ exists, then $L(a_i, a_j) \leqq \frac{1}{2}n(n - 1)$.*

Proof essentially repeats that given in [1]. We include it not only for the sake of completeness, but rather since the modifications of the method used here will be used several times in the sequel.

Denote $t = L(a_i, a_j)$ and consider the paths

$$a_i = a_i^{(0)} \to a_i^{(1)} \to \ldots a_i^{(t-1)} \to a_i^{(t)} = a\,,$$

$$a_j = a_j^{(0)} \to a_j^{(1)} \to \ldots a_j^{(t-1)} \to a_j^{(t)} = a\,.$$

Here $a \in a_i\varrho^t \cap a_j\varrho^t$, while for all $l < t$ we have $a_i\varrho^l \cap a_j\varrho^l = \emptyset$.

The sequence of pairs

$$(3) \qquad \begin{pmatrix} a_i^{(0)} \\ a_j^{(0)} \end{pmatrix}, \begin{pmatrix} a_i^{(1)} \\ a_j^{(1)} \end{pmatrix}, \ldots, \begin{pmatrix} a_i^{(t-1)} \\ a_j^{(t-1)} \end{pmatrix}, \begin{pmatrix} a \\ a \end{pmatrix}$$

cannot contain two equal pairs, since if

$$\begin{pmatrix} a_i^{(k)} \\ a_j^{(k)} \end{pmatrix} = \begin{pmatrix} a_i^{(k+u)} \\ a_j^{(k+u)} \end{pmatrix}$$

with $u \geqq 1$, we may omit in (3) the segment

$$\begin{pmatrix} a_i^{(k+1)} \\ a_j^{(k+1)} \end{pmatrix}, \ldots, \begin{pmatrix} a_i^{(k+u)} \\ a_j^{(k+u)} \end{pmatrix}$$

which yields $a_i\varrho^{t-u} \cap a_j\varrho^{t-u} \neq \emptyset$, a contradiction with the definition of $t$.

We next show that only one of the pairs of the form

$$\begin{pmatrix} a' \\ a'' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a'' \\ a' \end{pmatrix} \; [a' \in V,\ a'' \in V,\ a' \neq a'']$$

can appear in (3).

Indeed, suppose that

$$\begin{pmatrix} a_i^{(l)} \\ a_j^{(l)} \end{pmatrix} = \begin{pmatrix} a' \\ a'' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_i^{(l+u)} \\ a_j^{(l+u)} \end{pmatrix} = \begin{pmatrix} a'' \\ a' \end{pmatrix}$$

for some $u > 0$. The first equality implies that $v = t - l$ is the least integer for which $a'\varrho^v \cap a''\varrho^v \neq \emptyset$. The second one says that $a''\varrho^w \cap a'\varrho^w \neq \emptyset$ for $w = t - (l + u)$. Since $w < v$, we have a contradiction.

Now the number of all unordered pairs $\begin{pmatrix} a' \\ a'' \end{pmatrix}$ with $a' \neq a''$ $[a', a'' \in V]$ is equal to $\frac{1}{2}n(n-1)$. Hence in (3) we have $t \leqq \frac{1}{2}n(n-1)$. This proves Lemma 0,2.

A. Paz [1] remarks that it is not known whether the bound $\frac{1}{2}n(n-1)$ is sharp. He also remarks that the difference between the above bound and any sharper bound is of order of magnitude $\frac{1}{2}n$.

The purpose of this paper is to prove that the above bound is not sharp. We prove that for any binary relation $\varrho \in B_n(V)$ and $a_i, a_j \in V$, if $L(a_i, a_j)$ exists, then

$$(4) \qquad L(a_i, a_j) \leqq \tfrac{1}{2}n^2 - n + \varepsilon_n,$$

where

$$\varepsilon_n = \begin{cases} 1 & \text{if } n \text{ is even}, \\ \frac{3}{2} & \text{if } n \text{ is odd}. \end{cases}$$

Moreover, this result is the best possible.

The formulation of this result in terms of directed graphs or in terms of Boolean matrices is obvious.

Though we shall often use graph-theoretical methods, the use of binary relations seems to have many advantages. For instance, $a_i\varrho^l$ describes all vertices accessible from the vertex $a_i$ by a path of length $l$. Note also that we shall have to deal with "high" values of $l$ (i.e. values of order $\frac{1}{2}n^2$).

We prove our statement in several steps. We first prove it for primitive relations. This is the essential part. For irreducible but not primitive relations we find estimates which are better than (4). Finally we treat the case of a "general" relation.

The function $N(n) = \frac{1}{2}n^2 - n + \varepsilon_n$ defined for all integers $n \geqq 2$ plays an essential role in the following. Here we give the first eight values of $N(n)$:

$$\begin{aligned} N(2) &= 1\,, & N(6) &= 13\,,\\ N(3) &= 3\,, & N(7) &= 19\,,\\ N(4) &= 5\,, & N(8) &= 25\,,\\ N(5) &= 9\,, & N(9) &= 33\,. \end{aligned}$$

We now prove our statement in two rather trivial cases.

Suppose that $M(\varrho)$ contains a zero row, say the row corresponding to $a^* \in V$, i.e. $a^*\varrho = \emptyset$. We may suppose $n \geqq 3$. Clearly $L(a^*, a_j)$, $a_j \neq a^*$, does not exist. Let therefore $a_j \neq a^*$, $a_i \neq a^*$, $a_i \neq a_j$. The sequence arising from the paths joining $a_i$, $a_j$ with a common consequent $a^{(t)}$,

$$\binom{a_i}{a_j} = \binom{a_i^{(0)}}{a_j^{(0)}}, \binom{a_i^{(1)}}{a_j^{(1)}}, \ldots, \binom{a^{(t)}}{a^{(t)}}, \tag{5}$$

cannot contain a couple of the form

$$\binom{a^*}{a'} \quad \text{or} \quad \binom{a'}{a^*} \quad \text{with} \quad a' \neq a^*\,.$$

Suppose that (5) is the shortest possible sequence. We use the same argument as in Lemma 0,2. The number of all unordered pairs

$$\binom{a_i^{(v)}}{a_j^{(v)}} \quad \text{with} \quad a_i^{(v)} \neq a_j^{(v)}$$

in (5) is at most $\frac{1}{2}(n-1)(n-2) = \frac{1}{2}n^2 - \frac{3}{2}n + 1$, so that

$$L(a_i, a_j) \leqq \tfrac{1}{2}n^2 - \tfrac{3}{2}n + 1 = N(n) - (\tfrac{1}{2}n + \varepsilon_n - 1)\,.$$

We have proved:

**Lemma 0,3.** *Let $\varrho \in B_n(V)$, $n \geqq 3$, and suppose that $M(\varrho)$ contains a zero row. If $L(a_i, a_j)$ exists, we have*

$$L(a_i, a_j) \leqq \begin{cases} N(n) - \frac{1}{2}n & \text{if } n \text{ is even}\,,\\ N(n) - \dfrac{n+1}{2} & \text{if } n \text{ is odd}\,. \end{cases}$$

Remark. Here it may happen that the *c.c.* is the vertex $a^*$. As a simple example consider the relation $\varrho$ with

$$M(\varrho) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Here $L(a_1, a_3) = 1$ and the *c.c.* of $a_1, a_3$ is $a_2 = a^*$.

Suppose next that $M(\varrho)$ contains a zero column, say the column corresponding to $a^{**} \in V$. If $n = 2$ and $L(a_i, a_j)$ exists, then $L(a_i, a_j) = 1$. Suppose therefore in the following $n \geqq 3$.

If $a_i \neq a^{**}$, $a_j \neq a^{**}$, $a_i \neq a_j$, then the sequence (5) cannot contain a couple of the form

$$\begin{pmatrix} a^{**} \\ a' \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a' \\ a^{**} \end{pmatrix}$$

[since $a^{**}$ is inaccessible from any $a_i \in V$]. The number of pairs in (5) is again at most $\binom{n-1}{2}$.

If $a = a^{**}$, i.e. in (5) we begin with $\begin{pmatrix} a^{**} \\ a_j \end{pmatrix}$ [which is possible], we obtain a sequence containing at most $\binom{n-1}{2} + 1$ different terms so that

$$L(a_i, a_j) \leqq \frac{n^2}{2} - \frac{3}{2} n + 2 = N(n) - \left(\frac{n}{2} + \varepsilon_n - 2\right).$$

We have proved:

**Lemma 0,4.** *Let $\varrho \in B_n(V)$, $n \geqq 3$, and suppose that $M(\varrho)$ contains a zero column. If $L(a_i, a_j)$ exists, then*

$$L(a_i, a_j) \leqq \begin{cases} N(n) - \left(\dfrac{n}{2} - 1\right) & \text{if } n \text{ is even}, \\ N(n) - \frac{1}{2}(n+1) & \text{if } n \text{ is odd}. \end{cases}$$

Lemmas 0,3 and 0,4 [which show that (4) holds in the special cases just considered] will be used in Section 5.

Example 0,3. We close this section by giving a method how to find a *c.c.* (if it exists). Consider the relation $\varrho$ with

$$M(\varrho) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

To find $L(a_2, a_5)$ we write down (in an obvious notation) the sequence of "rows"

$$\begin{pmatrix} a_2\varrho^k \\ a_5\varrho^k \end{pmatrix}, \quad k = 1, 2, 3, \ldots :$$

$$\begin{pmatrix} a_2 \\ a_5 \end{pmatrix}, \begin{pmatrix} a_3 \\ a_1 \end{pmatrix}, \begin{pmatrix} a_4 \\ a_2 \end{pmatrix}, \begin{pmatrix} a_1, a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} a_2, a_3 \\ a_4 \end{pmatrix}, \begin{pmatrix} a_3, a_4 \\ a_1, a_2 \end{pmatrix}, \begin{pmatrix} a_4, a_1, a_2 \\ a_2, a_3 \end{pmatrix}.$$

Hence $a_2\varrho^6 \cap a_5\varrho^6 = \{a_2\}$, while $a_2\varrho^l \cap a_5\varrho^l = \emptyset$ for $l \leqq 5$. Here we have $L(a_2, a_5) = 6 = N(5) - 3$, which shows that the bound from Lemma 0,4 is achieved. The scheme "contains" any of the shortest paths joining $a_2$ and $a_5$ with the common consequent $\{a_2\}$. Of course some of the transitions (e.g. $a_2 \to a_2$ or $a_1 \to a_3$) are forbidden. In our example we have a unique possibility:

$$a_2 \to a_3 \to a_4 \to a_2 \to a_3 \to a_4 \to a_2 ,$$
$$a_5 \to a_1 \to a_2 \to a_3 \to a_4 \to a_1 \to a_2 .$$

## 1. PRELIMINARIES

Motivated by the terminology in the study of non-negative matrices we define:

A Boolean square matrix $A$ is called *reducible* if there is a permutation matrix $P$ such that $PAP^{-1}$ is of the form

$$\begin{pmatrix} B & O \\ C & D \end{pmatrix},$$

where $B$, $D$ are square matrices. Otherwise it is called irreducible.

A relation $\varrho \in B_n(V)$ is called reducible (irreducible) if $M(\varrho)$ is reducible (irreducible).

It follows that $1 \times 1$ matrices are irreducible.

In general, any Boolean square matrix is permutation cogredient to a matrix of the form

$$\begin{pmatrix} A_1 & 0 & \ldots & 0 \\ A_{21} & A_2 & \ldots & 0 \\ \vdots & & & \\ A_{k1} & A_{k2} & \ldots & A_k \end{pmatrix},$$

where $A_i$ are irreducible.

We now recall some results concerning irreducible relations. (See, e.g., [2].)

**Lemma 1,1.** *Let $\varrho \in B_n(V)$, $n \geqq 2$. The following statements are equivalent:*

a) *$\varrho$ is irreducible;*

b) $\varrho \cup \varrho^2 \cup \ldots \cup \varrho^n = V \times V$;

c) $a_i\varrho \cup a_i\varrho^2 \cup \ldots \cup a_i\varrho^n = V$ *for any* $a_i \in V$;

d) *$G(\varrho)$ is strongly connected.*

Note explicitly: If $\varrho$ is irreducible, then $M(\varrho^t)$ cannot have a zero row or a zero column (for any integer $t > 0$).

The statement c) implies:

**Lemma 1,2.** *Let $\varrho$ be irreducible, $\varrho \in B_n(V)$. Then for any $a_i \in V$ there is a least integer $h_i$, $1 \leqq h_i \leqq n$, such that $a_i \in a_i\varrho^{h_i}$.*

Given $\varrho$ we have defined a function $h: V \to \{1, 2, \ldots, n\}$ such that $h(a_i) = h_i$. This notation will be used throughout the paper.

**Definition.** *Let $\varrho$ be irreducible, $\varrho \in B_n(V)$. The greatest common divisor $d = d(\varrho) = (h_1, h_2, \ldots, h_n)$ is called the index of imprimitivity of $\varrho$. Clearly $d = d(\varrho) \leqq n$.*

Another characterization of $d(\varrho)$ is obtained by considering the matrix representation $M(\varrho)$.

**Lemma 1,3.** *If $\varrho$ is irreducible and $d = d(\varrho) > 1$, then the set $V = \Pi(\varrho)$ admits a decomposition into $d$ disjoint non-empty subsets $V = V_1 \cup V_2 \cup \ldots \cup V_d$ such that*

(7) $$\varrho \subset (V_1 \times V_2) \cup (V_2 \times V_3) \cup \ldots \cup (V_d \times V_1).$$

Hence $M(\varrho)$ is permutation cogredient to a matrix of the form

$$\begin{pmatrix} 0 & A_1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \ldots & 0 & A_{d-1} \\ A_d & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

If $|V_i| = v_i$ and $v_{d+1} = v_1$, the zero matrices in the main diagonal are $v_i \times v_i$ square matrices, while $A_i$ is a $v_i \times v_{i+1}$ Boolean rectangular matrix.

