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ARCHIMEDEAN EQUIVALENCE FOR STRICTLY POSITIVE LATTICE-ORDERED SEMIGROUPS

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1. INTRODUCTION

The concept of archimedean equivalence has been an important one in the study of partially ordered algebraic structures. Holder's characterization [10] of additive subgroups of the reals as totally ordered groups (o-groups) with only one non-trivial archimedean class was followed in 1907 by Hahn's representation [9] of any abelian o-group as a group of real-valued functions on its set of archimedean classes. Renewed interest in the topic in the 1950s led eventually to the Conrad-Harvey-Holland representation theorem [7] for abelian lattice-ordered groups (l-groups); archimedean equivalence has also been studied for non-abelian l-groups (see e.g. [6]).

In the last fifteen years T. Saito has studied archimedean equivalence for totally ordered semigroups (o-semigroups) ([14], [15], [16]); in this paper we shall consider this concept for lattice-ordered semigroups (l-semigroups).

By an l-semigroup we shall mean a semigroup equipped with a lattice order, so that multiplication distributes over both of the lattice operations, from both the left and the right (this definition is a bit stronger than that in [8]).

The concept of archimedean equivalence is easiest defined for positive elements, but there is some ambiguity about what is meant by a "positive element" in a partially ordered semigroup. We shall in this paper confine ourselves to the strongest possible definition (thus following Saito in [16]): an element a of an l-semigroup S is said to be *strictly positive* if $ab \wedge ba \geqq b$ for all b in S. An l-semigroup is said to be *strictly positive* if each of its elements is; all l-semigroups considered in this paper are assumed to have this property.

2. ARCHIMEDEAN EQUIVALENCE

Let S be a strictly positive l-semigroup. Elements a and b of S are said to be *archimedean equivalent* if there exist positive integers m and n for which

$$a \leqq b^m \quad \text{and} \quad b \leqq a^n .$$

Since the positive powers of a positive element form an ascending sequence, we may evidently assume that m and n are equal. We shall denote archimedean equivalence by writing $a \,\boldsymbol{a}\, b$; more specifically, if a and b are bounded above by nth powers of b and a, respectively, we shall write $a \,\boldsymbol{a}\, b$ via n. We shall call S *a-simple* in case all elements of S are archimedean equivalent.

The following results are all straightforward generalizations of Saito's work for the totally ordered case ([14], [16]); for completeness' sake, we will include the proofs:

1. *$\boldsymbol{a}$ is an equivalence relation.*

Proof. It is evident that $\boldsymbol{a}$ is reflexive and symmetric. If $a \,\boldsymbol{a}\, b$ via m and $b \,\boldsymbol{a}\, c$ via n, then we have

$$a \leqq b^m \leqq (c^n)^m = c^{mn},$$

and similarly, $c \leqq a^{mn}$.

2. *$\boldsymbol{a}$ is a semigroup congruence.*

Proof. If $a \,\boldsymbol{a}\, b$ via n, we have

$$ac \leqq (b^n)\, c \leqq (bc)^n, \quad \text{and} \quad bc \leqq (ac)^n.$$

The other side works the same way.

3. *$\boldsymbol{a}$ is a lattice congruence.*

Proof. If $a \,\boldsymbol{a}\, b$ via n, we have

$$a \wedge c \leqq b^n \wedge c \leqq (b^n \wedge b^{n-1}c \wedge \ldots \wedge c^n) = (b \wedge c)^n.$$

Three similar strings of inequalities together show that

$$(a \wedge c)\, \boldsymbol{a} (b \wedge c) \quad \text{and} \quad (a \vee c)\, \boldsymbol{a} (b \vee c).$$

4. *Each $\boldsymbol{a}$ class is an l-subsemigroup.*

Proof. It is obvious that each $\boldsymbol{a}$ class is a sublattice. To show an $\boldsymbol{a}$ class is a subsemigroup as well, suppose that $a \,\boldsymbol{a}\, b$ via n. Then $ab \leqq b^{n+1}$, and

$$b \leqq ab \leqq (ab)^{n+1},$$

and so $b \,\boldsymbol{a}\, ab$.

