Pavel Drábek; Milan Kučera

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EIGENVALUES OF INEQUALITIES OF REACTION-DIFFUSION TYPE AND DESTABILIZING EFFECT OF UNILATERAL CONDITIONS

PAVEL DRÁBEK, Plzeň, MILAN KUČERA, Praha

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0. INTRODUCTION

Let us consider a reaction-diffusion system of the type

$$\text{(RD)} \qquad \frac{\partial u}{\partial t} = d\,\Delta u + f(u, v)\,, \qquad \frac{\partial v}{\partial t} = \Delta v + g(u, v)$$

in a domain $\Omega \subset \mathbb{R}^n$, where f, g are real functions on $\mathbb{R}^2$, d is a real parameter (diffusion coefficient). Suppose that $\bar{u}, \bar{v}$ is a stationary and spatially homogeneous (constant) solution of (RD) with the Neumann boundary conditions, i.e. $\bar{u}, \bar{v}$ are constants such that $f(\bar{u}, \bar{v}) = g(\bar{u}, \bar{v}) = 0$. We shall study the linearized stability of $\bar{u}, \bar{v}$ as a solution of (RD) with the boundary conditions

$$\text{(BC)} \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N\,, \quad u = \bar{u}\,, \quad v = \bar{v} \text{ on } \Gamma_D\,,$$

and as a solution of (RD) with some unilateral conditions, e.g.

$$\text{(UC)} \qquad u = \bar{u}\,, \quad v = \bar{v} \text{ on } \Gamma_D\,, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_N\,,$$

$$\frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \Gamma_N \setminus \tilde{\Gamma}_N\,, \quad v \geqq \bar{v}\,, \quad \frac{\partial v}{\partial n} \geqq 0\,, \quad (v - \bar{v})\,\frac{\partial v}{\partial n} = 0 \text{ on } \tilde{\Gamma}_N\,.$$

(We suppose that Γ_D, Γ_N are subsets of the boundary $\partial\Omega$ of Ω, $\Gamma_D \cup \Gamma_N = \partial\Omega$, $\tilde{\Gamma}_N \subset \Gamma_N$.)

We shall consider the situation when there is $d_0 \in \mathbb{R}$ such that $\bar{u}, \bar{v}$ is a stable solution of (RD), (BC) for any $d > d_0$ and $\bar{u}, \bar{v}$ is an unstable solution of (RD), (BC) for any $d < d_0$. Such a situation occurs in applications and problems of this kind are studied e.g. in [13], [14]. In [11], it shown by simple examples that $\bar{u}, \bar{v}$ can be an unstable solution of (RD) with unilateral conditions also for the parameters $d > d_0$, i.e. that unilateral conditions can have a destabilizing effect. The aim of the present paper is to prove a general result of this type.

Under certain assumptions the stability of $\bar{u}, \bar{v}$ as a solution of (RD), (BC) is equivalent to the stability of the trivial solution of the corresponding linearized system (see e.g. [6]). In this case for the investigation of the stability of $\bar{u}, \bar{v}$ as a solution of (RD), (BC) it is sufficient to study the eigenvalues of the corresponding linearized problem

$$(\mathrm{RD}_\lambda) \qquad \begin{aligned} d\,\Delta u + b_{11}u + b_{12}v &= \lambda u\,, \\ \Delta v + b_{21}u + b_{22}v &= \lambda v\,, \end{aligned}$$

$$(\mathrm{BC}_0) \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N\,, \quad u = v = 0 \text{ on } \Gamma_D\,.$$

(The solution $\bar{u}, \bar{v}$ is automatically transformed to zero, i.e. we write u, v instead of $u - \bar{u}$, $v - \bar{v}$ in the linearized problem.)