**Definition.** *The sets $V_1, V_2, \ldots, V_d$ are called the sets of imprimitivity of $\varrho$.*

It follows from (7) that

$$\varrho^d \subset (V_1 \times V_1) \cup (V_2 \times V_2) \cup \ldots \cup (V_d \times V_d)$$

and it is well known that there is an integer $k_0 > 0$ such that for all $k \geqq k_0$,

$$\varrho^{dk} = (V_1 \times V_1) \cup (V_2 \times V_2) \cup \ldots \cup (V_d \times V_d).$$

This implies:

**Lemma 1,4.** *Let $\varrho$ be irreducible, $d \geqq 1$ and let $V'$ be one of the sets of imprimitivity of $\varrho$. If $a_i \in V'$, then there is an integer $k_0 > 0$ such that for any $k \geqq k_0$ we have $a_i\varrho^{kd} = V'$.*

To find explicitly the sets $V_1, V_2, \ldots, V_d$ we may use the following

**Lemma 1,5.** *Let $\varrho$ be irreducible, $d(\varrho) > 1$ and $a_i \in V = \Pi(\varrho)$.*

a) *The sets*

(8) $$a_i\varrho^l, a_i\varrho^{l+1}, \ldots, a_i\varrho^{l+d-1}$$

*are disjoint subsets of $V$ for any $l \geqq 1$.*

b) *There is an integer* $l_0 > 0$ *such that for any* $l \geqq l_0$ (*and any* $a_i \in V$) *the sets* (8) *constitute exactly all the sets* $V_1, V_2, \ldots, V_d$.

c) *If* $V'$ *is one of the sets of imprimitivity of* $\varrho$, *then* $V', V'\varrho, \ldots, V'\varrho^{d-1}$ *are all the sets of imprimitivity of* $\varrho$.

Remark. Bounds for $k_0$ and $l_0$ depending on $n$ and $d$ have been given in [3]. We shall not need them in this paper.

Example 1,1. Consider the relation $\varrho$ defined on $V = \{a_1, a_2, a_3, a_4\}$ with

$$M(\varrho) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$\varrho$ is irreducible since $G(\varrho)$ is strongly connected. We have, for instance,

$$a_2\varrho = \{a_3\}, \quad a_2\varrho^2 = \{a_1\}, \quad a_2\varrho^3 = \{a_2, a_4\}, \quad a_2\varrho^4 = \{a_3\}, \ \ldots$$

Here $l_0 = 1$ and $V_1 = \{a_3\}$, $V_2 = \{a_1\}$, $V_3 = \{a_2, a_4\}$. Hence $d = 3$. Also $h(a_i) = 3$ for $i = 1, 2, 3, 4$. Further

$$\varrho = (V_1 \times V_2) \cup (V_2 \times V_3) \cup (V_3 \times V_1),$$

and $M(\varrho)$ is permutation cogredient with the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

**Definition.** *A relation* $\varrho$ *is called primitive if there is an integer* $t > 0$ *such that* $\varrho^t = V \times V$.

A relation is primitive iff it is irreducible and $d(\varrho) = 1$.

Note also: If $\varrho$ is primitive, there is an integer $t_0 > 0$ such that for $t \geqq t_0$ we have $a_i\varrho^t = \{a_1, a_2, \ldots, a_n\} = V$ for any $a_i \in V$.

**Corollary 1,1.** *If* $\varrho \in B_n(V)$ *is primitive, then any two vertices* $a_i, a_j \in V$ *have a common consequent. That is,* $L(a_i, a_j)$ *exists for any pair* $(a_i, a_j)$ *where* $a_i \neq a_j$.

**Theorem 1,1.** *Let* $\varrho \in B_n(V)$. *Suppose that* $\varrho$ *is irreducible and* $d(\varrho) > 1$. *Then* $L(a_i, a_j)$ *exists iff* $a_i$ *and* $a_j$ *are contained in the same set of imprimitivity of* $\varrho$.

Proof. a) Suppose that $a_i \in V'$, $a_j \in V'$. Then (by Lemma 1,4) there is an integer $k_0$ such that for any $k \geqq k_0$ we have $a_i\varrho^{dk} = V' = a_j\varrho^{dk}$. Hence $L(a_i, a_j)$ exists.

b) Let $a_i \in V'$, $a_j \in V''$ and $V' \neq V''$. By Lemma 1,4 there is a $k_0 > 0$ such that $a_i\varrho^{dk} = V'$, $a_j\varrho^{dk} = V''$ for any $k \geqq k_0$. Suppose (for an indirect proof) that there is an $l > 0$ such that $a_i\varrho^l \cap a_j\varrho^l \neq \emptyset$. Choose $k$ such that $dk - l > 0$. By Lemma 0,1 (multiplying by $\varrho^{dk-l}$) we then have $a_i\varrho^{dk} \cap a_j\varrho^{dk} \neq 0$, $V' \cap V'' \neq \emptyset$, a contradiction with the fact that $V'$ and $V''$ are disjoint. Hence $L(a_i, a_j)$ does not exist.

The goal of this section is to prove Theorem 2,3. We begin with a series of lemmas.

**Lemma 2,1.** *Let $\varrho$ be irreducible, $\varrho \in B_n(V)$, $n \geqq 2$ and let $Q$ be a non-empty proper subset of $V$. Then $Q \,.\, \varrho$ contains at least one element of $V$ which is not contained in $Q$.*

Proof. Let $Q = \{a_\alpha, a_\beta, \dots, a_\nu\}$. Suppose for an indirect proof that

$$\{a_\alpha, a_\beta, \dots, a_\nu\} \,.\, \varrho \subset \{a_\alpha, a_\beta, \dots, a_\nu\} \,.$$

Let $(a_\varkappa, a_\lambda) \in \varrho$. If $a_\varkappa \in Q$, we necessarily have $a_\lambda \in Q$. Hence if $a_\varkappa \in Q$, $a_\lambda \in V - Q = B$, then $(a_\varkappa, a_\lambda) \notin \varrho$. Therefore

$$\varrho \subset (Q \times Q) \cup (B \times Q) \cup (B \times B)\,,$$

i.e. $\varrho$ is reducible, contrary to the assumption.

**Lemma 2,2.** *Let $\varrho$ be primitive, $\varrho \in B_n(V)$, $n \geqq 2$ and $a_i \in V$. If $a_i\varrho^s = a_i\varrho^t$ for some $1 \leqq s < t$, then $|a_i\varrho^s| = n$.*

Proof. The supposition $a_i\varrho^s = a_i\varrho^{s+(t-s)}$ implies $a_i\varrho^s = a_i\varrho^{s+l(t-s)}$ for any integer $l \geqq 1$. Since for a sufficiently large $l$ the right hand side is $V$, we have $a_i\varrho^s = V$, i.e. $|a_i\varrho^s| = n$.

Note that if $\varrho$ is primitive, $\varrho^t$ is primitive for any $t > 1$.

Let $\varrho$ be primitive, $\varrho \in B_n(V)$, $n \geqq 2$ and consider the chain:

$$a_i \in a_i\varrho^{h_i} \subset a_i\varrho^{2h_i} \subset \dots \subset a_i\varrho^{[n/2]h_i}\,. \tag{9}$$

Here (and in the following) $[x]$ is the largest integer $\leqq x$. By Lemma 2,1 $a_i\varrho^{h_i}$ contains at least two elements, by Lemma 2,1 and Lemma 2,2 $a_i\varrho^{2h_i}$ contains at least three elements of $V$, etc. Hence

$$|a_i\varrho^{[n/2]h_i}| \geqq \left[\frac{n}{2}\right] + 1\,.$$

If $a_j \neq a_i$, we analogously have $|a_j\varrho^{[n/2]h_j}| \geqq [n/2] + 1$.

If $h_i = h_j$, we have

$$a_i\varrho^{[n/2]h_i} \cap a_j\varrho^{[n/2]h_j} \neq \emptyset\,,$$

hence $L(a_i, a_j) \leqq [n/2]\, h_i$. If for instance $h_i < h_j$, multiply each term in (9) by $\varrho^{[n/2](h_j - h_i)}$. We obtain the following chain of length $[n/2] + 1$:

$$a_i\varrho^{[n/2](h_j-h_i)} \subset a_i\varrho^{h_i+[n/2](h_j-h_i)} \subset \dots \subset a_i\varrho^{[n/2]h_j}\,,$$

whence $|a_i\varrho^{[n/2]h_j}| \geqq [n/2] + 1$. Therefore $a_i\varrho^{[n/2]h_j} \cap a_j\varrho^{[n/2]h_j} \neq \emptyset$, and $L(a_i, a_j) \leqq [n/2]\, h_j$. Analogously if $h_j < h_i$.

We have proved:

**Lemma 2,3.** *Suppose that $\varrho \in B_n(V)$, $n \geqq 2$, and $\varrho$ is primitive. Then $L(a_i, a_j) \leqq \leqq [n/2] \max(h_i, h_j)$.*

Recall that $h_i \leqq n$. If $n = 3$, we have $L(a_i, a_j) \leqq 3$. This bound is achieved since in Example 0,1 we have $L(a_1, a_2) = 3$. If $n = 2$, there are only three primitive relations, namely those for which the matrix representations are:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Hence $L(a_1, a_2) = 1$.

We note this explicitly, since in the following we shall often suppose $n \geqq 4$.

**Corollary 2,3.** *Let $\varrho$ be primitive, $\varrho \in B_n(V)$. If $n = 2$, $L(a_1, a_2) = 1$. If $n = 3$, $L(a_i, a_j) \leqq 3$.*

Lemma 2,3 immediately implies:

**Theorem 2,1.** *Let $\varrho$ be primitive, $\varrho \in B_n(V)$ and $n \geqq 3$. If $h(a_i) \leqq n - 2$, $h(a_j) \leqq \leqq n - 2$, then*

$$L(a_i, a_j) \leqq \begin{cases} N(n) - 1 & \text{if } n \text{ is even}, \\ N(n) - \frac{1}{2}(n + 1) & \text{if } n \text{ is odd}. \end{cases}$$

Proof. For $n$ even we have $L(a_i, a_j) \leqq \frac{1}{2}n(n - 2) = N(n) - 1$. For $n$ odd we have $L(a_i, a_j) \leqq \frac{1}{2}(n - 1)(n - 2) = N(n) - \frac{1}{2}(n + 1)$.

There is a simple case which can be treated directly.

**Lemma 2,4.** *If $\varrho$ is primitive, $\varrho \in B_n(V)$ and $G(\varrho)$ contains a loop, then $L(a_i, a_j) \leqq \leqq n - 1$ (for any $a_i, a_j \in V$).*

Proof. Suppose that $a_l \in a_l\varrho$, and $a_i \neq a_l$, $a_j \neq a_l$, $a_i \neq a_j$. Since $G(\varrho)$ is strongly connected, there is a path of length $k_1$, $1 \leqq k_1 \leqq n - 1$, joining $a_i$ with $a_l$ and a path of length $k_2$, $1 \leqq k_2 \leqq n - 1$, joining $a_j$ with $a_l$, i.e., $a_l \in a_i\varrho^{k_1}$, $a_l \in a_j\varrho^{k_2}$. Using the loop (several times if necessary) we obtain $a_l \in a_i\varrho^{n-1}$, $a_l \in a_j\varrho^{n-1}$, hence $L(a_i, a_j) \leqq n - 1$. The same argument can be aplied also in the case if $a_i$ or $a_j$ coincides with $a_l$.

The next lemma may be considered a generalization of Lemma 2,4.

Note that if $\varrho$ is primitive, then $d(\varrho) = 1 = (h_1, h_2, \ldots, h_n)$ implies that not all $h_i$ can be equal to $n$. Hence the length of the shortest elementary circuit in $G(\varrho)$ is $\leqq n - 1$.

**Lemma 2,5.** *Let $\varrho$ be primitive, $\varrho \in B_n(V)$ and $n \geqq 4$. Let $h_0 \geqq 1$ be the length of the shortest elementary circuit in $G(\varrho)$. Denote $L_1 = ([n/2] - 1) h_0 + n$. Then for any $a_i \in V$ we have*

$$|a_i\varrho^{L_1}| \geqq \left[\frac{n}{2}\right] + 1.$$

Proof. Denote by $C = \langle u_1, u_2, \ldots, u_{h_0}, u_1 \rangle$ one of the elementary circuits of length $h_0$. [Here $h_0 \leqq n - 1$ and $u_i \in V$.] For each $u \in C$ we have $u \in u\varrho^{h_0}$. For any $a_i \in V - \{u_1, u_2, \ldots, u_{h_0}\}$ there is a path of length $k_i$, $1 \leqq k_i \leqq n - h_0$, joining $a_i$ with some $u_j \in C$. This means: there is a $u_j \in C$ such that $u_j \in a_i\varrho^{k_i}$, where $k_i \leqq \leqq n - h_0$. Consider the chain

$$u_j \in u_j\varrho^{h_0} \subset u_j\varrho^{2h_0} \subset \ldots \subset u_j\varrho^{[n/2]h_0}$$

and (for any integer $t \geqq 1$) the chain

$$u_j\varrho^{t} \subset u_j\varrho^{h_0+t} \subset u_j\varrho^{2h_0+t} \subset \ldots \subset u_j\varrho^{[n/2]h_0+t} .$$

For any $t \geqq 0$ we have $|u_j\varrho^{[n/2]h_0+t}| \geqq [n/2] + 1$.

Now, since $u_j \in a_i\varrho^{k_i}$ we have

$$\left[\frac{n}{2}\right] + 1 \leqq |u_j\varrho^{[n/2]h_0+t} \leqq |a_i\varrho^{[n/2]h_0+t+k_i}| .$$

Putting $t = n - h_0 - k_i \geqq 0$, we have

$$|a_i\varrho^{L_1}| \geqq \left[\frac{n}{2}\right] + 1 ,$$

which proves our lemma in the case that $a_i$ is outside of $C$.

If $u$ belongs to $C$, the chains

$$u \in u\varrho^{h_0} \subset u\varrho^{2h_0} \subset \ldots \subset u\varrho^{[n/2]h_0} ,$$

$$u\varrho^{t} \subset u\varrho^{h_0+t} \subset u\varrho^{2h_0+t} \subset \ldots \subset u\varrho^{[n/2]h_0+t} ,$$

show that for any $t \geqq 0$,

$$|u\varrho^{[n/2]h_0+t}| \geqq \left[\frac{n}{2}\right] + 1 .$$

In particular, with $t = n - h_0$ we obtain $|u\varrho^{L_1}| \geqq [n/2] + 1$.

This proves Lemma 2,5.

