Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 35 (1985), No. 3, 405–414

Persistent URL: http://dml.cz/dmlcz/102030

Czechoslovak Mathematical Journal, 35 (110) 1985, Praha

RECTANGULAR GROUPOIDS

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(Received March 2, 1984)

0. Introduction. A groupoid $(G, \cdot)$ is called a *rectangular (right) groupoid* if it satisfies the following two laws

(1) $$x^2 = x ,$$

(2) $$(xy) z = xz .$$

It is clear that every rectangular band is a rectangular groupoid.

A rectangular groupoid is called a *near-rectangular (right) groupoid* if it satisfies

(3) $$x(yx) = x .$$

We shall also deal with the following two identities:

(4) $$x(y(zx)) = x(z(yx)) ,$$

(5) $$x(y(zu)) = x(z(yu)) .$$

Denote by V_1 the variety of groupoids satisfying (1), (2), (3) and (4), and by V_2 the variety of groupoids satisfying (1), (2) and (5).

By $p_n(\mathfrak{A})$ we denote the number of all essentially n-ary polynomials over an algebra $\mathfrak{A}$ ([2]).

In this paper we prove the following:

Theorem 1. *For any rectangular groupoid $(G, \cdot)$ which is not a semigroup we have*

$$p_n((G, \cdot)) \geqq n^2 \quad for \quad n \geqq 3 .$$

Theorem 2. *Let $(G, \cdot)$ be a rectangular groupoid. Then the following conditions are equivalent:*

(i) $(G, \cdot)$ *is not a semigroup and* $(G, \cdot)$ *satisfies* $x(y(zu)) = x(z(yu))$;

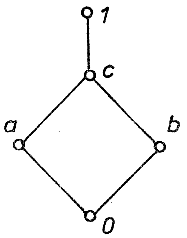
(ii) $p_4((G, \cdot)) = 16$;

(iii) $p_n((G, \cdot)) = n^2$ *for all* n.

Before formulating the next theorem we need some notations.

Let V be a variety of algebras. Then, by $\mathscr{L}(V)$ we denote the lattice of all subvarieties of V.

Theorem 3. *The lattice $\mathscr{L}(V_2)$ has the following diagram*:



where a is the variety of left zero semigroups, b is the variety or right zero semigroups and c is the variety of all rectangular bands. 0 *is the variety of one-element groupoids and* 1 *is the variety* V_2.

1. Examples. In this section we give some examples of rectangular groupoids which are not semigroups and belong to V_1 and or V_2.

1.1. Let $G_1 = \{1, 2, 3\}$ and $G_2 = \{0, 1, 2, 3, 4\}$. On G_1 and G_2 we define binary operations $\cdot$, $*$ by Cayley's tables

$\cdot$	1	2	3
1	1	3	3
2	2	2	2
3	1	3	3

$*$	0	1	2	3	4
0	0	1	3	3	1
1	0	1	3	3	1
2	2	4	2	2	4
3	0	1	3	3	1
4	2	4	2	2	4

Then $(G_1, \cdot)$ and $(G_2, *)$ are not semigroups since in the first case we have $1 = 31 = (32)\,1 \neq 3(21) = 32 = 3$ and in the other we have $3 = 0(40) \neq (04)\,0 = 0$. Nonetheless, both groupoids belong to V_2. It is easy to see (and useful in checking all identities of V_2) that $(\{1, 2, 3, 4\}, *)$ is a rectangular band. The groupoid $(G_1, \cdot)$ is the smallest rectangular groupoid which is not a semigroup. Let us mention that $(G_2, *)$ is a subgroupoid of the free groupoid in V_2 with two free generators.

1.2. Here we also give examples of two groupoids from V_1 which are not semigroups. If

$$G_3 = \{1, 2, 3, 4, 5, 6\}$$

and

$$G_4 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}\,,$$

then on the set G_3 and G_4 we define operations $\bigcirc$ and $\otimes$ by the tables

$\bigcirc$	1	2	3	4	5	6
1	1	3	3	1	1	3
2	4	2	2	4	2	4
3	1	3	3	1	1	3
4	4	2	2	4	2	4
5	5	5	6	6	5	6
6	5	5	6	6	5	6

(Notice that $(\{1, 2, 3, 4\}, \bigcirc)$ and $(\{5, 6\}, \bigcirc)$ are rectangular bands but $(G_3, \bigcirc)$ is not a semigroup since $6 = 6(12) \neq (61)\,2 = 5$.)