In the case of unilateral problems we shall deal with the linearized stability only, i.e. with the stability of the trivial solution of the linearized system

$$(\mathrm{RD}_\mathrm{L}) \qquad \frac{\partial u}{\partial t} = d\,\Delta u + b_{11}u + b_{12}v\,, \quad \frac{\partial v}{\partial t} = \Delta v + b_{21}u + b_{22}v\,,$$

with

$$b_{11} = \frac{\partial f}{\partial u}(\bar{u}, \bar{v})\,, \quad b_{12} = \frac{\partial f}{\partial v}(\bar{u}, \bar{v})\,, \quad b_{21} = \frac{\partial g}{\partial u}(\bar{u}, \bar{v})\,, \quad b_{22} = \frac{\partial g}{\partial v}(\bar{u}, \bar{v})$$

and with the corresponding unilateral conditions, i.e.

$$(\mathrm{UC}_0) \qquad u = v = 0 \text{ on } \Gamma_D\,, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N\,,$$

$$\frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_N \setminus \tilde{\Gamma}_N\,, \quad v \geqq 0\,, \quad \frac{\partial v}{\partial n} \geqq 0\,, \quad v\frac{\partial v}{\partial n} = \quad \text{on } \tilde{\Gamma}_N$$

in our model example. We shall show that under certain assumptions the problem (RD_λ), (UC_0) (in an abstract setting) has a positive eigenvalue for some $d_I > d_0$, i.e. for some parameter d_I lying in the domain of stability of (RD_L), (BC_0). The instability of the trivial solution of (RD_L), (UC_0) for such d_I will be an easy consequence.

Let us remark that under certain assumptions it is possible to prove the instability for any $d \in (d_0, d_0 + \eta)$ with some $\eta > 0$ (see [5]). These results are in a closed relation to [4] where it is proved that under certain assumptions there exists a bifurcation point $d_{BI} > d_0$ of the corresponding stationary (nonlinear) system with unilateral conditions, i.e. there are spatially nonhomogeneous stationary solutions of the unilateral problem for the parameters d lying in the domain of stability of the classical problem. Notice that on the other hand under certain other assumptions the greatest bifurcation point of a stationary unilateral problem is less than that of the classical problem (see [10]).

This paper is organized as follows. The aim of Section 1 is to explain the main results. It begins with the formulation of our problem in terms of abstract inequalities in a Hilbert space (analogous to those from [4]). The main results in their abstract form are formulated in Theorems 1.1, 1.2 and Remark 1.3. Remark 1.4 explains what they mean for the reaction-diffusion systems with unilateral conditions. Section 2 contains an elementary investigation of the corresponding abstract eigenvalue problems. Lemma 2.1 together with Remark 2.1 justify and further explain the sense of the results of Section 1, Lemmas 2.2, 2.3 are necessary for the proof of Theorem 1.1. A formal proof of the main results based on Theorem 2.1 from [4] is the subject of Section 3. Section 4 briefly explains some main ideas of the proof of Theorem 1.1. It is shown how the positive eigenvalues of the corresponding inequality can be obtained by a certain homotopy from the eigenvalues of the equation. This is a modification of the method developed in [7], [8], [9]. In sections 1–4 the assumption meas $\Gamma_D > 0$ is considered, i.e., the purely Neumann boundary conditions are excluded for the original classical problem. The case of Neumann conditions is more complicated and is briefly discussed in Section 5.

1. NOTATION, BASIC ASSUMPTIONS, FORMULATION OF THE MAIN RESULTS

We shall denote by $\mathbb{V}$ and $\mathbb{H}$ two Hilbert spaces such that

$$\text{(V, H)} \qquad \mathbb{V} \hookrightarrow\hookrightarrow \mathbb{H} \quad \text{(completely continuous imbedding)},$$

which are equipped with the inner product $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$, respectively. Let us denote the corresponding norms by $\|u\|^2 = \langle u, u\rangle$ and $|v|^2 = (v, v)$. Let $K \subset \mathbb{V}$ be a closed convex cone in $\mathbb{V}$ with its vertex at the origin. We shall denote by $\mathbb{V}^\sim$ and $\mathbb{H}^\sim$ the Hilbert spaces $\mathbb{V} \times \mathbb{V}$ and $\mathbb{H} \times \mathbb{H}$, respectively, with the inner products given by

$$\langle U, W\rangle_\sim = \langle u, w\rangle + \langle v, z\rangle\,, \quad (U, W)_\sim = (u, w) + (v, z)\,,$$

where $U = [u, v]$, $W = [w, z]$, and with the corresponding norms $\|U\|^2_\sim = \langle U, U\rangle_\sim$, $|V|^2_\sim = (V, V)_\sim$. The identity mapping in $\mathbb{V}$ ($\mathbb{H}$) and $\mathbb{V}^\sim(\mathbb{H}^\sim)$ will be denoted by I and $I^\sim$, respectively. We shall suppose that $K \neq \mathbb{V}$, $K^0 \neq \emptyset$ (the interior and the boundary of the set M are denoted by M^0 and ∂M, respectively). The symbols $\to$ and $\rightharpoonup$ will denote the strong and the weak convergence in the corresponding spaces, $\mathbb{R}$ and $\mathbb{R}^+$ will be the set of all reals and of all positive reals, respectively.