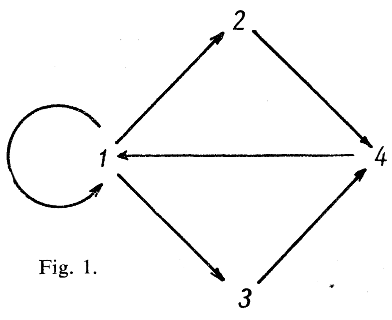


Fig. 1.

Remark 1. Some care is necessary when dealing with inequalities containing expressions as $|a_i\varrho^{l}|$ and $|a_i\varrho^{l+t}|$. It may happen that $|a_i\varrho^{l+1}| < |a_i\varrho^{l}|$. [This cannot occur if $\varrho$ is primitive and $l$ is sufficiently "large".] Consider for instance the primitive relation $\varrho$ for which the graph $G(\varrho)$ is given by Figure 1.

Here $a_1\varrho = \{a_1, a_2, a_3\}$, $a_1\varrho^2 = \{a_1, a_4\}$, hence $|a_1\varrho^2| < |a_1\varrho|$.

Remark 2. The exponent $L_1 = ([n/2] - 1)\,h_0 + n$ in Lemma 2,5 cannot be replaced by $L_1 - 1$. Consider for instance the primitive relation for which the graph $G(\varrho)$ is given by Figure 2.

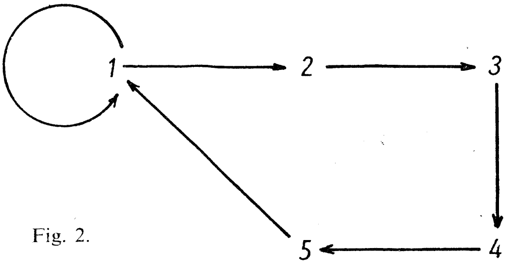


Fig. 2.

Here $a_2\varrho = \{a_3\}$, $a_2\varrho^2 = \{a_4\}$, $a_2\varrho^3 = \{a_5\}$, $a_2\varrho^4 = \{a_1\}$, $a_2\varrho^5 = \{a_1, a_2\}$. Further $L_1 = 6$, and $|a_2\varrho^5| = 2 < [n/2] + 1 = 3$.

**Lemma 2,6.** *Let $\varrho$ be primitive, $\varrho \in B_n(V)$ and $n \geqq 4$. Suppose that the length $h_0$ of the shortest elementary circuit in $G(\varrho)$ satisfies $h_0 \leqq n - 3$. Then*

$$L(a_i, a_j) \leqq \begin{cases} N(n) - \frac{1}{2}(n-4) & \text{if } n \text{ is even}, \\ N(n) - (n-3) & \text{if } n \text{ is odd}. \end{cases}$$

Proof. Denote $L_1 = [n/2]\,h_0 + n - h_0$. Since $|a_i\varrho^{L_1}| \geqq [n/2] + 1$ and $|a_j\varrho^{L_1}| \geqq$ $\geqq [n/2] + 1$, we have $a_i\varrho^{L_1} \cap a_j\varrho^{L_1} \neq \emptyset$ and $L(a_i, a_j) \leqq L_1$. Now

$$L_1 \leqq \left(\left[\frac{n}{2}\right] - 1\right)(n-3) + n = \begin{cases} \frac{1}{2}n^2 - \frac{3}{2}n + 3 & \text{if } n \text{ is even}, \\ \frac{1}{2}n^2 - 2n + \frac{9}{2} & \text{if } n \text{ is odd}. \end{cases}$$

Remark. For $n = 4$, $h_0 = 1$ and (by Lemma 2,4) $L(a_i, a_j) \leqq n - 1 = 3$. Hence for all $n \geqq 4$ we have $L(a_i, a_j) \leqq N(n) - 1$.

Lemma 2,6 shows that in order to prove (4) we have to deal only with those primitive relations for which $G(\varrho)$ contains only elementary circuits of length $n$ or $(n-1)$ or $(n-2)$.

We now divide our considerations into two parts. We first treat the case that $G(\varrho)$ has no elementary circuit of length $n$. Next we shall suppose that $G(\varrho)$ has at least one elementary circuit of length $n$.

## A

We sligthly improve Lemma 2,5 (of course, by imposing additional conditions).

**Lemma 2,7.** *Let $\varrho$ be primitive, $\varrho \in B_n(V)$, $n \geqq 4$, and suppose that $G(\varrho)$ has no loop and $G(\varrho)$ has no elementary circuit of length $n$. Let $h_0 > 1$ be the length of the shortest elementary circuit in $G(\varrho)$. Denote $L_1 = ([n/2] - 1)\,h_0 + n - 1$.*

*Then for any $a_i$ for which $h(a_i) = n - 1$ and any integer $t \geqq 0$ we have*

$$(10) \qquad |a_i\varrho^{L_1+t}| \geqq \left[\frac{n}{2}\right] + 1 .$$

Proof. Let $C$ be an elementary circuit of length $n - 1$ passing through the vertex $a_i$. This circuit contains all but one of the vertices. Denote this vertex by $a^*$.

We assert that $C$ contains a vertex, say $u$, such that $h(u) = h_0$. If $h(a^*) > h_0$, there is nothing to prove. If $h(a^*) = h_0 \geqq 2$, denote an elementary circuit which passes through $a^*$ by $C^*$. Then $C \cap C^* \neq \emptyset$. If $u \in C \cap C^*$, we have $h(u) = h_0$, which proves our assertion. Note that since $\varrho$ is primitive we have $h_0 \neq n - 1$, hence $h_0 < n - 1$ so that $a_i \neq u$.

[Remark. Our assertion need not be true if $G(\varrho)$ contains a loop. Consider for example the primitive relation $\varrho$ with $G(\varrho)$ given by the graph in Figure 3.

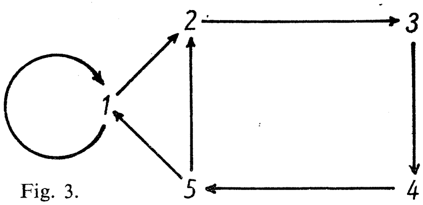


Fig. 3.

Here the circuit $C = \langle a_2, a_3, a_4, a_5, a_2 \rangle$ does not contain a vertex $u$ with $h(u) = 1$.]

Let $k_i < n - 1$ be the length of the (directed) path on $C$ joining the vertex $a_i$ with the vertex $u$, i.e., $u \in a_i\varrho^{k_i}$. Then there is a path of length $s_i = n - 1 - k_i$ (on $C$) joining $u$ with $a_i$, i.e., $a_i \in u\varrho^{s_i}$. Using $u \in u\varrho^{h_0}$ consider the chain with $[n/2]$ terms:

$$u\varrho^{s_i} \subset u\varrho^{h_0+s_i} \subset u\varrho^{2h_0+s_i} \subset \ldots \subset u\varrho^{([n/2]-1)h_0+s_i}$$

and (with $t \geqq 0$) the chain:

$$(11) \qquad u\varrho^{s_i+t} \subset u\varrho^{h_0+s_i+t} \subset \ldots \subset u\varrho^{([n/2]-1)h_0+s_i+t}.$$

a) We first show that $a_i = u\varrho^{s_i}$ cannot hold, so that $|u\varrho^{s_i}| \geqq 2$. Indeed, $a_i = u\varrho^{s_i}$ implies $a_i\varrho^{h_0} = u\varrho^{s_i+h_0}$; further, $a_i = u\varrho^{s_i} \subset u\varrho^{h_0}\varrho^{s_i} = a_i\varrho^{h_0}$, hence $h_0 \geqq n - 1$, a contradiction.

b) Suppose $a_i\varrho^t \subset u\varrho^{s_i+t}$, $a_i\varrho^t \neq u\varrho^{s_i+t}$ for $t = 0, 1, \ldots, t_0 - 1$, while $a_i\varrho^{t_0} = u\varrho^{s_i+t_0}$, where $t_0 > 0$. Since $\varrho$ is primitive such a $t_0 > 0$ exists. [We have denoted $a_i\varrho^0 = a_i$.]

b 1) For $t < t_0$, the first term in (11) contains at least two elements of $V$ so that

$$\left[\frac{n}{2}\right] + 1 \leqq |u\varrho^{([n/2]-1)h_0+s_i+t}| .$$

Since $u \in a_i\varrho^{k_i}$, we conclude

$$\left[\frac{n}{2}\right] + 1 \leqq \left|a_i\varrho^{([n/2]-1)h_0+s_i+t+k_i}\right| .$$

The exponent on the right hand side is $([n/2] - 1) h_0 + (n - 1 - k_i) + t + k_i = L_1 + t$. This proves (10) for all $t < t_0$.

b 2) The equality $a_i\varrho^{t_0} = u\varrho^{s_i+t_0}$ implies $a_i\varrho^t = u\varrho^{s_i+t}$ for all $t \geqq t_0$.

The chain (11) with $[n/2] + 1$ terms can be now written (for $t \geqq t_0$) in the form

$$a_i\varrho^t \subset a_i\varrho^{t+h_0} \subset \ldots \subset a_i\varrho^{[n/2]h_0+t} ,$$

and multiplying by $\varrho^{n-1-h_0}$ we obtain

$$a_i\varrho^{t+(n-1-h_0)} \subset a_i\varrho^{t+h_0+(n-1-h_0)} \subset \ldots \subset a_i\varrho^{t+[n/2]h_0+(n-1-h_0)} .$$

Since $t + [n/2] h_0 + n - 1 - h_0 = L_1 + t$, we conclude

$$\left|a_i\varrho^{L_1+t}\right| \geqq \left[\frac{n}{2}\right] + 1 .$$

This proves (10) for all $t \geqq t_0$. Lemma 2,7 is completely proved.

Remark. It is essential for our further purposes that (10) holds not only for $t = 0$ but also for all $t > 0$.

We now prove:

**Theorem 2,2.** *Let $\varrho$ be primitive, $\varrho \in B_n(V)$, $n \geqq 4$. Suppose that $G(\varrho)$ has no elementary circuit of length $n$. Then for any $a_i, a_j \in V$ we have*

$$L(a_i, a_j) \leqq \begin{cases} N(n) & \textit{for } n \textit{ even}\,, \\ N(n) - \frac{1}{2}(n-1) & \textit{for } n \textit{ odd}\,. \end{cases}$$

Proof. a) If $h_0 = 1$, then (by Lemma 2,4) $L(a_i, a_j) \leqq n - 1$. For $n \geqq 4$ we have $n - 1 < N(n) - \frac{1}{2}(n - 1)$. Hence our statement holds.

b) If $h_i \leqq n - 2$, $h_j \leqq n - 2$ our statement holds by Theorem 2,1.

In the following we may suppose that at least one of the numbers $h_i, h_j$ is equal to $n - 1$. Note again that since $\varrho$ is primitive not all elementary circuits can be of length $n - 1$, hence $h_0 \leqq n - 2$.

c) If $h_0 > 1$ and $h_i = h_j = n - 1$, we have $a_i\varrho^{L_1} \cap a_j\varrho^{L_1} \neq \emptyset$ by Lemma 2,7, where $L_1 = ([n/2] - 1) h_0 + (n - 1)$. Hence

$$L(a_i, a_j) \leqq \left(\left[\frac{n}{2}\right] - 1\right)(n-2) + (n-1) = \begin{cases} N(n) & \text{if } n \text{ is even}\,, \\ N(n) - \frac{1}{2}(n-1) & \text{if } n \text{ is odd}\,. \end{cases}$$

d) Suppose $h_0 > 1$ and $h_i = n - 1$, $h_j \leqq n - 2$. By Lemma 2,7 we have $\left|a_i\varrho^{L_1+t}\right| \geqq [n/2] + 1$ for any $t$ ant $L_1 = ([n/2] - 1) h_0 + n - 1$.

By considering the chain [see (9)]

$$a_j\varrho^t \subset a_j\varrho^{t+h_j} \subset \ldots \subset a_j\varrho^{t+[n/2]h_j} ,$$

we obtain

$$|a_j\varrho^{L_2+t'}| \geqq \left[\frac{n}{2}\right] + 1$$

for any $t' \geqq 0$. Here $L_2 = [n/2]\, h_j$.

If $L_2 < L_1$, choose $t' = t_1 > 0$ such that $L_2 + t_1 = L_1$. Then $a_i\varrho^{L_1} \cap a_j\varrho^{L_1} \neq \emptyset$ and this gives the same estimate as in c).

If $L_2 \geqq L_1$, choose $t = t_2 \geqq 0$ so that $L_1 + t_2 = L_2$. We then have $a_i\varrho^{L_2} \cap \cap\, a_j\varrho^{L_2} \neq \emptyset$. This implies

$$L(a_i, a_j) \leqq \left[\frac{n}{2}\right] h_j \leqq \left[\frac{n}{2}\right](n-2) = \begin{cases} N(n) - 1 & \text{if } n \text{ is even}, \\ N(n) - \frac{1}{2}(n+1) & \text{if } n \text{ is odd}. \end{cases}$$

Theorem 2,2 is proved.

In the foregoing considerations we have used a simple principle. If both $a_i\varrho^{L_0}$ and $a_j\varrho^{L_0}$ contain more than a half of all vertices, then $L(a_i, a_j) \leqq L_0$.

We cannot expect that this method will give the best possible results since a common consequent may exist even if one (or both) of the sets $a_i\varrho^{L_0}$, $a_j\varrho^{L_0}$ contain a half or less than a half of all vertices. This is shown by Example 2,1 below. Nevertheless, the result obtained in Theorem 2,2 is sufficient for our purposes, since in the "worst" case, namely the case in which there is a vertex $a_i$ with $h(a_i) = n$, we obtain sharp estimates in which the bounds are not smaller than those given by Theorem 2,2.

Example 2,1. Consider the relation $\varrho$ for which $G(\varrho)$ is given by Figure 4. Here $h(a_1) = 4$, $h(a_2) = h(a_3) = h(a_4) = 3$.