$\otimes$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1	2	3	4	5	6	2	1	4	3	6	5	3	1	5	2	6	4
2	1	2	3	4	5	6	2	1	4	3	6	5	3	1	5	2	6	4
3	1	2	3	4	5	6	2	1	4	3	6	5	3	1	5	2	6	4
4	1	2	3	4	5	6	2	1	4	3	6	5	3	1	5	2	6	4
5	1	2	3	4	5	6	2	1	4	3	6	5	3	1	5	2	6	4
6	1	2	3	4	5	6	2	1	4	3	6	5	3	1	5	2	6	4
7	8	7	10	9	12	11	7	8	9	10	11	12	9	11	7	12	8	10
8	8	7	10	9	12	11	7	8	9	10	11	12	9	11	7	12	8	10
9	8	7	10	9	12	11	7	8	9	10	11	12	9	11	7	12	8	10
10	8	7	10	9	12	11	7	8	9	10	11	12	9	11	7	12	8	10
11	8	7	10	9	12	11	7	8	9	10	11	12	9	11	7	12	8	10
12	8	7	10	9	12	11	7	8	9	10	11	12	9	11	7	12	8	10
13	14	16	13	18	15	17	15	17	13	18	14	16	13	14	15	16	17	18
14	14	16	13	18	15	17	15	17	13	18	14	16	13	14	15	16	17	18
15	14	16	13	18	15	17	15	17	13	18	14	16	13	14	15	16	17	18
16	14	16	13	18	15	17	15	17	13	18	14	16	13	14	15	16	17	18
17	14	16	13	18	15	17	15	17	13	18	14	16	13	14	15	16	17	18
18	14	16	13	18	15	17	15	17	13	18	14	16	13	14	15	16	17	18

$3 = 2\ 13 = (1\ 7)\ 13 \neq 1(7\ 13) = 1\ 9 = 4.$
It can be checked that $(G_3, \bigcirc)$ and $(G_4, \otimes)$ belong to V_1 and $(G_4, \otimes)$ is isomorphic to the free groupoid in V_1 with three free generators.

1.3. Here we give some examples of infinite rectangular groupoids. Let X be a set with $\operatorname{card} X \geqq 2$. Take $G = X^{\aleph_0}$. On the set G we define "multiplication" as follows. If $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ and $x, y \in G$, then we put

$$xy = \begin{cases} (x_1, y_2, y_3, \ldots) & \text{if } x_1 = y_1\,, \\ (x_1, y_1, y_2, \ldots) & \text{if } x_1 \neq y_1\,. \end{cases}$$

It can be easily checked that this groupoid is a rectangular groupoid which is not a semigroup. Similarly, an infinite groupoid can be constructed in the following manner. Take a set A such that $\operatorname{card} A \geqq 2$. For the underlying set G we put $\bigcup_{n=1}^{\infty} A^n$. If $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$, then we put

$$xy = \begin{cases} (x_1, y_2, \ldots, y_n) & \text{if } x_1 = y_1\,, \\ (x_1, y_1, y_2, \ldots, y_n) & \text{if } x_1 \neq y_1\,. \end{cases}$$

Analogously as in the previous case, one can verify that $(G, \cdot)$ satisfies the required identities (1) and (2) and does not satisfy the identity (3). It should be mentioned that there exist nonidempotent groupoids satisfying $(xy)\,z = xz$. For example, take $G = X^{\aleph_0}$, where card $X \geqq 2$, and define

$$xy = (x_1, x_2, \ldots)(y_1, y_2, \ldots) = (x_1, y_1, y_2, \ldots)\,.$$

2. Polynomials in rectangular groupoids. The main aim of this section is to present all lemmas we need for the proof of Theorem 1.