In what follows we shall suppose that

(A) A is a linear completely continuous symmetric positive[1]) operator in $\mathbb{V}$. Particularly, this is fulfilled for the operator defined by

$$(1.1) \qquad \langle Au, \varphi\rangle = (u, \varphi)\,, \quad \text{for all} \quad u, \varphi \in \mathbb{V}\,,$$

by the assumption (V, H).

Let $b_{ij} \in \mathbb{R}$ $(i, j = 1, 2)$ be given and suppose

$$\text{(B)} \qquad b_{11} > 0\,, \quad b_{12} < 0\,, \quad b_{21} > 0\,, \quad b_{22} < 0\,, \quad b_{11} + b_{22} < 0\,.$$

[1]) We mean $\langle Au, u\rangle > 0$ for any $\|u\| \neq 0$.

We shall denote

$$B = \begin{pmatrix} b_{11}, & b_{12} \\ b_{21}, & b_{22} \end{pmatrix}, \quad D(d) = \begin{pmatrix} d, & 0 \\ 0, & 1 \end{pmatrix},$$

$$\tilde{A}U = [Au, Av], \quad \text{for all} \quad U = [u, v] \in \mathbb{V}^{\sim}.$$

Then we have

$$B\tilde{A}U = [b_{11}Au + b_{12}Av, \; b_{21}Au + b_{22}Av],$$

$$D(d)\,U = [du, v] \quad \text{for all} \quad U = [u, v] \in \mathbb{V}^{\sim}.$$

Further, introduce a cone $\tilde{K}$ in $\mathbb{V}^{\sim}$ by

$$\tilde{K} = \{U \in \mathbb{V}^{\sim};\; U = [u, v],\; v \in K\}.$$

We shall investigate the stability of the trivial solution of the abstract inequality

$$\text{(AI)} \quad \begin{cases} u(t) \in \mathbb{V}, \quad v(t) \in K, \\ \left(\dfrac{\partial u}{\partial t}(t), \varphi\right) + \langle d\,u(t) - b_{11}A\,u(t) - b_{12}A\,v(t), \varphi\rangle = 0, \\ \left(\dfrac{\partial v}{\partial t}(t), \psi - v(t)\right) + \langle v(t) - b_{21}A\,u(t) - b_{22}A\,v(t), \psi - v(t)\rangle \geqq 0, \\ \qquad\qquad \text{for all} \quad \varphi \in \mathbb{V}, \quad \psi \in K \quad \text{and a.a.} \quad t \geqq 0 \end{cases}$$

and of the corresponding equation

$$\text{(AE)} \quad \begin{cases} \left(\dfrac{\partial u}{\partial t}(t), \varphi\right) + \langle d\,u(t) - b_{11}A\,u(t) - b_{12}A\,v(t), \varphi\rangle = 0, \\ \left(\dfrac{\partial v}{\partial t}(t), \psi\right) + \langle v(t) - b_{21}A\,u(t) - b_{22}A\,v(t), \psi\rangle = 0, \\ \qquad\qquad \text{for all} \quad \varphi, \psi \in \mathbb{V} \quad \text{and a.a.} \quad t \geqq 0. \end{cases}$$

More precisely see Remark 1.1 below.