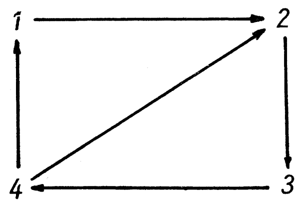


Fig. 4.

$$M(\varrho) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

A simple computation shows that the matrix representations of $\varrho^2$, $\varrho^3$, $\varrho^4$ and $\varrho^5$ are, successively,

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Here $L(a_1, a_3) = 5$, though $|a_1\varrho^5| = 2 < 3$. Also $L(a_1, a_2) = 4$ though both

$|a_1\varrho^4| = |a_2\varrho^4| = 2$. It can be immediately checked that $a_i\varrho^5 \cap a_j\varrho^5 \neq \emptyset$, while $a_1\varrho^4 \cap a_3\varrho^4 = \emptyset$. This implies $L(a_i, a_j) = 5 = N(4)$.

This example will be used in proving Corollary 2,4 below.

## B

We shall now suppose that $G(\varrho)$ contains a vertex $a_1$ with $h(a_1) = n$ and $n \geqq 4$. With respect to Lemma 2,6 it is sufficient to consider only such $\varrho$ for which $G(\varrho)$ has no elementary circuits of length $\leqq n - 3$.

For brevity we introduce the following

Notation. We shall say that $\varrho$ satisfies Condition $H$ if $n \geqq 4$ and all elementary circuits contained in $G(\varrho)$ are of length $h \geqq n - 2$.

We first find the possible types of primitive relations satisfying Condition $H$.

Consider the equality

$$a_1\varrho \cup a_1\varrho^2 \cup \ldots \cup a_1\varrho^n = V.$$

We shall use several times Lemma 2,1 in the following form: Let $Q \subset V$ and $|Q| = s < n$. Then $|Q \cup Q\varrho| = s_1 > s$. If $s_1 < n$, then $|(Q \cup Q\varrho) \cup (Q \cup Q\varrho)\varrho| = |Q \cup Q\varrho \cup Q\varrho^2| > s_1$, etc.

The set $a_1\varrho$ cannot contain more than one element of $V$ since otherwise (by successively applying Lemma 2,1) we should obtain that $a_1\varrho \cup a_1\varrho^2 \cup \ldots \cup a_1\varrho^{n-1} = V$, contrary to the supposition $a_1 \notin a_1\varrho^h$ if $h < n$. Denote $a_2 = a_1\varrho$.

Analogously, each segment $a_1\varrho \cup a_1\varrho^2 \cup \ldots \cup a_1\varrho^r$, $r \leqq n - 1$, contains $r$ different elements and $a_1\varrho^r$ contains exactly one element which is not contained in $a_1\varrho \cup \ldots \cup a_1\varrho^{r-1}$. Denote this element by $a_{r+1}$, so that $a_{r+1} \in a_1\varrho^r$.

If no further condition is imposed, $a_1\varrho^2$ may contain $\{a_3, a_2\}$. If $a_2 \in a_1\varrho^2$, i.e. $a_2 \in a_2\varrho$, we have a loop in $a_2$. Hence if we suppose that Condition $H$ is fulfilled we necessarily have $a_3 = a_1\varrho^2$.

Again if no condition is imposed, $a_1\varrho^3$ may contain a subset of $\{a_4, a_3, a_2\}$. If $a_2 \in a_1\varrho^3$, i.e. $a_2 \in a_2\varrho^2$, then $\varrho$ has a circuit of length 2. If $a_3 \in a_1\varrho^3$, i.e., $a_3 \in a_3\varrho$, then we have a loop in $a_3$. Hence if $n \geqq 5$ and Condition $H$ is fulfilled we necessarily have $a_4 = a_1\varrho^3$.

In this manner, supposing that Condition $H$ is satisfied, we obtain the following sequence:

$$a_2 = a_1\varrho,\ a_3 = a_1\varrho^2, \ldots, a_{n-2} = a_1\varrho^{n-3},\ a_{n-1} = a_1\varrho^{n-2}.$$

Even the last term $a_1\varrho^{n-2}$ is necessarily a one point set since, for instance, $a_2 \in a_1\varrho^{n-2}$ would imply $a_2 \in a_1\varrho \,.\, \varrho^{n-3}$, i.e. $a_2 \in a_2\varrho^{n-3}$.

The situation changes in the next step. The set $a_{n-1}\varrho = a_1\varrho^{n-1}$ contains a new element $a_n$, does not contain $a_1$, may contain $a_2$, but cannot contain any element of the set $\{a_{n-1}, a_{n-2}, \ldots, a_3\}$. [The inclusion $a_2 \in a_1\varrho^{n-1}$ is possible since $a_2 \in a_2\varrho^{n-2}$ indicates the existence of a circuit of length $n - 2$ which is not forbidden.]

Finally, $a_1\varrho^n$ contains $a_1$, may contain $a_2$ and $a_3$, but contains no element of the set $\{a_n, a_{n-1}, \ldots, a_5, a_4\}$. [If for instance $a_4 \in a_1\varrho^n$, we should have $a_4 \in a_4\varrho^{n-3}$, hence a circuit of length $n - 3$.]

The whole situation is illustrated in Figure 5. Here the dotted edges are possible edges (not excluded by Condition $H$).

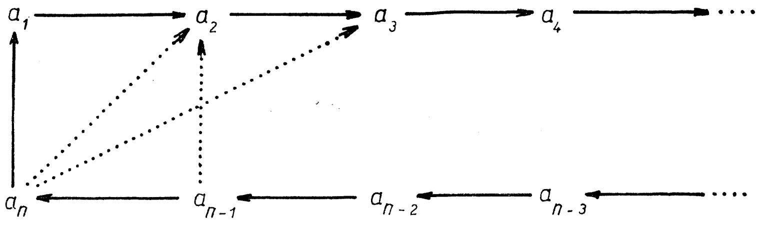


Fig. 5.

Note that if $G(\varrho)$ contains the edge $\langle a_{n-1}, a_2 \rangle$, i.e. $a_2 \in a_{n-1}\varrho$, then $a_3 = a_2\varrho \subset \subset a_{n-1}\varrho^2 = a_1\varrho^n$, hence $G(\varrho)$ contains also the edge $\langle a_n, a_3 \rangle$.

Denote by $C$ the elementary circuit $\langle a_1, a_2, \ldots, a_n, a_1 \rangle$ passing through $a_1$. We then have the following 5 possible types of graphs $G(\varrho_i)$:

α) $G(\varrho_1)$, consisting of $C$ and the edge $\langle a_n, a_2 \rangle$;
β) $G(\varrho_2)$, consisting of $C$ and the edges $\langle a_n, a_2 \rangle$ and $\langle a_n, a_3 \rangle$;
γ) $G(\varrho_3)$, consisting of $C$ and the edges $\langle a_n, a_2 \rangle$, $\langle a_n, a_3 \rangle$, $\langle a_{n-1}, a_2 \rangle$;
δ) $G(\varrho_4)$, consisting of $C$ and the edge $\langle a_n, a_3 \rangle$;
ε) $G(\varrho_5)$, consisting of $C$ and the edges $\langle a_n, a_3 \rangle$, $\langle a_{n-1}, a_2 \rangle$.

For our purposes the case β) and γ) can be reduced to the study of the case α). The graph $G(\varrho_1)$ is a (directed) subgraph of $G(\varrho_2)$ and $G(\varrho_3)$. The relation $\varrho_1$ is primitive (since it contains elementary circuits of lengths $n$ and $n - 1$). Hence $\varrho_2$ and $\varrho_3$ are primitive.

Suppose that we know the number $L(a_i, a_j) = L_\varrho(a_i, a_j)$ for a given $\varrho$. If we form a new relation $\varrho'$ by adding some new edges to $\varrho$, we have $L_{\varrho'}(a_i, a_j) \leqq L_\varrho(a_i, a_j)$. In our case $L_{\varrho_2}(a_i, a_j) \leqq L_{\varrho_1}(a_i, a_j)$ and $L_{\varrho_3}(a_i, a_j) \leqq L_{\varrho_1}(a_i, a_j)$. Hence it is sufficient to find an estimate of $L(a_i, a_j)$ for the graph $G(\varrho_1)$.

Analogously $G(\varrho_4)$ is a subgraph of $G(\varrho_5)$. They both contain circuits only of lengths $n$ and $n - 2$. Such graphs are primitive iff $n$ is odd. Since (under this supposition) $L_{\varrho_5}(a_i, a_j) \leqq L_{\varrho_4}(a_i, a_j)$, it is sufficient for our purposes to deal only with the graph $G(\varrho_4)$.

Summarily: We have to treat only the cases α) and δ).

I) We begin with the second case, i.e. consider the relation $\varrho$ with the graph given in Figure 6, where $n$ is odd (hence $n \geqq 5$).

**Lemma 2,8.** *For the relation $\varrho$ given by the graph in Figure 6 (where $n \geqq 5$ is odd) we have*

$$L(a_i, a_j) \leqq N(n) - \tfrac{1}{2}(n - 3).$$

Proof. Consider the chains

$$a_3 \in a_3\varrho^{n-2} \subset a_3\varrho^{2(n-2)} \subset \ldots \subset a_3\varrho^{[n/2](n-2)}$$

and

(12) $$a_3\varrho^{t} \subset a_3\varrho^{n-2+t} \subset a_3\varrho^{2(n-2)+t} \subset \ldots \subset a_3\varrho^{[n/2](n-2)+t},$$

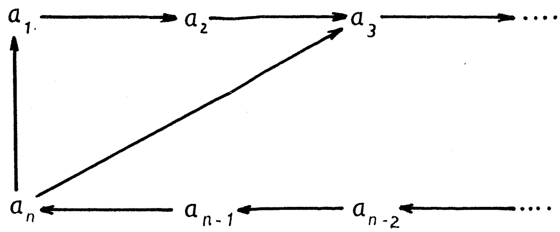


Fig. 6.

and denote $L_1 = [n/2]\,(n-2) = \frac{1}{2}n^2 - \frac{3}{2}n + 1 = N(n) - \frac{1}{2}(n+1)$. (12) implies (for any integer $t \geqq 0$) $|a_3\varrho^{L_1+t}| \geqq [n/2] + 1 = \frac{1}{2}(n+1)$. Since $a_3 = a_1\varrho^2$, $a_3 = a_2\varrho$, the last inequality (with $t = 0$ and $t = 1$) implies

$$|a_1\varrho^{L_1+2}| \geqq \frac{n+1}{2}, \quad |a_2\varrho^{L_1+2}| \geqq \frac{n+1}{2}.$$

Further, for $3 < i \leqq n$ we have $a_i = a_3\varrho^{i-3}$, whence

$$|a_3\varrho^{L_1+t}| = |a_3\varrho^{i-3}\varrho^{L_1-(i-3)+t}| = |a_i\varrho^{L_1-(i-3)+t}| \geqq \frac{n+1}{2}.$$

Putting $t = i - 1$, we obtain $|a_i\varrho^{L_1+2}| \geqq \frac{1}{2}(n+1)$. [Note that for $n \geqq 5$, $l_1 - (i-3) \geqq L_1 - (n-3) = \frac{1}{2}n^2 - \frac{5}{2}n + 4 > 0$.]

Summarily: $|a_i\varrho^{L_1+2}| \geqq \frac{1}{2}(n+1)$ for any $a_i \in V$. This implies $L(a_i, a_j) \leqq L_1 + 2 = N(n) - \frac{1}{2}(n-3)$. Lemma 2,8 is proved.

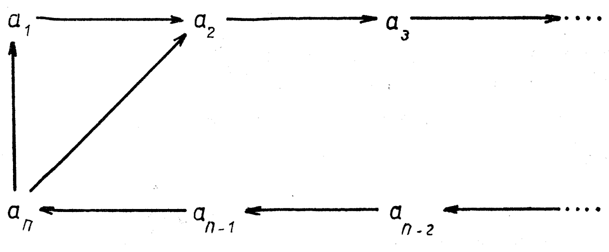


Fig. 7.

II. We now turn to the last case, namely to the relation $\varrho$ defined by the graph in Figure 7.

We first consider the set $V' = \{a_2, a_3, \ldots, a_n\}$. Let $M_0$ be the least integer $m$ such that $a_i\varrho^m \cap a_j\varrho^m \neq \emptyset$ holds for all $a_i, a_j \in V'$. [Clearly, $M_0 = \max L(a_i, a_j)$, where $a_i, a_j$ run through all $a_i, a_j \in V'$.]

Since $a_k = a_2\varrho^{k-2}$ $(k = 2, 3, \ldots, n)$ we have to find the least $m$ such that

$$a_2\varrho^{m+i-2} \cap a_2\varrho^{(m+i-2)+j-i} \neq \emptyset \tag{13}$$

holds for all couples $(i, j)$ with $2 \leqq i < j \leqq n$. Another way to write (13) is

$$a_2\varrho^{m+i-2} \cap a_2\varrho^{(m+i-2)+s} \neq 0 \tag{14}$$

for $i = 2, \ldots, n - 1$ and $s = 1, 2, \ldots, n - 2$. This is equivalent to finding the least integer $m > 0$ such that

$$a_2\varrho^m \cap a_2\varrho^{m+s} \neq \emptyset \tag{15}$$

for $s = 1, 2, \ldots, (n - 2)$. [For, if (15) is satisfied, then multiplying (15) by $\varrho^{i-2}$ we obtain (14).]

Note that for $s = n - 1$ we have $a_2\varrho^{m+s} = a_2\varrho^{n-1} . \varrho^m = \{a_1, a_2\} \varrho^m$, hence $a_2\varrho^m \cap a_2\varrho^{m+n-1} \neq \emptyset$ for any $m > 0$.

Next let $M_1$ be the least integer $m > 0$ such that $a_1\varrho^m \cap a_i\varrho^m \neq \emptyset$ for $i = 2, 3, \ldots$ $\ldots, n$. [Clearly $M_1 = \max_i L(a_1, a_i)$.] Since $a_i = a_2\varrho^{i-2}$, $a_2 = a_1\varrho$, this can be rewritten as

$$a_2\varrho^{m-1} \cap a_2\varrho^{m-1+(i-1)} \neq \emptyset \tag{16}$$

for $i = 2, 3, \ldots, n$. As noted above, (16) is always satisfied if $i = n$. Hence we have: The number $M_1$ is the least integer $m$ for which

$$a_2\varrho^{m-1} \cap a_2\varrho^{m-1+s} \neq \emptyset$$

for all $s = 1, 2, \ldots, (n - 2)$. This immediately implies $M_1 - 1 = M_0$.