Lemma 2.1. *If $(G, \cdot)$ is a rectangular groupoid, then $(G, \cdot)$ satisfies the identities*

$$(xy)\,x = x\,, \quad (xy)\,y = xy$$

and

$$xy = x(xy)\,.$$

The proof is obvious.

An idempotent algebra of a fixed type is called *proper* if all fundamental polynomials are mutually different and depend on all variables (for a groupoid $(G, \cdot)$ this means that xy is essentially binary).

Lemma 2.2. *If $(G, \cdot)$ is a proper rectangular groupoid, then*

$$x(yx) \notin \{y, xy, yx\}\,.$$

Proof. Let $x(yx) = y$. Then by the identity (2) we get $y = yy = (x(yx))\,y = xy$ – a contradiction since $(G, \cdot)$ is proper. Assuming $xy = x(yx)$ and putting xy for y in this identity, we get, by the previous lemma,

$$xy = x(xy) = x((xy)\,x) = xx = x$$

which is impossible. If $x(yx) = yx$, then

$$y = (yx)\,y = (x(yx))\,y = xy \quad (\text{by } (2))$$

which is also a contradiction.

Lemma 2.3. *Let $(G, \cdot)$ be a proper rectangular groupoid. Then $(G, \cdot)$ satisfies the identity (3) if and only if $p_2((G, \cdot)) = 2$.*

Proof. If $(G, \cdot)$ is a proper rectangular groupoid satisfying (3), then using Marczewski's description of the set $\mathbb{A}^{(n)}(\mathfrak{A})$ of a given algebra $\mathfrak{A}$ ([6]) we infer that xy, yx are the only essentially binary and distinct polynomials over $(G, \cdot)$ (This fact can be verified directly.) The converse follows from Lemma 2.2.

Lemma 2.4. *If $(G, \cdot)$ is a proper rectangular groupoid, then the polynomial $x \circ y = x(yx)$ is not commutative.*

Proof. If $x(yx) = y(xy)$, then $xy = (x(yx))\,y = (y(xy))\,y = y$ and hence $y(xy) = y$ which proves $x = y$.

Lemma 2.5. *If* $(G, \cdot)$ *is a rectangular groupoid, then* $(G, \circ)$, *where* $x \circ y = x(yx)$, *satisfies the identities*

$$x \circ x = x\,, \quad x \circ (y \circ z) = x \circ y\,.$$

Proof. We have

$$x \circ (y \circ z) = x \circ (y(zy)) = x((y(zy))\,x) = x(yx) = x \circ y\,.$$

Note that idempotent groupoids satisfying $x \circ (y \circ z) = x \circ y$ were considered by J. Płonka ([8]).

Further on, we shall consider two sequences of n-ary polynomials over a (proper rectangular) groupoid $(G, \cdot)$, namely

$$g_n = g_n(x_1, x_2, \ldots, x_{n-1}, x_n) = x_1(x_2(\ldots(x_{n-1}x_n))\ldots)$$

and

$$g_n^* = g_n^*(x_1, x_2, \ldots, x_{n-1}, x_n) = g_{n+1}(x_1, x_2, \ldots, x_{n-1}, x_n, x_1) = \\ = x_1(x_2(\ldots(x_{n-1}(x_nx_1)))\ldots)$$

for all $n \geqq 2$.

Lemma 2.6. *If* $(G, \cdot)$ *is a rectangular groupoid which is not a semigroup, then the polynomial* g_n *is essentially n-ary for all* $n \geqq 2$.

Proof follows from Lemma 3 of [1].

To formulate the next lemma we need some more notations and definitions.

Let f be an n-ary function on a set A. We say that f admits a permutation $\sigma \in S_n$, where S_n is the symmetry group of an n-element set, if $f = f^\sigma$, where $f^\sigma(x_1, \ldots, x_n) = f(x_{\sigma 1}, \ldots, x_{\sigma n})$. By $G(f)$ we denote the group of all admissible permutations of f ([4]).