5. *For all a, b in S, $ab \,\boldsymbol{a}\, ba \,\boldsymbol{a} (a \vee b)$.*

Proof. We have

$$ab \leqq b(ab)\, a = (ba)^2,$$

and similarly for ba. Now, $(a \vee b) \leqq ab$, while

$$ab \leqq (a \vee b)^2.$$

Putting 1. through 5. together gives us the following:

Theorem. *Let S be a strictly positive l-semigroup. Then $S/\boldsymbol{a}$ is a commutative* *band where* $[a][b] = [ab] = [a \vee b]$; *that is, S is a lattice of a-simple l-semigroups.*

This means that in order to understand the structure of strictly positive l-semigroups, it suffices to consider those which are a-simple. The following result, which is also a straightforward generalization of Saito's result [14] in the totally ordered case, shows that the a-simple l-semigroups fall into two distinct classes. In order to state this classification, we need a little terminology: a semigroup is a *nilsemigroup* if it has a zero element 0, and some finite power of every other element equals 0 (see [13] for a discussion of totally ordered nilsemigroups). An element of a semigroup is *torsion-free* if all of its positive powers are distinct, and a semigroup is *torsion-free* if all of its elements are.

Theorem. *Let S be an a-simple l-semigroup. Then either S contains a unique idempotent and S is a nilsemigroup, or else every element of S is torsion-free.*

Proof. Suppose first that S contains an idempotent e. Then for any other a in S, there exists some integer n for which a **a** e via n. But $e^n = e$, and so e is necessarily the largest element of S, and hence the unique idempotent of S and a zero for S; furthermore, $a^n = e$. On the other hand, if S contains no idempotents, obviously

$$a < a^2 < a^3 \dots ,$$

and so S is torsion free.

In the next section we shall consider the a-simple case.

In section 5 we include examples to show that neither the set of torsion elements, nor the set of torsion-free elements, need form l-subsemigroups of S.

3. A-SIMPLE NIL-L-SEMIGROUPS

In this section we shall obtain structural results for a-simple l-semigroups which are also nilsemigroups. We shall call such semigroups *a-simple nil-l-semigroups*; throughout this section S will denote such a semigroup.

We shall begin by considering the left annihilators of S. For a in S, let

$$L(a) = \{b\colon ba = 0\},$$

the *left annihilator of a*. The reader may easily verify that $L(a \wedge b) = L(a) \cap L(b)$ and that $L(a)$ is a sublattice of S. Furthermore, because 0 is the largest element of S, $L(a)$ is also a dual lattice ideal (that is, if $b > c$ and c is in $L(a)$, then so is b).

We shall now impose one further condition on S; we shall assume that 0 is *finitely join irreducible*: that is, if $a \vee b = 0$, then a or b is 0. We shall call an a-simple nil-l-semigroup with this additional assumption a *step*. In section 4 of the paper we shall become acquainted with the reason for this terminology, and better understand that this is not an unnatural assumption to make at this point. For now, we observe that this assumption obviously implies that $L(a \vee b) = L(a) \cup L(b)$, thus giving us the dual of the equation mentioned above.

We now define the relation e_1 on S:

$$a\ e_1\ b \quad \text{if and only if} \quad L(a) = L(b)\,.$$

This is obviously a lattice congruence. In fact, it is also a left semigroup congruence, because if $L(a) = L(b)$, then $L(ca) = L(cb)$. Furthermore, S/e_1 is a totally ordered set. This can be verified directly, but in fact follows from (the dual of) Lemma 1 of [3], since $\{0\}$ is a $\vee$-prime dual lattice ideal of S, and e_1 is just the left Dubreil congruence for the set $\{0\}$ (see [5], page 182).

We shall now inductively define a sequence of equivalence relations e_n on S, as follows:

$$a\ e_n\ b \quad \text{if and only if} \quad a\ e_{n-1}\ b \quad \text{and} \quad L(a^n) = L(b^n)\,.$$

This is obviously an equivalence relation; unfortunately, e_n need not be a lattice congruence on all of S; however, it is such a congruence on the equivalence classes of its predecessor, as the next proposition asserts. Denote by $[a]_n$ the e_n equivalence class for a in S.

Proposition. *e_n is a lattice congruence on the lattice $[a]_{n-1}$, and $[a]_{n-1}/e_n$ is totally ordered.*