We have proved:

**Lemma 2,9.** *If $M_0$ is the least integer $m > 0$ such that $a_2\varrho^m \cap a_2\varrho^{m+s} \neq \emptyset$ for $s = 1, 2, \ldots, (n - 2)$, then $\max_{i,j} L(a_i, a_j) = M_0 + 1$, where $a_i, a_j \in V$.*

To find $M_0$ we describe $a_2\varrho^l$ explicitly:

$$\begin{aligned} &(17) && a_2\varrho^{n-1} = \{a_2, a_1\}, \\ & && a_2\varrho^{2(n-1)} = \{a_1\varrho^{n-1}, a_2\varrho^{n-1}\} = \{a_2, a_1, a_n\}, \\ & && a_2\varrho^{3(n-1)} = \{a_n\varrho^{n-1}, a_1\varrho^{n-1}, a_2\varrho^{n-1}\} = \{a_2, a_1, a_n, a_{n-1}\}, \\ &(18) && a_2\varrho^{k(n-1)} = \{a_2, a_1, a_n, a_{n-1}, \ldots, a_{n-(k-2)}\}. \end{aligned}$$

The last equality holds for those $k$ for which $n - (k - 2) \geqq 3$ and $k \geqq 2$, i.e. $2 \leqq$ $\leqq k \leqq n - 1$. [For $k = 1$ we have (17).]

From now on we consider the cases $n$ even and $n$ odd separately.

$\alpha$) Let $n$ be even. The case $n = 4$ has been settled in Example 2,1.

If $n \geqq 6$, then $k = \frac{1}{2}n - 1$ satisfies $2 \leqq k \leqq n - 1$. In (18) put $k = \frac{1}{2}n - 1$ and denote $L_1 = (\frac{1}{2}n - 1)(n - 1)$. The following set contains $\frac{1}{2}n$ elements of $V$:

$$a_2\varrho^{L_1} = \{a_2, a_1, a_n, a_{n-1}, \ldots, a_{n/2+3}\} .$$

Denote further $L_0 = L_1 + \frac{1}{2}n - 1 = \frac{1}{2}n^2 - n$. The last equality implies

$$\begin{aligned} a_2\varrho^{L_0} &= a_2\varrho^{L_1+n/2-1} = \{a_{n/2+1}, a_{n/2}, \ldots, a_3, a_2\} ,\\ a_2\varrho^{L_0+1} &= a_2\varrho^{L_1+n/2} = \{a_{n/2+2}, a_{n/2+1}, \ldots, a_4, a_3\} . \end{aligned}$$

We assert that $a_2\varrho^{L_0} \cap a_2\varrho^{L_0+s} \neq \emptyset$ for $s = 1, 2, \ldots, n - 2$ (hence $M_0 \leqq L_0$). For $s = 1, 2, \ldots, \frac{1}{2}n - 1$, each of the „shifted" sets $a_2\varrho^{L_0+1}, a_2\varrho^{L_0+2}, \ldots$ $\ldots, a_2\varrho^{L_0+n/2-1}$ contains the element $a_{n/2+1}$ so that our statement is true.

For $s \geqq \frac{1}{2}n$ we use the usual argument. Since

$$a_2 \in a_2\varrho^{n-1} \subset \ldots \subset a_2\varrho^{(n/2)(n-1)} ,$$

we have $|a_2\varrho^{(n/2)(n-1)}| \geqq \frac{1}{2}n + 1$ and also $|a_2\varrho^{(n/2)(n-1)+t}| \geqq \frac{1}{2}n + 1$ for any $t \geqq 0$. Now $\frac{1}{2}n(n - 1) + t = L_0 + \frac{1}{2}n + t$, so that $|a_2\varrho^{L_0+s}| \geqq \frac{1}{2}n + 1$ for $s = \frac{1}{2}n, \frac{1}{2}n + 1, \ldots, n - 2$. Since $|a_2\varrho^{L_0}| = \frac{1}{2}n$ we have $a_2\varrho^{L_0} \cap a_2\varrho^{L_0+s} \neq \emptyset$ for $s = \frac{1}{2}n, \frac{1}{2}n + 1, \ldots, (n - 2)$. This proves our statement.

To prove that $M_0 = L_0$ it is sufficient to show that for some $s \in \{1, 2, \ldots, n - 2\}$ we have $a_2\varrho^{L_0-1} \cap a_2\varrho^{L_0-1+s} = \emptyset$.

Now

$$a_2\varrho^{L_0-1} = \{a_{n/2}, a_{n/2-1}, \ldots, a_2, a_1\}$$

and for $s = \frac{1}{2}n$,

$$a_2\varrho^{L_0-1+n/2} = \{a_n, a_{n-1}, \ldots, a_{n/2+1}\} .$$

Hence

$$a_2\varrho^{L_0-1} \cap a_2\varrho^{L_0-1+n/2} = \emptyset . \tag{19}$$

Lemma 2,9 implies $\max_{i,j} L(a_i, a_j) = L_0 + 1 = N(n)$. Since $a_2 = a_1\varrho$, $a_2\varrho^{n/2-1} = a_{n/2+1}$, (19) can be written in the form $a_1\varrho^{L_0} \cap a_{n/2+1}\varrho^{L_0} = \emptyset$.

**Corollary 2,4.** *If $n$ is even, then* $\max_{i,j} L(a_i, a_j) = N(n)$ *and the value $N(n)$ is achieved for the couple $(a_1, a_{n/2+1})$.*

β) Let now $n$ be odd (hence $n \geqq 5$). Denote $L_0 = [\frac{1}{2}n](n - 1) = \frac{1}{2}n^2 - n + \frac{1}{2}$. Since

$$a_2 \subset a_2\varrho^{n-1} \subset \ldots \subset a_2\varrho^{[n/2](n-1)} .$$

we conclude that $|a_2\varrho^{L_0}| \geqq [\frac{1}{2}n] + 1 = \frac{1}{2}(n + 1)$ and also for any $s > 0$, $|a_2\varrho^{L_0+s}| \geqq \frac{1}{2}(n + 1)$. Hence $a_2\varrho^{L_0} \cap a_2\varrho^{L_0+s} \neq \emptyset$ for $s = 1, 2, \ldots, n - 2$. This implies $M_0 \leqq L_0$.

To show that $M_0 = L_0$ it is sufficient to show that for some $s \in \{1, 2, \ldots, n - 2\}$ we have $a_2\varrho^{L_0-1} \cap a_2\varrho^{L_0-1+s} = \emptyset$.

For $n \geqq 5$ and $k = [\frac{1}{2}n]$ we have $2 \leqq k \leqq n - 1$ so that we can use (18) by which

$$a_2\varrho^{L_0} = \{a_2, a_1, a_n, \ldots, a_{(n+5)/2}\},$$

while

$$a_2\varrho^{L_0-1} = \{a_n, a_{n-1}, \ldots, a_{(n+3)/2}\}. \tag{20}$$

Multiplying (20) by $\varrho^{(n-1)/2}$ we obtain

$$a_2\varrho^{L_0-1+(n-1)/2} = \{a_{(n+1)/2}, a_{(n-1)/2}, \ldots, a_2, a_1\}.$$

Hence for $s = \frac{1}{2}(n-1)$ we have $a_2\varrho^{L_0-1} \cap a_2\varrho^{L_0-1+s} = \emptyset$. This proves $M_0 = L_0$.

Lemma 2,9 again implies $\max L(a_i, a_j) = L_0 + 1 = N(n)$. We further have $a_2\varrho^{L_0-1} \cap a_2\varrho^{L_0-1+(n-1)/2} = a_1\varrho^{L_0} \cap a_{(n+1)/2}\varrho^{L_0} = \emptyset$. Hence:

**Corollary 2,5.** *If $n$ is odd, $\max_{i,j} L(a_i, a_j) = N(n)$ and the value $N(n)$ is achieved for the couple $(a_1, a_{(n+1)/2})$.*

**Lemma 2,10.** *If $\varrho$ is the relation defined by the graph in Figure 7, then $L(a_i, a_j) \leqq N(n)$. The bound is sharp since there is a pair $(a_1, a_j)$ for which $L(a_1, a_j) = N(n)$.*

Taking into account Corollary 2,3, Lemma 2,6, Theorem 2,2 Lemma 2,8 and Lemma 2,10 we finally have:

**Theorem 2,3. (The main result.)** *Let $\varrho$ be primitive, $\varrho \in B_n(V)$, $n \geqq 2$. Then for any $a_i, a_j \in V$ we have $L(a_i, a_j) \leqq N(n)$. This result is the best possible.*

The goal of the following sections is to prove that the estimate of Theorem 2,3 holds not only for primitive relations but for any $\varrho \in B_n(V)$.

## 3. IRREDUCIBLE RELATIONS WITH $d(\varrho) > 1$

Let now $\varrho \in B_n(V)$, $\varrho$ irreducible but not primitive. Since in this case $L(a_1, a_2)$ does not exist for $n = 2$, we may suppose $n \geqq 3$.

Without loss of generality we shall suppose that the matrix representation of $\varrho$ is of the form

$$M(\varrho) = \begin{pmatrix} 0 & B_1 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \ldots & 0 & B_{d-1} \\ B_d & 0 & \ldots & 0 & 0 \end{pmatrix}.$$

Here $d = d(\varrho) > 1$ is the index of imprimitivity of $\varrho$. In this case we have

$$M(\varrho^d) = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ 0 & 0 & \ldots & A_d \end{pmatrix},$$

where $A_k$ are primitive $v_k \times v_k$ Boolean matrices, $\Pi(A_k) = V_k$ are the sets of imprimitivity of $\varrho$, and $V_1 \cup \ldots \cup V_d = V$, $v_1 + \ldots + v_d = n$.

By Theorem 1,1, $L(a_i, a_j)$ exists iff $a_i, a_j$ are contained in the same set of imprimitivity, say $V_k$. Suppose that this is the case and $v_k \geqq 2$. We may use Theorem 2,3 by which

$$L(a_i, a_j) \leqq d \,.\, (\tfrac{1}{2}v_k^2 - v_k + \varepsilon_{v_k})\,.$$

We essentially improve this straightforward result. Denote $\min\limits_{k=1,\ldots,d} |V_k| = \beta$ and denote by $V_0$ a fixed chosen set of imprimitivity with $|V_0| = \beta$.

a) Suppose first $|V_0| = \beta = 1$. If $|V_k| = 1$ for all $k = 1, \ldots, d$, no two elements of $V$ have a *c.c.* In any $V_k$ with $|V_k| \geqq 2$ choose two vertices $a_i, a_j$. Since $V_0 = V_k\varrho^u$ for some $u$, $1 \leqq u \leqq d - 1$, we have $a_i\varrho^u = a_j\varrho^u$, i.e. $L(a_i, a_j)$ exists and $L(a_i, a_j) \leqq$ $\leqq d - 1$.

b) Next suppose $|V_0| = \beta \geqq 2$. For any $a_i, a_j \in V_0$ we have $L(a_i, a_j) \leqq$ $\leqq d(\frac{1}{2}\beta^2 - \beta + \varepsilon_\beta) = L_0$, i.e. $a_i\varrho^{L_0} \cap a_j\varrho^{L_0} \neq \emptyset$. Let $V_k \neq V_0$ be any set of imprimitivity and $a_h, a_l \in V_k$. Since $V_0 = V_k\varrho^u$ with some $u$, $1 \leqq u \leqq d - 1$, both $a_h\varrho^u$, $a_l\varrho^u$ are contained in $V_0$. Therefore $a_h\varrho^u \,.\, \varrho^{L_0} \cap a_l\varrho^u \,.\, \varrho^{L_0} \neq \emptyset$. Hence $L(a_h, a_l) \leqq$ $\leqq u + L_0 \leqq d - 1 + d(\frac{1}{2}\beta^2 - \beta + \varepsilon_\beta)$.

We have proved:

**Theorem 3,1.** *Suppose that* $\varrho \in B_n(V)$, $n \geqq 3$, $\varrho$ *is irreducible and* $d(\varrho) > 1$. *Denote* $\min\limits_k |V_k| = \beta$.

*a) If* $\beta = 1$ *and* $L(a_i, a_j)$ *exists, then* $L(a_i, a_j) \leqq d - 1$.

*b) If* $\beta > 1$ *and* $L(a_i, a_j)$ *exists, then*

$$L(a_i, a_j) \leqq d - 1 + d(\tfrac{1}{2}\beta^2 - \beta + \varepsilon_\beta)\,. \tag{21}$$

We now transform (21) into another form which enables us to obtain estimations in which $\beta$ does not appear explicitly. This will be worse than (21), but sufficient for our purposes.

Write $n = \alpha d + \alpha_1$, where $\alpha \geqq 1$ is an integer and $0 \leqq \alpha_1 \leqq d - 1$. At least one of the numbers $|V_1|, \ldots, |V_d|$ is $\leqq \alpha$. [If all of them were $\geqq \alpha + 1$ this would imply $d(\alpha + 1) = \alpha d + d > n$.]

We have $2 \leqq \beta \leqq \alpha = (n - \alpha_1)/d$. Since $N(\beta)$ is an increasing function of $\beta$ we have (for any $a_i, a_j$ for which $L(a_i, a_j)$ exists)

$$L(a_i, a_j) \leqq d - 1 + (\tfrac{1}{2}\alpha^2 - \alpha + \varepsilon_\alpha)\, d = d - 1 + d\left[\frac{1}{2}\left(\frac{n - \alpha_1}{d}\right)^2 - \left(\frac{n - \alpha_1}{d}\right) + \varepsilon_\alpha\right] =$$

$$= \frac{1}{2d}(n - d - \alpha_1)^2 + d(\varepsilon_\alpha + \tfrac{1}{2}) - 1\,.$$

Putting here $\alpha_1 = 0$, $\varepsilon_\alpha = \frac{3}{2}$ we obtain:

**Corollary 3,1.** *Let* $\varrho \in B_n(V)$, $\varrho$ *irreducible*, $n \geqq 3$, $d(\varrho) > 1$. *If* $L(a_i, a_j)$ *exists, then*

$$L(a_i, a_j) \leqq \frac{1}{2d}(n - d)^2 + 2d - 1\,.$$

A still weaker estimate in which even $d$ does not appear explicitly is obtained as follows.