Lemma 2.7. *The assumption as above. If* $g_n = g_n^\sigma$ *for some* $n > 2$ *and some (non-identical permutation)* $\sigma \in S_n$, *then* $\sigma(1) = 1$ *and* $\sigma(n) = n$.

Proof. Assume that $g_n = g_n^\sigma$ and $n \geqq 3$. Then we have

$$x_1x = (g_n(x_1, \ldots, x_n))\,x = (g_n^\sigma(x_1, \ldots, x_n))\,x = x_{\sigma 1}x\,.$$

Hence $\sigma(1) = 1$. Suppose that $\sigma(n) \neq n$ in the identity $g_n = g_n^\sigma$. Putting x_nx_{n+1} for x_n in this identity and using $(xy)\,z = xz$ we infer that

$$g_n(x_1, \ldots, x_{n-1}, x_nx_{n+1}) = g_{n+1}(x_1, \ldots, x_n, x_{n+1}) = \\ = g_n(x_1, x_{\sigma 1}, \ldots, x_nx_{n+1}, \ldots, x_{\sigma n}) = g_n(x_1, \ldots, x_n, x_{\sigma k}, \ldots, x_{\sigma n})\,.$$

This proves that the polynomial g_{n+1} does not depend on the variable x_{n+1}, which contradicts the previous lemma.

Lemma 2.8. *Let* $(G, \cdot)$ *be a rectangular groupoid. Then the following conditions are equivalent*:

(c_1) $(G, \cdot)$ *is a rectangular band*;

(c_2) *the polynomial* $x(y(zx))$ *is not essentially ternary*;

(c_3) *the identity* $x(y(zx)) = x$ *holds in* $(G, \cdot)$.

Proof. Let us remark that if $(G, \cdot)$ is a semigroup of left (right) zeros, then all these conditions are equivalent. So, further on, we may assume that $(G, \cdot)$ is proper. The implications $(c_1) \Rightarrow (c_2)$ and $(c_1) \Rightarrow (c_3)$ are obvious. Now we prove $(c_2) \Rightarrow (c_3)$. First of all observe that $xu = (x(y(zx)))\,u$ proves that $x(y(zx))$ depends on x. One can prove that the polynomial $x(y(zx))$ does not depend on y if and only if it does not depend on z. From this fact and (c_2) we get $x(y(zx)) = x$. Now we prove $(c_3) \Rightarrow$ $\Rightarrow (c_1)$. First of all, putting $x = z$ in (c_3) we have $x(yx) = x$. Setting uv for x in the identity $x = x(y(zx))$, we obtain $uv = (uv)\,(y(z(uv)))$. If $z = v$, we deduce that

$$uv = u(y(v(uv))) = u(yv)\,, \quad \text{which proves} \quad (c_1)\,.$$

Lemma 2.9. *If* $(G, \cdot)$ *is a proper rectangular groupoid, then the polynomial* $g_n^*(x_1, \ldots, x_n)$ *depends on* x_1 *for* $n = 2, 3, \ldots$.

Proof follows from $x_1 u = g_n^*(x_1, \ldots, x_n)\,u$.

Lemma 2.10. *If* $(G, \cdot)$ *is a rectangular groupoid which is not a semigroup, then the polynomials* g_n^* *are essentially n-ary for all* $n \geqq 3$.

Proof. The proof proceeds by induction on n. If $n = 3$, then using Lemma 2.8(c_2), we infer that g_3^* is essentially ternary. Assume now that for all k, where $4 \leqq k < n$, g_k^* is essentially k-ary. First of all observe that g_n^* depends on x_2. Indeed, if g_n^* does not depend on x_2, then putting $x_3 = x_4$ in the polynomial g_n^* we deduce that $g_n^*(x_1, x_2, x_4, x_4, x_5, \ldots, x_n) = g_{n-1}^*(x_1, x_2, x_4, x_5, \ldots, x_n)$ does not depend on x_2, either. This contradicts the inductive assumption. Now suppose that g_n^* does not depend on x_t, where $2 \leqq t \leqq n$. Then, of course, the polynomial

$$g_n^*(x_2, x_2, \ldots, x_t, \ldots, x_n) = g_{n-1}^*(x_2, x_3, \ldots, x_t, \ldots, x_n)$$

does not depend on x_t, where $3 \leqq t \leqq n$, which is a contradiction. Thus g_n^* is essentially n-ary for all $n \geqq 3$.