Proof. Suppose that a, b and c are all in $[a]_{n-1}$, and that $a\ e_n\ b$. If w is a word with positive exponents in a and c with exponents adding to n, we claim that $L(w)$ is equal to either $L(a^n)$ or $L(c^n)$. Let v be the word w with the leftmost entry deleted. We have by inductive hypothesis that $L(v)$ equals either $L(a^{n-1})$ or $L(c^{n-1})$. But since e_1 is a left congruence, $L(w)$ must equal one of the following (depending on what the leftmost entry of w is, and what $L(v)$ equals): $L(a^n)$, $L(c^n)$, $L(ca^{n-1})$ or $L(ac^{n-1})$. The first two cases are what we desire. But inductively we have that $L(a^{n-1}) = L(c^{n-1})$, meaning that the second two cases reduce to the first. Application of the distributive law now gives us $L((a \wedge b)^n) = L(a^n \wedge b^n)$ and likewise for joins, which shows that e_n is in fact a lattice congruence on $[a]_{n-1}$. To see that $[a]_{n-1}/e_n$ is totally ordered, we need only observe that $[a]_n \leqq [b]_n$ exactly when $L(a^n) \subseteq L(b^n)$, and that S/e_1 is totally ordered.

We intend to use the congruences e_n to study the structure of S. We will apply them to certain subsemigroups of S defined as follows: for a positive integer n let

$$S(n) = \{a : a^n = 0\}\,.$$

It is evident that each $S(n)$ is a $\vee$-subsemigroup of S (an example in Section 5 shows that $S(n)$ need not be an l-semigroup); furthermore, S is the union of this ascending chain of subsemigroups. Note that $S(1)$ is of course just $\{0\}$.

We shall now inductively provide a structural decomposition for each of the $S(n)$'s, by making use the the relations e_n. For a in $S(n)$, denote by $\{a\}_n$ the intersection of $[a]_n$ with $S(n)$. For the sake of clarity, we will consider the case $n = 2$ separately.

Theorem. $S(2)$ *is a chain of zero l-semigroups, with multiplication left zero to the right.*

Proof. For a and b in $\{a\}_2$, $L(a) = L(b)$; but because a is in $L(a)$, this means that $ab = 0$. Thus, $\{a\}_2 \cup \{0\}$ is a zero semigroup. Because $[a]_2$ is lattice and $S(2)$ is a $\vee$-semigroup, we need only check that $\{a\}_2 \cup \{0\}$ is closed under taking meets to verify that it is an l-semigroup. But for a and b in $\{a\}_2$,

$$(a \wedge b)^2 = a^2 \wedge ab \wedge ba \wedge b^2 = 0,$$

and so $a \wedge b$ is an element of $S(2)$.

Now obviously $S(2)$ is a disjoint union of the totally ordered set of e_2 equivalence classes, each of which forms a zero l-semigroup when 0 is added. By the last phrase in the statement of the theorem we mean that if $\{a\}_2 \leqq \{b\}_2$, then $ab = 0$. But this is clear, because $a \in L(a) \subseteq L(b)$.

Theorem. $S(n)$ *is a chain of l-semigroups all of which contain* $S(n-1)$, *with multiplication left* $S(n-1)$ *to the right.*

Proof. For a in $S(n)$, we show that $\{a\}_n \cup S(n-1)$ is an l-semigroup. Since $S(n-1)$ is upper directed, to show that this is a semigroup we clearly need only check that if b is in $\{a\}_n$, then so is ab. But we know that $L(a^{n-1}) = L(b^{n-1})$ and so $ab^{n-1} = 0$. But then $(ab)^{n-1} \geqq ab^{n-1} = 0$; that is, ab is in $S(n-1)$. To show that we have an l-semigroup, we must show that $a \vee b$ and $a \wedge b$ belong to $\{a\}_n$. But

$$L((a \vee b)^n) = L(a^n \vee b^n) = L(a^n),$$

and similarly for meets.

Now, $S(n)$ is of course the disjoint union of its e_n equivalence classes, and each of these together with $S(n-1)$ is an l-semigroup. These l-semigroups form a chain under the lexicographic order, where we consider successively the orders from $e_1, e_2, \ldots, e_n$. It remains to show that if a and b are in $S(n) - S(n-1)$ and $\{a\}_n \leqq \{b\}_n$, then ab is in $S(n-1)$. If these classes are equal, this is clear. If not, suppose that e_i is the first stage at which $\{a\}_i < \{b\}_i$. Then $a^{n-i} \in L(b^i)$, and so

$$0 = a^{n-i}b^i \leqq (ab)^{n-1};$$

thus, ab is in $S(n-1)$. (Actually, in this case ab is in $S(k)$, where k is the minimum of $n - i$ and i.)

Thus, to summarize, each step S is an ascending union of the $\vee$-semigroups $S(n)$; at each stage, the outgrowth $S(n) - S(n-1)$ can be decomposed as a totally ordered set of classes in which multiplication reverts to the previous level. We shall apply and interpret these results in the case of an important example in the following section.