We shall write them usually in the vector form

$$(\text{AI}^{\sim}) \quad \begin{cases} U(t) \in \tilde{K}, \\ \left(\dfrac{\partial U}{\partial t}(t), \Phi - U(t)\right)_{\sim} + \langle D(d)\,U(t) - B\tilde{A}\,U(t), \Phi - U(t)\rangle_{\sim} \geqq 0 \\ \qquad\qquad \text{for all} \quad \Phi \in \tilde{K}, \quad \text{a.a.} \quad t \geqq 0, \end{cases}$$

$$(\text{AE}^{\sim}) \quad \begin{cases} \left(\dfrac{\partial U}{\partial t}(t), \Phi\right)_{\sim} + \langle D(d)\,U(t) - B\tilde{A}\,U(t), \Phi\rangle_{\sim} = 0, \\ \qquad\qquad \text{for all} \quad \Phi \in \mathbb{V}^{\sim}, \quad \text{a.a.} \quad t \geqq 0. \end{cases}$$

The key role will be played by the following eigenvalue problem for the inequality

$$(\text{EI}^{\sim}) \qquad U \in \tilde{K},$$

$$\langle D(d)\,U - B\tilde{A}U + \lambda\tilde{A}U, \Phi - U\rangle_{\sim} \geqq 0 \quad \text{for all} \quad \Phi \in \tilde{K},$$

and the corresponding eigenvalue problem for the equation

$$(\mathrm{EE}^{\sim}) \qquad D(d)\,U - B\tilde{A}U + \lambda\tilde{A}U = 0\,.$$

Remark 1.1. We shall not discuss the existence and the smoothness of the solutions to $(\mathrm{AI}^{\sim})$. Our aim will be to show the existence of a solution of the type $U(t) = \exp(\lambda t)\, W_0$ of $(\mathrm{AI}^{\sim})$ with $\lambda > 0$ for a suitable parameter d_I only, which has the derivative $(\partial U/\partial t)(t) \in \mathbb{H}^{\sim}$ for any $t \in \mathbb{R}^+$ and $(\mathrm{AI}^{\sim})$ is fulfilled for all $t \in \mathbb{R}^+$. If we wanted to give a general correct definition of the solution on $\langle 0, T)$ we could consider for instance

$u, v \in L_2(0; T; \mathbb{V})$ such that $\partial u/\partial t$, $\partial v/\partial t \in L_2(0, T; \mathbb{V}^*)$ and (AI) (or (AE)) is fulfilled for a.a. $t \in (0, T)$.

(The derivative $\partial u/\partial t$ of $u \in L_2(0, T; \mathbb{V})$ exists as a distribution with values in $\mathbb{V}$, i.e. also in $\mathbb{H}$ by (V, H); by $\partial u/\partial t \in L_2(0, T; \mathbb{V}^*)$ we mean that this distribution can be represented by an integrable function with values in $\mathbb{H}$ and

$$\int_0^T \Bigg[\sup_{\substack{\varphi \in \mathbb{V} \\ \|\varphi\| = 1}} \left(\frac{\partial u}{\partial t}, \varphi \right) \Bigg]^2 \mathrm{d}t$$

is finite; cf. e.g. [2].)

Remark 1.2. Consider the linearized reaction-diffusion system $(\mathrm{RD_L})$, $(\mathrm{BC_0})$ from Introduction. Let Ω be a bounded domain in $\mathbb{R}^n$ with the lipschitzian boundary $\partial\Omega$. Suppose that Γ_D, Γ_N are disjoint open sets in $\partial\Omega$ such that meas $[\partial\Omega \setminus (\Gamma_D \cup \Gamma_N)] = 0$ and

$$(1.2) \qquad \operatorname{meas} \Gamma_D > 0\,.$$

Introduce the space

$$\mathbb{V} = \{u \in W_2^1(\Omega);\ u = 0 \ \text{ on } \ \Gamma_D \ \text{in the sense of traces}\}$$

with the inner product

$$\langle u, \varphi \rangle = \int_\Omega \sum_{i=1}^n u_{x_i} \varphi_{x_i}\, \mathrm{d}x\,, \quad \text{for all} \quad u, \varphi \in \mathbb{V}\,.$$

The corresponding norm $\|\cdot\|$ is equivalent on $\mathbb{V}$ to the usual norm of the Sobolev space $W_2^1(\Omega)$ (see e.g. [12]). Further, denote by $\mathbb{H}$ the Lebesgue space $L_2(\Omega)$ with the usual inner product $(\cdot, \cdot)$ and introduce the operator A by (1.1), i.e.

$$\langle Au, \varphi \rangle = (u, \varphi) = \int_\Omega u\varphi\, \mathrm{d}x\,, \quad \text{for all} \quad u, \varphi \in \mathbb{H}\,.$$

Hence, the conditions (V, H), (A) are fulfilled.