Lemma 2.11. *If* $(G, \cdot)$ *is a proper rectangular groupoid, then* $g_n^* = g_n^{*\sigma}$ *implies* $\sigma(1) = 1$ $(n = 2, 3, \ldots)$.

Proof follows from the fact that $x_1 x = g_n^* x = g_n^{*\sigma} x = x_{\sigma 1} x$ and the fact that $(G, \cdot)$ is proper.

Lemma 2.12. *If* $(G, \cdot)$ *is a rectangular groupoid which is not a semigroup, then* $\operatorname{card}(G(g_n)) \leqq (n-2)!$ *and* $\operatorname{card}(G(g_n^*)) \leqq (n-1)!$ *for every* $n \geqq 3$.

Proof follows from Lemmas 2.6, 2.7, 2.10 and 2.11.

Lemma 2.13. *The assumption as above. The polynomials* g_n^* *and* g_n^σ *are distinct for all* $\sigma \in S_n$ *and all* $n \geqq 2$.

Proof. For $n = 2$, the proof follows from Lemma 2.2. Let now

$$g_n^*(x_1, \ldots, x_n) = g_n(x_{\sigma 1}, \ldots, x_{\sigma(n-1)}, x_{\sigma n})$$

for $n \geqq 3$ and for a certain $\sigma \in S_n$. From this identity, as in the proof of Lemma 2.7, we infer that $\sigma(1) = 1$. Now, putting xy for $x_{\sigma n}$, where $\sigma(n) = k$, in the identity $g_n^* = g_n^\sigma$ and using the identity $(xy)\, z = xz$, we infer that

$$g_n^*(x_1, \ldots, x_{k-1}, x, \ldots, x_n) = g_n^*(x_1, \ldots, x_{k-1}, xy, \ldots, x_n) =$$
$$= g_n(x_1, x_{\sigma 2}, \ldots, x_{\sigma(n-1)}, xy) = g_{n+1}(x_1, x_{\sigma 2}, \ldots, x_{\sigma(n-1)}, x, y)$$

is not essentially $n + 1$-ary. This contradicts Lemma 2.6.

Lemma 2.14. *Let $(G, \cdot)$ be a rectangular groupoid satisfying $x(yx) = x$. Then $(G, \cdot)$ is a rectangular band if and only if $(G, \cdot)$ satisfies*

$$x_1(x_2(x_3x_4)) = x_1(x_3(x_2x_4)) .$$

Proof. If $(G, \cdot)$ is a rectangular band, then this statement is obvious. To prove the converse, observe that

$$x_1(x_2x_3) = x_1(x_2(x_3x_3)) = x_1(x_3(x_2x_3)) = x_1x_3 .$$

Lemma 2.15. *If $(G, \cdot)$ is a rectangular groupoid which is not a semigroup, then the following conditions are equivalent:*

(α) $p_4((G, \cdot)) = 16$;

(β) *the identity $x_1(x_2(x_3x_4)) = x_1(x_3(x_2x_4))$ holds in $(G, \cdot)$.*

Proof. Assume that $p_4((G, \cdot)) = 16$. Consider the polynomial $g_4(x_1, x_2, x_3, x_4) = x_1(x_2(x_3x_4))$. By Lemma 2.6 we infer that g_4 is essentially 4-ary. If this polynomial does not admit any nontrivial permutation, then by permuting variables in g_4 we get 24 essentially 4-ary polynomials, which contradicts the assumption $p_4((G, \cdot)) = 16$. If g_4 admits a nontrivial permutation of its variables, then according to Lemma 2.7 we get the required result.