Suppose $n = 3$. If $d = 3$, then $M(\varrho)$ is a permutation matrix and $L(a_i, a_j)$ does not exist. If $d = 2$, then $\beta = 1$ and (by Theorem 3,1) $L(a_i, a_j) = 1 < N(3) = 3$. Suppose $n = 4$. If $d = 4$, $L(a_i, a_j)$ does not exist. If $d = 3$, then $\beta = 1$ and (by Theorem 3,1) $L(a_i, a_j) \leqq 2 < N(4) = 5$. If $d = 2$, then $\beta = 2$ and (by Theorem 3,1), $L(a_i a_j) \leqq \leqq 2d - 1 = 3 < N(4) = 5$.

Suppose $n \geqq 5$, and note that for $d = n$ the matrix $M(\varrho)$ is a permutation matrix, so that $L(a_i, a_j)$ does not exist. The function

$$f(d) = \frac{1}{2d}(n - d)^2 + 2d - 1 = \frac{1}{2d} n^2 - n + \frac{5}{2} d - 1$$

is a decreasing function of $d$ in $(1, n/\sqrt{5})$ and an increasing function in $(n/\sqrt{5}, n)$. Hence it assumes the largest value either for $d = 2$ or $d = n - 1$. We have

$$f(2) = \frac{1}{4} n^2 - n + 4\,, \quad f(n - 1) = 2n - 3 + \frac{1}{2(n - 1)}\,.$$

Hence

$$L(a_i, a_j) \leqq \max\left[\frac{1}{4} n^2 - n + 4,\ 2n - 3 + \frac{1}{2(n - 1)}\right].$$

For $n \geqq 5$, $\frac{1}{4}n^2 - n + 4 < N(n)$ and $2n - 3 + 1/(2(n - 1)) < N(n)$. This implies:

**Corollary 3,2.** *Let $\varrho \in B_n(V)$, $n \geqq 3$, $\varrho$ irreducible but not primitive. If $L(a_i, a_j)$ exists, we have $L(a_i, a_j) < N(n)$.*

This together with Theorem 2,3 implies:

**Theorem 3,2.** *Let $\varrho$ be irreducible, $\varrho \in B_n(V)$, $n \geqq 2$. If $L(a_i, a_j)$ exists, we have $L(a_i, a_j) \leqq N(n)$.*

Remark. Though it is irrelevant for our purposes, we remark that Corollary 3,2 can be easily sharpened. Since

$$\left(\frac{n^2}{4} - n + 4\right) - \left[2n - 3 + \frac{1}{2(n - 1)}\right] = \frac{1}{4}\left[(n - 6)^2 - 8\right] - \frac{1}{2(n - 1)}$$

and the right hand side is positive iff $n \geqq 9$, we have:

**Corollary 3,3.** *Suppose that $\varrho \in B_n(V)$, $n \geqq 3$, $\varrho$ is irreducible but not primitive. If $L(a_i, a_j)$ exists, we have*

$$L(a_i, a_j) \leqq \begin{cases} 1 & \text{for } n = 3\,, \\ 3 & \text{for } n = 4\,, \\ 2n - 3 & \text{for } n = 5, 6, 7, 8\,, \\ \frac{1}{4}n^2 - n + 4 & \text{for } n \geqq 9\,. \end{cases}$$

[Even these results are not the best ones.]

We now have to consider the case

$$M(\varrho) = \begin{pmatrix} A_1 & & & \\ A_{21} & A_2 & & \\ \dots & \dots & \dots & \dots \\ A_{k1} & A_{k2} & \dots & A_k \end{pmatrix},$$

where $A_i$ are irreducible.

If $|\Pi(M)| = n$, we wish to prove that for any $x, y \in \Pi(M)$ for which $L(x, y)$ exists, we have $L(x, y) < N(n)$.

With respect to Lemmas 0,3 and 0,4 we may suppose that $M(\varrho)$ has no zero rows nor zero columns.

Though such a result may be intuitively expected the detailed proof is rather long.

We begin by considering the case $\varrho \in B_n(V)$, where

$$M(\varrho) = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix}. \tag{22}$$

Here $A$ is any $r \times r$ Boolean matrix without a zero row, $B$ is an $s \times s$ irreducible Boolean matrix and $B \neq 0$. For our convenience we write $\Pi(A) = \{a_1, a_2, \dots, a_r\}$, $\Pi(B) = \{b_1, b_2, \dots, b_s\}$, $r > 0$, $s > 0$, and $r + s = n$.

Further, let $\Pi(B) = T_1 \cup T_2 \cup \dots \cup T_d$ be the decomposition of $\Pi(B)$ into the sets of imprimitivity of $B$. Put $|T_i| = s_i$, so that $s_1 + \dots + s_d = s$.

In this section we shall find some estimates for $L(a, b)$, where $a \in \Pi(A)$, $b \in \Pi(B)$, and for $L(b_i, b_j)$, where $b_i, b_j \in \Pi(B)$ [provided they exist].

If $L(a, b)$ exists, the $s \times r$ matrix $C$ cannot be the zero matrix. Denote $\varrho_C = \varrho \cap$ $\cap [B \times A]$ and let

$$\varrho_C = \{(b'_1, a^*_1), (b'_2, a^*_2), \dots, (b'_v, a^*_v)\}. \tag{23}$$

If $b_i, b_j \in \Pi(B)$ have a common consequent, two cases are possible:

a) There is a *c.c.* of $b_i, b_j$ which is contained in $\Pi(B)$. This is the case iff both $b_i, b_j$ are contained in the same set of imprimitivity. By Theorem 3,2 we then have $L(b_i, b_j) \leq N(s) < N(n)$.

b) If $L(b_i, b_j)$ exists but the *c.c.* is contained in $\Pi(A)$, then $\varrho_C \neq \emptyset$ [and we shall again suppose that $\varrho_C$ is given by (23)]. In this case we necessarily have $b_i \in T_u$, $b_j \in T_v$ and $T_u \neq T_v$.

We first give two examples to show various possibilities which may occur. (Both examples will be used in the proof of Lemma 4,5.)

Example 4,1. Consider the relation $\varrho$ given by the matrix

$$\begin{array}{c|cccc} & a_1 & a_2 & b_1 & b_2 \\ \hline a_1 & 0 & 1 & 0 & 0 \\ a_2 & 1 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 & 1 \\ b_2 & 0 & 0 & 1 & 0 \end{array}.$$

Here $\varrho_C = \{(b_1, a_1)\}$. Recall the definition of $L(A)$ (before Lemma 0,1). Here $L(a_2, b_1) = 1$ while $L(A) = 0$. Next, $L(a_1, b_2) = 2$ as the following diagram of paths shows:

$$a_1 \to a_2 \to a_1 ,$$
$$b_2 \to b_1 \to \{a_1, b_2\} .$$

$L(b_1, b_2)$ does not exist since the diagram is of the form

$$b_1 \to \{a_1, b_2\} \to \{a_2, b_1\} \to \{a_1, b_2\} \to \dots ,$$
$$b_2 \to \{b_1\} \quad \to \{a_1, b_2\} \to \{a_2, b_1\} \to \dots .$$

In this simple case the situation becomes clear when considering the graph $G(\varrho)$. [See Fig. 8.]

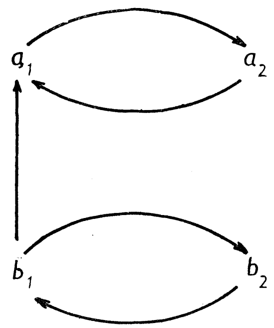

Fig. 8.

Clearly (up to an isomorphism) the same situation takes place if $\varrho_C$ is any couple $(b_i, a_j)$.

For further purposes consider more generally all Boolean matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

If $a_{11} = 1$, $L(b_1, b_2) = 2$. There are only 3 matrices $A$ for which $a_{11} = 0$ and $A$ does not have a zero row, namely

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} .$$

In the first case $L(b_1, b_2)$ does not exist, in the second and the third case $L(b_1, b_2) = 3$.

Hence in any case either $L(b_1, b_2)$ does not exist or $L(b_1, b_2) \leqq 3$.

Example 4.2. Let $\varrho$ be given by the $5 \times 5$ Boolean matrix

$$\begin{array}{c} \\ a_1 \\ a_2 \\ b_1 \\ b_2 \\ b_3 \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & b_1 & b_2 & b_3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array} .$$

By considering the graph [see Fig. 9]

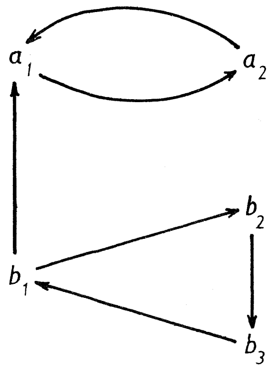


Fig. 9.

we easily find that $L(a_1, b_2) = 6$. The vertex $a_1$ is reached from $a_1$ by paths of length $2k_1$, from $b_2$ by paths of length $3k_2 + 3 + 2k_3$ ($k_i$ nonnegative integers). Now $s = 2k_1 = 3k_2 + 3 + 2k_3$ has a solution with the smallest $s > 0$ if $k_1 = 3$, $k_2 = 1$, $k_3 = 0$. The corresponding sequence is (we omit the arrows):

$$\binom{a_1}{b_2}, \binom{a_2}{b_3}, \binom{a_1}{b_1}, \binom{a_2}{b_2}, \binom{a_1}{b_3}, \binom{a_2}{b_1}, \binom{a_1}{a_1}. \tag{24}$$

The sequence (24) shows that $L(a_i, b_j)$ exists for any couple $\binom{a_i}{b_j}$. In our example $\varrho_C = \{(b_1, a_1)\}$ but it is clear that an analogous situation takes place if $\varrho_C$ is equal to any couple $(b_i, a_j)$.

Consider now the couple $\binom{b_2}{b_3}$. The vertex $a_1$ is reached from the vertex $b_2$ by paths of length $3k_1 + 3 + 2k_2$, from the vertex $b_3$ by paths of length $3k_3 + 2 + 2k_4$. The positive minimum of $s = 3k_1 + 3 + 2k_2 = 3k_3 + 2 + 2k_4$ is achieved by putting $k_1 = 0$, $k_2 = 1$, $k_3 = 1$, $k_4 = 0$. The corresponding sequence is

$$\binom{b_2}{b_3}, \binom{b_3}{b_1}, \binom{b_1}{b_2}, \binom{a_1}{b_3}, \binom{a_2}{b_1}, \binom{a_1}{a_1}.$$

Hence $L(b_2, b_3) = 5$ and $L(b_i, b_j) \leqq 5$.

Again for further purposes, consider the case that

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is replaced by any $2 \times 2$ Boolean matrix without a zero row (and not changing $\varrho_C$ and $B$). If there is a loop in $a_1$, then $L(b_i, b_j) \leqq 3$. The case

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has been settled above. In the remaining cases

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

we easily find $L(b_i, b_j) \leqq 4$.

Summarily: In any case in which $A$ does not have a zero row $L(b_i, b_j)$ exists, and $L(b_i, b_j) \leqq 5$.

We have obtained this result by supposing that $|\varrho_C| = 1$; it holds the more if $|\varrho_C| > 1$.

### ESTIMATIONS FOR $L(a, b)$, $a \in \Pi(A)$, $b \in \Pi(B)$

We suppose $M(\varrho)$ in the form (22). Our aim is to find estimations for $L(a, b)$, $a \in \Pi(A)$, $b \in \Pi(B)$.

If $n = 2$, then

$$M(\varrho) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

is the single case for which $L(a, b)$ exists and $L(a, b) = 1$. In the following we shall suppose $n \geqq 3$.

1) We first settle the case $r = 1$, $s \geqq 2$. (By supposition, $A \neq 0$.) Any sequence which leads to a common consequent is of the form

$$\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b^{(1)} \end{pmatrix}, \begin{pmatrix} a \\ b^{(2)} \end{pmatrix}, \dots$$

Hence after at most $s$ steps we obtain the couple $\begin{pmatrix} a \\ a \end{pmatrix}$. Therefore $L(a, b) \leqq s = n - 1 < N(n)$.

From now on we suppose $r \geqq 2$.

2) Next we settle the case $s = 1$. Write $\Pi(B) = \{b\}$. [Recall that $B = 0$ has been settled by Lemma 0,4.]

Since $b \in b\varrho$, we have $b \in b\varrho \subset b\varrho^2 \subset \dots \subset b\varrho^{n-1} = b\varrho^n = \dots$. If $L(a_j, b)$ exists, then there is an $l > 0$ such that $a_j\varrho^l \cap b\varrho^l \neq \emptyset$. Let $a' \in a_j\varrho^l \cap b\varrho^l$. Since $a_j\varrho^l$ is contained in the transitive closure $\{a_j\varrho \cup a_j\varrho^2 \cup \dots \cup a_j\varrho^{n-1}\}$, there is a $v$, $1 \leqq v \leqq n - 1$, such that $a' \in a_j\varrho^v$. Hence $a' \in a_j\varrho^v \cap b\varrho^l \subset a_j\varrho^v \cap b\varrho^{n-1}$. Multiplying by $\varrho^{n-1-v}$ we have $a'\varrho^{n-1-v} \in a_j\varrho^{n-1} \cap b\varrho^{n-1}$. [Here $a'\varrho^0$ denotes $a'$.] Hence $L(a_j, b) \leqq r = n - 1 < N(n)$. [Note that we have used that $A$ has no zero row.]

3) Suppose that $s \geqq 2$, $r \geqq 2$ and $\varrho_C$ is given by (23). Then any one-step transition from a vertex in $\Pi(B)$ to a vertex in $\Pi(A)$ is of the form $b_i' \to a_i^*$ ($i = 1, 2, \dots, v$).

The shortest sequence which leads to a common consequent is of the form

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a^{(0)} \\ b^{(0)} \end{pmatrix}, \begin{pmatrix} a^{(1)} \\ b^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} a^{(l)} \\ b_i' \end{pmatrix}, \begin{pmatrix} a^{(l+1)} \\ a_i^* \end{pmatrix}, \dots \tag{25}$$

for some $i \in \{1, 2, \dots, v\}$.