Let T be a totally ordered set and $S(T)$ the set of order preserving functions from T into itself. Then $S(T)$ is an l-semigroup under functional composition and the pointwise order; see [2] for a discussion of this semigroup. The underlying lattice of this semigroup is certainly distributive; we call l-semigroups with this property *distributive*. The problem of representing distributive l-semigroups in some $S(T)$ (thus obtaining an analogue of Holland's representation theorem for l-groups [11]) has been considered in [3] and [1]. In particular, any such l-semigroup to which an identity can be adjoined admits such a representation. It is obvious that we can alway adjoin an identity (if not already present) as least element to a strictly positive l-semigroup; consequently every distributive strictly positive l-semigroup can be l-embedded into some $S(T)$ (and in fact in $S(T)^+$, the set of positive elements of $S(T)$).

Now, an idempotent in $S(T)^+$ is a function which is the identity on its range [2]. If we think for a moment of the case when T is the set of real numbers, this means that the graph of such a function lies above the line $y = x$, but that points on the graph strictly above this line are connected to it by horizontal segments to the right; there may of course be infinitely many of these "steps". If we now consider such an idempotent with only a single step, it is clear that the archimedean class of this idempotent is a step in the sense of the previous section, because the idempotent is clearly finitely join irreducible.

So, consider now the archimedean class S of an idempotent 0 in $S(T)^+$ which maps all elements of the interval $(s, t]$ to t, and leaves all other elements of T fixed. S is then a step, and the equivalence relations e_n of the previous section can be interpreted as follows. For a in S, let

$$M(a) = \{x \in (s, t]: xa = t\}.$$

In order that some finite power of a be 0, it follows that $M(a)$ must contain more elements than just t; it is in fact a final segment of $(s, t]$. Furthermore, $a = 0$ exactly when $M(a) = (s, t]$. The reader may verify that $a\ e_1\ b$ means exactly that $M(a) = M(b)$, and that inductively, $a\ e_n\ b$ means that

$$M(a) = M(b),\ M(a^2) = M(b^2),\ \ldots,\ M(a^n) = M(b^n).$$

We can now use this representation to describe all distributive a-simple nil-l-semigroups in terms of steps:

Theorem. *Each distributive a-simple nil-l-semigroup is a subdirect product of steps.*

Proof. Let S be such an l-semigroup with unique idempotent 0. We may suppose that S is an l-subsemigroup of $S(T)$, for some totally ordered set T. To eliminate a special case in the argument which follows, we may as well assume that T has a least element (if not, just adjoin one, and specify that each element of S maps it to itself). Then a maximal interval of T on which 0 is constant is always of the form $(s, t]$.

Now consider the function p from S into $S((s, t])$ defined as the restriction of elements of S to the interval $(s, t]$. Because for each a in S, $a \leqq 0$, and some power of a equals 0, we know that $(s, t]\, a \subseteq (s, t]$, and so this is well-defined. It is then easy to see that p is an l-homomorphism; furthermore, $p(S)$ is obviously a step.

We can now define such a function p for each non-singleton maximal interval on which 0 is constant; it is evident that for any distinct elements a and b of S, there exists at least one such p for which $p(a) \neq p(b)$. Thus, S is a subdirect product of the steps $p(S)$.

Question. Is the preceding theorem true for non-distributive l-semigroups?

Question. The proof of the theorem above uses the fact that a distributive a-simple nil-l-semigroup which has a representation in which 0 has only a single interval on which it is not the identity, is necessarily a step (that is, 0 is finitely join irreducible). Is the converse of this statement true?

We can summarize what we now know about distributive strictly positive l-semigroups: Each is a lattice of a-simple l-semigroups, and each of the torsion archimedean classes is a subdirect product of steps, which can be described completely by the theory of the preceding section. Consequently, complete understanding of the structure of such l-semigroups is reduced to this:

Question. What can be said about the structure of strictly positive (distributive) a-simple torsion-free l-semigroups?