Now assume that $(G, \cdot)$ is a rectangular groupoid which is not a semigroup and let the identity

$$x_1(x_2(x_3x_4)) = x_1(x_3(x_2x_4))$$

hold in $(G, \cdot)$. Applying Lemmas 2.6 and 2.10 we deduce that g_4 and g_4^* are essentially 4-ary. By permuting variables in g_4 we get 12 different essentially 4-ary polynomials over $(G, \cdot)$. Applying the same for g_4^* and Lemma 2.11 we get 4 essentially 4-ary polynomials. Now, using Lemma 2.13 we get at least 16 mutually different and essentially 4-ary polynomials. Applying Marczewski's description of $\mathbb{A}^{(n)}(\mathfrak{A})$ for $\mathfrak{A} = (G, \cdot)$ and $n = 4$ we compute that the only essentially 4-ary polynomials are the 16 polynomials obtained from g_4 and g_4^* by permuting their variables.

Lemma 2.16. *If $(G, \cdot)$ is a rectangular groupoid that is not a semigroup, then $p_4((G, \cdot)) = 16$ implies $p_n((G, \cdot)) = n^2$ for all n.*

Proof. According to the previous lemma we may assume that $(G, \cdot)$ satisfies the identity

$$x_1(x_2(x_3x_4)) = x_1(x_3(x_2x_4)) .$$

Now, using this identity, Lemmas 2.6, 2.7, 2.10 and 2.11 (analogously as in the previous lemma) and Marczewski's description of $\mathbb{A}^{(n)}((G, \cdot))$ we infer that the only essentially n-ary polynomials are

$$x_t(x_{\sigma 2}(x_{\sigma 3}(\ldots (x_{\sigma(n-1)}x_n)) \ldots))$$

and

$$x_r(x_{\tau 2}(x_{\tau 3}(\ldots (x_{\tau(n-1)}(x_{\tau n}x_r) \ldots)) ,$$

where $\sigma \in S_{n-2}$, $\tau \in S_{n-1}$ and $t, r = 1, 2, \ldots, n$.

Thus we have

$$p_n((G, \cdot)) = \frac{n!}{(n-2)!} + \frac{n!}{(n-1)!} = n^2 .$$

3. Proofs of theorems.

Proof of Theorem 1. Let $(G, \cdot)$ be a rectangular groupoid which is not a semigroup. Using Lemmas 2.6 and 2.10 we infer that g_n and g_n^* are essentially n-ary for every $n \geqq 3$. Now, applying Lemmas 2.12 and 2.13, we deduce

$$p_n((G, \cdot)) \geqq \frac{n!}{\operatorname{card} G(g_n)} + \frac{n!}{\operatorname{card} G(g_n^*)} \geqq \frac{n!}{(n-2)!} + \frac{n!}{(n-1)!} = n^2 .$$

The proof of the theorem is completed.

Proof of Theorem 2. The proof follows from Lemmas 2.15 and 2.16.

Proof of Theorem 3. It is clear that the varieties of a – left zero semigroups, b – right zero semigroups and c – the variety of rectangular bands, are subvarieties of V_2. Thus $\{0, a, b, c, 1\} \subseteq \mathscr{L}(V_2)$. To complete the proof it suffices to show that any subvariety of V_2 belongs to $\{0, a, b, c, 1\}$. Now, let U be a subvariety of V_2 and let $(G, \cdot)$ be a free groupoid in U with $\aleph_0$ generators. Then $u = v$ is an identity in $(G, \cdot)$ if and only if $u = v$ is an identity in U. Assume that $u = v$ is not a consequence of the identities of V_2. Obviously, if u and v are variables, then $U = 0$. If $(G, \cdot)$ is a rectangular band, then $U \in \{0, a, b, c, 1\}$. Now assume that $(G, \cdot)$ is a rectangular groupoid which is not a band and u and v are at least binary. Therefore the identity $u = v$ is one of the following:

$$g_n = g_m^{\sigma} , \quad g_n = g_m^{*\sigma} , \quad g_n^* = g_m^{*\sigma} ,$$

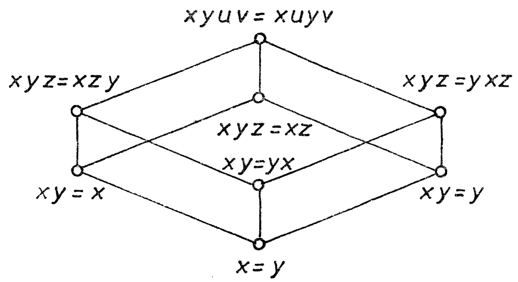
where $\sigma \in S_n$.