The necessary steps to obtain such a sequence are described by the diagram

$$
(26)\qquad
\begin{array}{l}
a \xrightarrow{k_0} a^{[1]} \xrightarrow{k_1} a^{[2]} \xrightarrow{k_2} \dots \xrightarrow{k_{n-1}} a^{[u]} \xrightarrow{1} a^{(l+1)} \to \dots, \\
b \xrightarrow{k_0} b'_i \xrightarrow{k_1} b'_i \xrightarrow{k_2} \dots \xrightarrow{k_{n-1}} b'_i \xrightarrow{1} a_i^* \to \dots .
\end{array}
$$

Here $a^{[u]} = a^{(l)}$, $0 \leqq k_0 \leqq s - 1$, $1 \leqq k_i \leqq s$ (for $i > 0$).

Since no repetitions are allowed, there are at most $r$ terms of the form $\binom{a^{[v]}}{b'_i}$ so that the length of the paths by which $\binom{a^{(l)}}{b'_i}$ is reached is at most $k_0 + (k_1 + \dots + k_{r-1}) \leqq (s - 1) + (r - 1) s = rs - 1$. This result is trivial since there are eaxactly $rs$ different couples of the form $\binom{a}{b}$. But the method by which (26) has been constructed allows to easily sharpen the result obtained.

a) Suppose that the column corresponding to $a_i^*$ in $A$ has a non-zero entry. Then there is a vertex $a'_i \in \Pi(A)$ such that $(a'_i, a_i^*) \in \varrho$. We then have $a^* \in a'_i\varrho \cap b'_i\varrho$, i.e. $L(a'_i, b'_i) = 1$.

If $\binom{a'_i}{b'_i}$ is contained in (25), then it is reached in at most $rs - 1$ steps and $L(a, b) \leqq \leqq (rs - 1) + 1 = rs$. (Note that in this case $a^{(l+1)} = a_i^*$.)

If (25) does not contain the couple $\binom{a'_i}{b'_i}$, then in (26) we have at most $r - 1$ terms of the form $\binom{a^{[k]}}{b'_i}$. The couple $\binom{a^{[u]}}{b'_i}$ is reached by a path of length at most $s - 1 + s(r - 2)$, so that after $s - 1 + s(r - 2) + 1 = rs - s$ steps the both „coordinates" are contained in $\Pi(A)$. Hence

$$L(a, b) \leqq rs - s \quad \text{if} \quad a^{(l+1)} = a_i^*\,, \quad \text{and}$$

$$L(a, b) \leqq rs - s + L(A) \quad \text{if} \quad a^{(l+1)} \neq a_i^*\,.$$

[The first possibility occurs in particular if $L(a, b)$ exists but $L(A) = 0$.]

b) Suppose that the column corresponding to $a_i^*$ in $A$ consists entirely of zeros. Then the couple $\binom{a_i^*}{b'_i}$ is not contained in (26) so that $\binom{a^{[u]}}{b'_i}$ is reached by path of length at most $s - 1 + s(r - 2)$. We certainly have $a^{(l+1)} \neq a_i^*$ and

$$L(a, b) \leqq (s - 1) + s(r - 2) + 1 + L(a^{(l+1)}, a^*) \leqq rs - s + L(A)\,.$$

We have proved:

**Lemma 4,1.** *Suppose that $M(\varrho)$ is of the form (22), where $A$ is any $r \times r$ Boolean matrix without a zero row and $B \neq 0$ is an irreducible $s \times s$ Boolean matrix. Denote $n = r + s$ and suppose $n \geqq 3$. Let $a \in \Pi(A)$, $b \in \Pi(B)$ and suppose that $L(a, b)$ exists. We then have:*

1) *If $r = 1$, $s \geqq 2$, then $L(a, b) \leqq n - 1 < N(n)$.*

2) *If* $r \geqq 2$, $s = 1$, *then* $(a, b) \leqq n - 1 < N(n)$.
3) *If* $r \geqq 2$, $s \geqq 2$, *then* $L(a, b) \leqq \max [rs, rs - s + L(A)]$.

Remark. The proof shows that Lemma 4,1 holds also without the supposition that $B$ is irreducible.

The next lemma will be useful in some forthcoming computations.

**Lemma 4,2.** *If* $n = r + s$, $(r \geqq 2, s \geqq 2)$, *then*

$$N(r) + N(s) = N(n) - rs + \delta_{rs}, \tag{27}$$

*where*

$$\delta_{rs} = \varepsilon_r + \varepsilon_s - \varepsilon_{r+s} = \begin{cases} 2 \textit{ if both } r,s \textit{ are odd}, \\ 1 \textit{ in all the other cases}. \end{cases}$$

Proof:

$$N(r) + N(s) = \tfrac{1}{2}r^2 - r + \varepsilon_r + \tfrac{1}{2}s^2 - s + \varepsilon_s =$$
$$= \tfrac{1}{2}(r + s)^2 - rs - (r + s) + \varepsilon_r + \varepsilon_s =$$
$$= \tfrac{1}{2}n^2 - n + \varepsilon_n - rs + (\varepsilon_r + \varepsilon_s - \varepsilon_n) = N(n) - rs + \delta_{rs}.$$

We now apply Lemma 4,1 to the case of $A \neq 0$ irreducible. First, $rs \leqq \frac{1}{4}n^2 < N(n)$. Next, $L(A) \leqq N(r)$ so that

$$rs - s + L(A) \leqq rs - s + N(r) = N(n) - N(s) + \delta_{rs} - s.$$

Note that $\delta_{r2} = 1$, so that $\delta_{rs} - s < 0$ for any $s \geqq 2$. Hence $rs - s + L(A) \leqq \leqq N(n) - N(s) - 1 \leqq N(n) - 2 < N(n)$. We have proved:

**Corollary 4,1.** *Suppose that the suppositions of Lemma* 4,1 *are satisfied and* $A \neq 0$ *is irreducible. Then* $L(a, b) < N(n)$.

## THE ESTIMATES FOR $L(b_i, b_j)$, $b_i \in T_i$, $b_j \in T_j$, $T_i \neq T_j$

Suppose that $M(\varrho)$ is of the form (22) and $s \geqq 2$. Suppose that $b_i \in T_i$, $b_j \in T_j$, $T_i \neq T_j$ and $L(b_i, b_j)$ exists. The *c.c.* is then necessarily contained in $\Pi(A)$ and $\varrho_C \neq \emptyset$.

The diagram of paths which lead to a common consequent $a^{(t)} \in \Pi(A)$ has the following form:

$$\begin{aligned} b_i &= b_i^{(0)} \to b_i^{(1)} \to \ldots \to a = a^{(0)} \to a^{(1)} \to \ldots \to a^{(t)}, \\ b_j &= b_j^{(0)} \to b_j^{(1)} \to \ldots \to b = b^{(0)} \to b^{(1)} \to \ldots \to a^{(t)}. \end{aligned} \tag{28}$$

Denote by $g(b_i, b_j) = g[M(\varrho), b_i, b_j]$ the number of couples in (28) which are of the form $\begin{pmatrix} b_i^{(v)} \\ b_j^{(v)} \end{pmatrix}$, $v \geqq 0$. After the terms of this form a certain number of couples of the form $\begin{pmatrix} a^{(v)} \\ b^{(v)} \end{pmatrix}$ follow (possibly none) before enterring with both "coordinates" into $\Pi(A)$.

If in (28) no term of the form $\binom{a^{(v)}}{b^{(v)}}$ occurs, then either

$$L(b_i, b_j) \leqq g(b_i, b_j) \quad \text{or} \quad L(b_i, b_j) \leqq g(b_i, b_j) + L(a^{(u)}, a^{(v)})$$

for some $a^{(u)} \neq a^{(v)}$. Hence in both cases $L(b_i, b_j) \leqq g(b_i, b_j) + L(A)$.

If $\binom{a}{b}$ occurs, then $L(b_i, b_j) \leqq g(b_i, b_j) + L(a, b)$.

In any case we have to find an eastimate for $g(b_i, b_j)$. We suppose that (28) describes a shortest sequence leading to a common consequent.

In a shortest sequence no repetitions of a couple $\binom{b'}{b''}$ are possible. Further, analogously as in the proof of Lemma 0,1, if $\binom{b'}{b''}$ occurs in (28), then $\binom{b''}{b'}$ cannot occur. This reduces the number of pairs $\binom{b'}{b''}$ to at most $\frac{1}{2}s(s-1)$. But here a further reduction can be made. No pair $b_i^{(k)}, b_j^{(k)}$ can be contained in the same say, any $T_l$, since otherwise there would exist a common consequent in $\Pi(B)$, contrary to our assumption. Denote as above $|T_k| = s_k$, so that $s_1 + s_2 + \ldots + s_d = s$. The set $T_k$ contains exactly $\frac{1}{2}s_k(s_k - 1)$ unordered pairs with different coordinates which are included in the number $\frac{1}{2}s(s-1)$ and which have to be subtracted. Hence

$$g(b_i, b_j) \leqq \tfrac{1}{2}s(s-1) - \sum_{i=1}^{d} \tfrac{1}{2}s_i(s_i - 1) = \tfrac{1}{2}s^2 - \tfrac{1}{2}\sum_{i=1}^{d} s_i^2 .$$

We first settle the case that for all $s_i$ we have $s_i \geqq 2$ (hence $s \geqq 4$). In this case $s_i \geqq 2$ implies $s_i^2 \geqq 2s_i$ and $\sum_{i=1}^{d} s_i^2 \geqq 2\sum_{i=1}^{d} s_i = 2s$. Hence $g(b_i, b_j) \leqq \frac{1}{2}s^2 - \frac{1}{2} \cdot 2s = \frac{1}{2}s^2 - s < N(s)$.

It remains to deal with the case that at least one of the sets of imprimitivity of $B$ is a one-point set.

Suppose first $d = 2$ and put $T_1 = \{b_1\}$, $T_2 = \{b_2, b_3, \ldots, b_s\}$. Recall that $T_1\varrho = T_2$, $T_2\varrho = T_1$. When beginning with the couple $\binom{b_1}{b_j}$, $j \in \{2, 3, \ldots, s\}$, any admissible sequence is of the form

$$\binom{b_1}{b_j}, \binom{\cdot}{b_1}, \binom{b_1}{\cdot}, \binom{\cdot}{b_1}, \ldots .$$

No such sequence without repetitions can contain more than $s - 1$ terms. Hence $g(b_i, b_j) \leqq s - 1 < N(s)$ for $s \geqq 3$. For $s = 2$ we have $g(b_i, b_j) = 1$.

Suppose now $d \geqq 3$ (hence $s \geqq 3$) and $b_i \in T_i$, $b_j \in T_j$, $T_i \neq T_j$. Consider the sequences

(29)
$$T_i, T_i\varrho, T_i\varrho^2, \ldots, T_i\varrho^{d-1} ,$$
$$T_j, T_j\varrho, T_j\varrho^2, \ldots, T_j\varrho^{d-1} .$$

Let $\{b_0\}$ be one of the sets of imprimitivity containing exactly one element. Then there is integer $k_0$, $0 \leqq k_0 \leqq d - 1$, and an integer $l_0$, $0 \leqq l_0 \leqq d - 1$, such that $T_i\varrho^{k_0} = \{b_0\}$, $T_j\varrho^{l_0} = \{b_0\}$. [Here $T_i\varrho^0$ denotes $T_i$.] Denote $T_i\varrho^{l_0} = T''$, $T_j\varrho^{k_0} = T'$. The equality $k_0 = l_0$ cannot hold since the couple $(b_i, b_j)$ has no common consequent in $\Pi(B)$. Now (29) can be written in the following form:

$$\begin{aligned} &T_i,\ T_i\varrho,\ \dots,\ \{b_0\},\ \dots,\ T'',\ \dots,\ T_i\varrho^{d-1}\,, \\ &T_j,\ T_j\varrho,\ \dots,\ T',\ \dots,\ \{b_0\},\ \dots,\ T_j\varrho^{d-1}\,. \end{aligned} \tag{30}$$

[It may happen that $\{b_0\}$ appears in the second row earlier than in the first row, but this has no influence on the following considerations.] Denote $T' = \{\beta_1, \dots, \beta_u\}$, $T'' = \{\gamma_1, \dots, \gamma_v\}$ and consider the paths

$$\begin{aligned} &b_i \xrightarrow{k_0} b_0 \xrightarrow{d} b_0 \xrightarrow{d} \dots \xrightarrow{d} b_0\,, \\ &b_j \xrightarrow{k_0} \beta_1 \xrightarrow{d} \beta_2 \xrightarrow{d} \dots \xrightarrow{d} \beta_u\,. \end{aligned}$$

Another application of a path of length $d$ necessarily leads to a repetition, while an addition of a path of length $d - 1$ cannot be excluded.

If $b_i \neq b_0$ (i.e. $k_0 \neq 0$), we have

$$g(b_i, b_j) \leqq (1 + k_0) + [|T'| - 1]\,d + (d - 1) \leqq [|T'| + 1]\,d - 1\,.$$

If $b_i = b_0$, we obtain

$$g(b_i, b_j) \leqq 1 + [|T'| - 1]\,d + (d - 1) = |T'|\,.\,d\,.$$

We may proceed analogously with the second row and consider the paths

$$\begin{aligned} &b_i \xrightarrow{l_0} \gamma_1 \xrightarrow{d} \gamma_2 \xrightarrow{d} \dots \xrightarrow{d} \gamma_v\,, \\ &b_j \xrightarrow{l_0} b_0 \xrightarrow{d} b_0 \xrightarrow{d} \dots \xrightarrow{d} b_0\,, \end{aligned}$$

whence

$$g(b_i, b_j) \leqq d[|T''| + 1] - 1 \quad \text{if} \quad b_j \neq b_0 \quad \text{and} \quad g(b_i, b_j) \leqq |T''|\,.\,d$$

if $b_j = b_0$.

We have proved:

**Lemma 4,3.** *If $d \geqq 3$, then using the notations introduced above we have*

$$g(b_i, b_j) \leqq d\,.\,\min\,(|T'|, |T''|) + d - 1\,. \tag{31}$$

We now prove:

**Lemma 4,4.** *Let $M(\varrho)$ be of the form* (22) *and $s \geqq 4$. If $b_i, b_j \in \Pi(B)$ have a common consequent in $\Pi(A)$, then $g(b_i, b_j) \leqq N(s)$.*

Remark. For $s = 2$ we have $g(b_1, b_2) = 1$, and for $s = 3$ we have $g(b_i, b_j) \leqq 3$.