Notice that the theorem of section 2 gives us a semigroup analogue of the Hahn [9] theorem for abelian o-groups, and the Conrad-Harvey-Holland theorem [7] for abelian l-groups: arbitrary l-semigroups described in terms of a-simple ones. However, we do lose an important part of the group-theoretic results, as we shall now describe. If G is an l-subgroup of an l-group H, and every element of H is archimedean equivalent to some element of G, then H is an *a-extension* (see [6] or [4]) of G. Hahn's theorem not only provides a representation for abelian o-groups; it demonstrates the existence of a maximal a-extension (or *a-closure*) in that class of groups. Conrad [6] calculates a cardinality bound on a-extensions of abelian l-groups, and consequently infers the existence of a-closures for such l-groups; Khuon [12] obtains the corresponding bound in the non-abelian case. As we show below, no such theory of a-closure for l-semigroups is possible:

Theorem. *A distributive a-simple nil-l-semigroup admits a-extensions of arbitrarily large cardinality.*

Proof. Let S be such a semigroup, with idempotent 0; we may assume that S has been l-embedded into some $S(T)$. Notice that the archimedean class of the idempotent 0 in $S(T)$ is of course an a-extension of S. Suppose that $(s, t]$ is some non-singleton maximal interval on which 0 is constant; choose $x \neq t$ in this interval. Let K be a totally ordered set of cardinality strictly larger than that of S. Let U be the totally ordered set obtained from T by replacing $\{x\}$ by K. Choose some element j

in K. Then, for a in $S(T)$, we can define an element a^* in $S(U)$, by specifying that $ya^* = j$ whenever $ya = x$, $ya^* = z$ whenever $ya = z$ and $y \neq x \neq z$, and $ka^* = xa$, for all k in K. This gives us an l-embedding of $S(T)$ into $S(U)$. But for each k in K, we can define an element z_k as follows:

$$mz_k = \begin{cases} m0 & \text{if} \quad m \notin (s, t] \\ k & \text{if} \quad s < m < k \\ t & \text{if} \quad k \leqq m \leqq t\,. \end{cases}$$

It is evident that $\{z_k\colon k \in K\}$ is a subset of the a-class of 0 in $S(U)$, and consequently, the cardinality of this a-extension of S is strictly larger than the cardinality of S; thus, we can clearly make the cardinality of an a-extension of S as large as we please.

5. EXAMPLES

In this section we will provide some examples promised earlier in the paper. All of the examples considered here are l-subsemigroups of the semigroup $S(\boldsymbol{R})^+$ of order-preserving functions on the real numbers $\boldsymbol{R}$.

1. *The set of torsion elements need not be a subsemigroup.*

Define idempotents e, f in $S(\boldsymbol{R})$ as follows:

$$xe = \begin{cases} x & (-\infty, 0] \\ 1/2 & (0, 1/2] \\ 3/4 & (1/2, 3/4] \\ 7/8 & (3/4, 7/8) \\ \ldots\ldots & \\ x & [1, \infty) \end{cases}$$

and

$$xf = \begin{cases} m\ x & (-\infty, 1/4] \\ 5/8 & (1/4, 5/8] \\ 13/16 & (5/8, 13/16] \\ 29/32 & (13/16, 29/32] \\ \ldots\ldots & \\ x & [1, \infty) \end{cases}$$

Now, we know that ef and $e \vee f$ are archimedean equivalent. Explicitly, $e \vee f$ is defined as follows:

$$x(e \vee f) = \begin{cases} x & (-\infty, 0] \\ 1/2 & (0, 1/4] \\ 5/8 & (1/4, 1/2] \\ 3/4 & (1/2, 5/8] \\ 13/16 & (5/8, 3/4] \\ \ldots\ldots\ldots & \\ x & [1, \infty). \end{cases}$$

It is easily seen that this is torsion free, because successive applications of this function to $1/4$ yields:

$$1/4,\ 1/2,\ 5/8,\ 3/4,\ \ldots.$$

2. *The set of torsion free elements need not be a subsemigroup.*

Let e be an idempotent in $S(\boldsymbol{R})^+$ with just two steps, say on intervals $(0, 1]$ and $(2, 3]$. Then construct two torsion free elements, both with support equal to that of e: one which agrees with e on $(0, 1]$ and maps the interval $(2, 3)$ onto itself; and the other with the roles of the two intervals interchanged. Then the product (and join) of these elements is e.

3. *$S(n)$ need not be a $\wedge$-subsemigroup of a step.*

Let e be the one step idempotent with support $(0, 1)$, and S its archimedean class. Define elements a, b of S as follows:

$$xa = \begin{cases} 1 & [1/2, 1] \\ 1/2 & (0, 1/2] \end{cases}$$

$$xb = \begin{cases} 1 & [3/4, 1] \\ 7/8 & (0, 3/4] \end{cases}$$

Then clearly $a^2 = b^2 = e$; however, $(a \wedge b)^2 < e$. Thus, $S(2)$ is clearly not a $\wedge$-subsemigroup of S; similar examples can be constructed for any integer n.

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