If $n \neq m$, we infer that g_n or g_n^* is not essentially n-ary since there is a variable appearing on only one side of the identity. This contradicts Lemmas 2.6, 2.10 for $n \geqq 3$ or $m \geqq 3$. If n and m are less than 3, then applying Lemma 2.14 we have a contradiction.

If $n = m$, then using Lemma 2.13 we get a contradiction.

4. Final remarks. In this section we give two remarks.

Remark 1. Let us note that there exist bands $(G, \cdot)$ for which $p_n((G, \cdot)) = n^2$ for all n. Such a band is a medial band, i.e. $(G, \cdot)$ satisfies $xyuv = xuyv$ (abelian law in [5]). This fact can be proved by applying the method of the proof of Lemma 2.15. Denote by V_3 the variety of all medial bands. Then the lattice $\mathscr{L}(V_3)$ is of the following form (cf. [7]):



It is clear that the lattice $\mathscr{L}(V_2)$ is a sublattice of $\mathscr{L}(V_3)$ and the free groupoids $F_{V_2}(n)$ and $F_{V_3}(n)$ have the same number of elements for all n. It is also worth noticing, by virtue of Theorem 3, that $V_2 = HSP((G_1, \cdot))$, where $(G_1, \cdot)$ is the groupoid considered in 1.1.

Remark 2. Observe that any nontrivial distributive lattice $(L, +, \cdot)$ satisfies $p_2((L, +, \cdot)) = 2$ and $p_3((L, +, \cdot)) = 9$. Groupoids $(G, \cdot)$ from V_1 which are not bands also have the same property. The fact that $p_2((G, \cdot)) = 2$ is obvious. Using the description of the set $A^{(3)}((G, \cdot))$ one can prove that the polynomials

$$x(yz),\ y(zx),\ z(xy),\ y(xz),\ z(yx),\ x(zy),\ x(y(zx)),\ y(z(xy))$$

and $z(x(yz))$ are the only essentially ternary polynomials over $(G, \cdot)$ and, of course, they are all mutually different. It is also clear that such a groupoid $(G, \cdot)$ can be treated as a proper algebra (G, xy, yx) of type (2,2). Thus we infer that there exists a proper idempotent algebra $\mathfrak{B}$ of type (2,2) for which $p_2(\mathfrak{B}) = 2$ and $p_3(\mathfrak{B}) = 9$.

We should mention here that this fact is connected with a Płonka's problem. To formulate this problem we need a concept of the minimal extension property of sequences ([3]).

Following Grätzer [3] we say that a sequence $a = (a_0, a_1, \ldots, a_n)$ of cardinals has the minimal extension property if there exists an algebra $\mathfrak{A}_0$ such that $p_k(\mathfrak{A}_0) = a_k$ for $k \leqq n$ and, if any algebra $\mathfrak{A}$ satisfies $p_k(\mathfrak{A}) = a_k$ for $k \leqq n$, then $p_k(\mathfrak{A}_0) \leqq p_k(\mathfrak{A})$ for all k.

Let l_n denote $p_n(\updownarrow)$, where $\updownarrow$ is a two-element lattice.

Recently J. Płonka has asked whether the sequence $(l_0, l_1, \ldots)$ is the minimal extension of the sequence $(0, 1, 2, 9)$. We should also mention here that numbers $l_n = p_n(\updownarrow)$ for $n \geqq 8$ are unknown ([2]).

If Płonka's question has an affirmative answer, then $p_n((G, \cdot)) \geqq l_n$ for all n and any $(G, \cdot) \in V_1$ which is not a semigroup.

We even do not know whether $p_n((G_3, \cdot)) \geqq l_n$ for all n, where $(G_3, \cdot)$ is the five-element groupoid of Section 1.

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