Proof. With respect to the results obtained above it is sufficient to suppose that $d \geqq 3$ and at least one of the sets of imprimitivity is a one-point set.

1) Suppose that $s$ is even (hence $s \geqq 4$). Then both $|T'|$, $|T''|$ cannot be $\geqq \frac{1}{2}s$

simultaneously since then the cardinality of $\{b_0\} \cup T' \cup T''$ would be $>s$. We may suppose $|T'| \leqq \frac{1}{2}s - 1$ and $|T''| \geqq |T'|$.

a) If $|T'| \leqq \frac{1}{2}s - 2$ (hence $s \geqq 6$), then (since $d \leqq s$) we have by (31) $g(b_i, b_j) \leqq s(\frac{1}{2}s - 2) + s - 1 = \frac{1}{2}s^2 - s - 1 < N(s)$.

b) Suppose that $|T'| = \frac{1}{2}s - 1$ (hence $s \geqq 4$). Since $|\{b_0\}| + |T'| + |T''| \geqq \geqq 1 + (\frac{1}{2}s - 1) + (\frac{1}{2}s - 1) = s - 1$, there is at most one further class of imprimitivity, so that $d \leqq 4$ and by (31)

$$g(b_i, b_j) \leqq d[|T'| + 1] - 1 = 4[(\tfrac{1}{2}s - 1) + 1] - 1 = 2s - 1 < N(s) \quad \text{for} \quad s \geqq 6.$$

If $s = 4, d = 3$, we have $g(b_i, b_j) \leqq 5 = N(s)$.

It remains to assume $s = 4$, in the case all sets of imprimitivity are one-point sets. It is immediately seen that in this case $g(b_i, b_j) = 4 < N(4) = 5$.

2) Suppose that $n$ is odd. Then both $|T'|$ and $|T''|$ cannot be greater than $\frac{1}{2}(s - 1)$ simultaneously. We suppose $|T'| \leqq |T''|$ in what follows

a) If $|T'| = |T''| = \frac{1}{2}(s - 1)$, then $d = 3$, and by (31) we have $g(b_i, b_j) \leqq \leqq 3 \cdot \frac{1}{2}(s - 1) + (3 - 1) = \frac{1}{2}(3s + 1) < N(s)$ for $s \geqq 5$.

b) If $|T'| \leqq \frac{1}{2}(s - 5)$ (hence $s \geqq 7$), then by (31) we have $g(b_i, b_j) \leqq \leqq s \cdot \frac{1}{2}(s - 5) + s - 1 = \frac{1}{2}s^2 - \frac{3}{2}s - 1 < N(s)$.

c) Let $|T'| = \frac{1}{2}(s - 3)$ (hence $s \geqq 5$). Since $|\{b_0\}| + |T'| + |T''| \geqq 1 + + \frac{1}{2}(s - 3) + \frac{1}{2}(s - 3) = s - 2$. there are at most two further sets of imprimitivity so that $d \leqq 5$ and by (31) for $s > 5$,

$$g(b_i, b_j) \leqq 5\left[\frac{s - 3}{2} + 1\right] - 1 = \tfrac{1}{2}(5s - 7) < N(s).$$

If $s = 5$, then $|T'| = 1$ and $d$ is either 3 or 4 or 5. If $d = 3$, then (by (31)) $g(b_i, b_j) \leqq \leqq 5 < N(5) = 9$. If $d = 4$, then (again by (31)) $g(b_i, b_j) \leqq 7 < N(5)$. If $d = 5$, we immediately have $g(b_i, b_j) = 5 < N(5)$.

This proves Lemma 4,4.

We are finally able to find estimates for $L(b_i, b_j)$ if the *c.c.* is contained in $\Pi(A)$.

1) Suppose $r = 1$, $s \geqq 2$. This case can be treated directly. By supposition there is a $b'_i \in \Pi(B)$ such that $(b'_i, a) \in \varrho$, i.e. $a = a\varrho \cap b'_i\varrho$ (since $a = a\varrho = a\varrho^2 = \ldots$). Since $B$ is irreducible, for any $b_k$ there exists an integer $l_k$ $(0 \leqq l_k \leqq s - 1)$ such that $b'_i \in b_k\varrho^{l_k}$, hence $a \in b_k^{l_k+1}$. This implies $a \in b_k\varrho^s$ (for any $k$). In particular $a \in b_i\varrho^s \cap \cap b_j\varrho^s$, whence $L(b_i, b_j) \leqq s = n - 1 < N(n)$.

2) Suppose $r \geqq 2$, $s \geqq 2$. Returning to (28) recall that if no term of the form $\begin{pmatrix} a^{(v)} \\ b^{(v)} \end{pmatrix}$ occurs in (28) we have $L(b_i, b_j) \leqq g(b_i, b_j) + L(A)$.

3) Suppose that $r \geqq 2$, $s \geqq 2$ and (18) contains a term of the form $\begin{pmatrix} a^{(v)} \\ b^{(v)} \end{pmatrix}$. Then $L(b_i, b_j) \leqq g(b_i, b_j) + L(a, b)$ with some $a \in \Pi(A)$, $b \in \Pi(B)$. Hence

$$L(b_i, b_j) \leqq g(b_i, b_j) + \max\,[rs, rs - s + L(A)]\,.$$

If $L(A) < s$, $\max\,[rs, rs - s + L(A)] = rs > s > L(A)$. If $L(A) \geqq s$,

$$\max\,[rs, rs - s + L(A)] = rs - s + L(A) = (r - 1)\,s + L(A) > L(A)\,.$$

Hence we always have $L(A) < \max[rs, rs - s + L(A)]$, so that the case 2) may be omitted (giving bounds smaller than the case 3)).

a) Suppose $L(A) > s$, then $L(b_i, b_j) \leqq g(b_i, b_j) + rs - s + L(A)$. By Lemma 4,2

$$L(b_i, b_j) \leqq [g(b_i, b_j) - N(s)] + N(n) - [N(r) - L(A)] - (s - \delta_{rs}).$$

Since $g(b_i, b_j) - N(s) \leqq 0$, $s - \delta_{rs} \geqq 1$, we have $L(b_i, b_j) < N(n) - N(r) + L(A)$.

b) Suppose $L(A) \leqq s$, then $L(b_i, b_j) \leqq g(b_i, b_j) + rs = [g(b_i, b_j) - N(s)] + N(n) - [N(r) - \delta_{rs}]$.

For $r \geqq 3$ we have $N(r) - \delta_{rs} \geqq 1$, and since $g(b_i, b_j) - N(s) \leqq 0$ we obtain $L(b_i, b_j) < N(n)$.

If $r = 2$ and $s > 4$, we have $g(b_i, b_j) - N(s) < 0$ and $N(r) - \delta_{rs} = 0$, hence $L(b_i, b_j) < N(n)$. For $r = 2, s = 4$, we find directly $L(b_i, b_j) \leqq 6 < N(n)$.

Now two cases remain, namely $r = 2$, $s = 2$ and $r = 2$, $s = 3$. The case $r = 2$, $s = 2$ has been settled in Example 4,1 by which $L(b_1, b_2) \leqq 3 < N(4) = 5$. The case $r = 2$, $s = 3$ and $d_B = 3$ has been considered in Example 4,2 by which $L(b_i, b_j) \leqq 5 < N(5) = 9$. If $r = 2$, $s = 3$ and $d_B = 2$, we have seen above that $g(b_i, b_j) \leqq s - 1 = 2$, so that $L(b_i, b_j) \leqq 2 + rs = 8 < N(5) = 9$. [As a matter of fact by considering the corresponding graph we obtain in the last case that either $L(b_i, b_j)$ does not exist or $L(b_i, b_j) \leqq 3$.]

We have proved:

**Lemma 4,5.** *Suppose that $M(\varrho)$ is of the form (22), where $A$ is an $r \times r$ Boolean matrix without a zero row and $B$ is an irreducible Boolean matrix with $s \geqq 2$. Put $r + s = n$. Suppose that $b_i, b_j \in \Pi(B)$ have a common consequent contained in $\Pi(A)$. Then the following holds*:

a) *If $r = 1$, $s \geqq 2$, then $L(b_i, b_j) \leqq n - 1 < N(n)$.*

b) *If $r \geqq 2$, $s \geqq 2$ and $L(A) \leqq s$, then $L(b_i, b_j) < N(n)$.*

c) *If $r \geqq 2$, $s \geqq 2$ and $L(A) > s$, then $L(b_i, b_j) < N(n) - N(r) + L(A)$.*

In the special case that also $A$ is irreducible we have $L(A) \leqq N(r)$ (for $r \geqq 2$), so that $L(b_i, b_j) < N(n)$. This enables us to prove:

**Theorem 4,1.** *Suppose that $M(\varrho)$ is of the form (22) and both $A \neq 0$, $B \neq 0$ are irreducible. Let $x, y \in \Pi(A) \cup \Pi(B)$. If $L(x, y)$ exists, then $L(x, y) < N(n)$.*

Proof. We have to consider the following cases: a) $x, y \in \Pi(A)$ (if $r > 1$); b) $x \in \Pi(A)$, $y \in \Pi(B)$; c) $x, y \in \Pi(B)$ (if $s > 1$). In the first case (by Theorem 3,2) $L(x, y) \leqq N(r) < N(n)$. In the second case $L(x, y) < N(n)$ (by Corollary 4,1). In the third case if the *c.c.* of $x, y$ is contained in $\Pi(B)$, $L(x, y) \leqq N(s) < N(n)$. If the *c.c.* is contained in $\Pi(A)$, then $L(x, y) < N(n)$ (by the remark made just before the statement of our theorem). This proves Theorem 4,1.

**Theorem 5,1.** *Let $M(\varrho)$ be a $n \times n$ Boolean matrix of the form*

$$M(\varrho) = \begin{pmatrix} A_1 & & & \\ A_{21} & A_2 & & \\ \vdots & & & \\ A_{l1} & A_{l2} & \dots & A_l \end{pmatrix},$$

*where $l \geqq 2$ and $A_i$ $(i = 1, 2, \dots, l)$ are irreducible Boolean matrices. If $x, y \in \Pi(M)$ and $L(x, y)$ exists, then $L(x, y) < N(n)$.*

Proof. We may suppose $A_1 \neq 0$ since otherwise the statement is true by Lemma 0,3. We proceed by induction. For our convenience we denote

$$A^{[k]} = \begin{pmatrix} A_1 & & & \\ A_{21} & A_2 & & \\ \vdots & & & \\ A_{k1} & A_{k2} & \dots & A_k \end{pmatrix}$$

and $A_{k+1} = B$, so that

$$A^{[k+1]} = \begin{pmatrix} A^{[k]} & 0 \\ C & B \end{pmatrix},$$

where $C$ has the obvious meaning. Denote $|\Pi(A^{[k]})| = r$, $|\Pi(B)| = s$, so that $r + s = n$

The statement is true for the matrix $A^{[2]}$. Indeed, if $A_2 = 0$ it follows from Lemma 0,4, if $A_2 \neq 0$, it is the statement of Theorem 4,1.

Let $k \geqq 2$ and suppose that for any $x, y \in \Pi(A^{[k]})$ we have $L(x, y) < N(r)$. Our statement will be proved if we are able to show that then $L(x, y) < N(n)$ for all pairs $x, y \in \Pi(A^{[k+1]})$ (provided $L(x, y)$ exists).

We may suppose $r \geqq 2$. Then we have to consider several cases. When writing $L(x, y)$ we suppose that it exists.

1) If $x, y \in \Pi[A^{[k]}]$ and $L(A^{[k]}) > 0$, we have by supposition $L(x, y) \leqq N(r) < < N(n)$.

2) If $A^{[k]}$ has a zero row, we have (by Lemma 0,3) $L(x, y) < N(n)$ for any $x, y \in \in \Pi(A^{[k+1]})$.

3) If $s = 1$, $B = 0$, we have (by Lemma 0,4) $L(x, y) < N(n)$ for any $x, y \in \in \Pi(A^{[k+1]})$.

4) If $s = 1$, $B \neq 0$, $x \in \Pi(A^{[k]})$, $y \in \Pi(B)$, we have (by Lemma 4,1) $L(x, y) \leqq \leqq n - 1 < N(n)$.

5) Suppose $s \geqq 2$, $A^{[k]}$ has not a zero row, and $x \in \Pi(A^{[k]})$, $y \in \Pi(B)$.

By Lemma 4,1 we have $L(x, y) \leqq \max(rs, rs - s + L(A^{[k]}))$. Now $rs \leqq \frac{1}{4}n^2 < < N(n)$. Next, since $L(A^{[k]}) < N(r)$, by the induction hypothesis, we have

$$rs - s + L(A^{[k]}) < rs - s + N(r) = N(n) - N(s) - s + \delta_{rs} < N(n).$$

Hence $L(x, y) < N(n)$.

6) Suppose $s \geqq 2$, $A^{[k]}$ has not a zero row, and $x, y \in \Pi(B)$. If the *c.c.* of $x$ and $y$ is in $\Pi(B)$, then (by Theorem 3,2) $L(x, y) \leqq N(s) < N(n)$. Suppose therefore that the *c.c.* is contained in $\Pi(A^{[k]})$.

If $L(A^{[k]}) \leqq s$, then (by Lemma 4,5b) $L(x, y) < N(n)$.

If $L(A^{[k]}) > s$ [and, of course, $L(A^{[k]}) < N(r)$ by supposition], we have (by Lemma 4,5c) $L(x, y) < N(n) - N(r) + L(A^{[k]}) < N(n)$. This proves Theorem 5,1.

*References*


[1] *Paz, A.:* Introduction to probabilistic automata, Academic Press, New York, 1971.

[2] *Schwarz, Š.:* On the semigroup of binary relations on a finite set. Czech. Math. J. *20* (1970), 632—679.

[3] *Schwarz, Š.:* On a sharp estimation in the theory of binary relations on a finite set. Czech. Math. J. *20* (1970), 703—714.



*Author's address:* 814 73 Bratislava, Obrancov mieru 49, Czechoslovakia (Matematický ústav SAV).