In this case (AE) is an abstract formulation of $(\mathrm{RD_L})$, $(\mathrm{BC_0})$ [2]. A couple of func-

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[2]) Cf.e.g. [2], [3]; (AE) and (AI) is obtained from $(\mathrm{RD_L})$ by multiplying by a test function, integrating by parts and using the boundary conditions $(\mathrm{BC_0})$ and $(\mathrm{UC_0})$, respectively.

tions $U = [u, v]$ is a classical solution of $(\mathrm{RD_L})$, $(\mathrm{BC_0})$ if and only if U satisfies $(\mathrm{AE}^\sim)$ (for instance in the sense of Remark 1.1).

If we set

(1.3) $\quad K = \{\psi \in \mathbb{V};\ \psi \geqq 0$ on $\tilde{\Gamma}_N$ in the sense of traces$\}$ (where $\tilde{\Gamma}_N \subset \Gamma_N$)

then $(\mathrm{AI}^\sim)$ is an abstract formulation of $(\mathrm{RD_L})$, $(\mathrm{UC_0})$ [2]) (cf. [4]).

Further, if K is an arbitrary closed convex cone in $\mathbb{V}$ with its vertex at the origin then we can introduce the solution of $(\mathrm{RD_L})$ with unilateral conditions given by $\mathbb{V}$, K as a couple $U = [u, v]$ satisfying $(\mathrm{AI}^\sim)$. Of course, in the general case the connection with some classical formulation need not be clear.

Analogously, $(\mathrm{EE}^\sim)$ is a weak formulation of (RD_λ), $(\mathrm{BC_0})$ and $(\mathrm{EI}^\sim)$ for K from (1.3) is a weak formulation of (RD_λ), $(\mathrm{UC_0})$ (cf. [3], [4]). For a general cone K in $\mathbb{V}$ with its vertex at the origin we can define a weak solution of (RD_λ) with unilateral conditions given by $\mathbb{V}$, K as $U \in \mathbb{V}^\sim$ satisfying $(\mathrm{EI}^\sim)$.

Definition 1.1. Let $d > 0$ be given. If λ is such that there is a nontrivial solution U of $(\mathrm{EI}^\sim)$ or of $(\mathrm{EE}^\sim)$ then λ and U is called an *eigenvalue* and the *eigenvector of* $(\mathrm{EI}^\sim)$ or *of* $(\mathrm{EE}^\sim)$, respectively, *with the parameter d*. The set of all solutions of $(\mathrm{EI}^\sim)$ and $(\mathrm{EE}^\sim)$ will be denoted by $E_I(d, \lambda)$ and $E_B(d, \lambda)$, respectively. We shall say that an eigenvalue λ of $(\mathrm{EE}^\sim)$ is *simple* if $\dim E_B(d, \lambda) = 1$.

Definition 1.2. A point $d > 0$ is called a *critical point of* $(\mathrm{EI}^\sim)$ or $(\mathrm{EE}^\sim)$ if $\lambda = 0$ is an eigenvalue of $(\mathrm{EI}^\sim)$ or $(\mathrm{EE}^\sim)$, respectively. A critical point d of $(\mathrm{EE}^\sim)$ is *simple* if $\lambda = 0$ is a simple eigenvalue of $(\mathrm{EE}^\sim)$.

(Hence, we consider the geometrical simplicity, but it will be shown later that under our assumptions it is equivalent to the algebraic simplicity – see Remarks 2.2, 2.3.)

Theorem 1.1. *Let the assumptions* (A), (B) *be fulfilled and let d_0 be the greatest critical point of* $(\mathrm{EE}^\sim)$ [3]). *Suppose that d_0 is simple and $E_B(d_0, 0) \cap \tilde{K}^0 \neq \emptyset$. Then there is $d_I > d_0$ such that* $(\mathrm{EI}^\sim)$ *(with $d = d_I$) has a positive eigenvalue λ and $E_I(d_I, \lambda) \subset \partial\tilde{K}$.*

Theorem 1.2. *Let the assumptions* (1.1), (B), (V, H) *be fulfilled. Suppose that the greatest critical point d_0*[3]) *of* $(\mathrm{EE}^\sim)$ *is simple and $E_B(d_0, 0) \cap \tilde{K}^0 \neq \emptyset$. Then there are $d_I > d_0$, $\lambda > 0$ and $W_I \in \partial\tilde{K} \setminus \{0\}$ such that the abstract function $U(t) = \exp(\lambda t)\, W_I$ satisfies* $(\mathrm{AI}^\sim)$.

Proof follows directly from Theorem 1.1 (more precisely see [11]).

Remark 1.3. We can say that the trivial solution of $(\mathrm{AI}^\sim)$ (or of $(\mathrm{AE}^\sim)$) is stable with respect to the norm $\|\cdot\|_\sim$ if for any $r_1 > 0$ there exists $r_0 > 0$ such that any solution U of $(\mathrm{AI}^\sim)$ (or of $\mathrm{AE}^\sim$), respectively) in the sense of Remark 1.1 satisfying $U(0) \in B(0, r_0)$ (the open ball in $\mathbb{V}^\sim$ with the radius r_0 centered at the origin) has the

[3]) The existence of d_0 will be proved in Section 2 (Lemma 2.1).

property $U(t) \in B(0, r_1)$ for all $t \in \langle 0, +\infty)$. The trivial solution is said to be unstable if it is not stable.

We have $U(0) = W_I$ and $\|U(t)\|_\sim \to +\infty$, for $t \to +\infty$ if $U(t)$ is the solution from Theorem 1.2. This means, in particular, that the trivial silution of $(\mathrm{AI}^\sim)$ for $d = d_I$ is unstable (because the solutions tending to infinity start in an arbitrary small initial condition τW_I). Moreover, it is clear that it implies the unstability in an arbitrary reasonable sense (with respect to an arbitrary norm). On the other hand it will be seen in Section 2 (see Remark 2.1) that under our assumptions the trivial solution of $(\mathrm{AE}^\sim)$ is stable for any $d > d_0$. Hence unilateral conditions of the type considered have a destabilizing effect.

Remark 1.4. Consider the linearization $(\mathrm{RD_L})$ of the reaction-diffusion system from Introduction under the same assumptions as in Remark 1.2. Particularly, let meas $\Gamma_D > 0$. Consider the corresponding space $\mathbb{V}$ from Remark 1.2. It follows that Theorems 1.1, 1.2 and Remark 1.3 give the following assertion in this special case:

Let (B) be fulfilled and let the greatest critical point d_0 of $(\mathrm{RD_L})$, $(\mathrm{BC_0})$[4]) be simple, i.e. (RD_λ), $(\mathrm{BC_0})$ with $\lambda = 0$, $d = d_0$ has a one-dimensional space of solutions. Suppose that K is a closed convex cone in $\mathbb{V}$ and let $E_B(d_0, 0) \cap \tilde{K}^0 \neq \emptyset$. Then there exists $d_I > d_0$ such that (RD_λ) with $d = d_I$ and with the unilateral conditions given by $\mathbb{V}, K$ has a positive eigenvalue λ_I. For an arbitrary eigenvector $[u_I, v_I]$ corresponding to λ_I we have $v_I \in \partial K$ and $u(t) = \exp(\lambda t)\, u_1$, $v(t) = \exp(\lambda t)\, v_I$ is a solution of $(\mathrm{RD_L})$ with the unilateral conditions given by $\mathbb{V}$, K. Particularly, it follows that the trivial solution of $(\mathrm{RD_L})$ with unilateral conditions given by $\mathbb{V}$, K is unstable for some $d_I > d_0$ (cf. Remark 1.3). Simultaneously it will follow from Lemma 2.1 and Remark 1.2 that the trivial solution of $(\mathrm{RD_L})$, $(\mathrm{BC_0})$ is stable for any $d > d_0$ and unstable for $d < d_0$ only (see Remark 2.1).

Notice that for the cone (1.3) the assumption $E_B(d_0, 0) \cap \tilde{K}^0 \neq \emptyset$ is fulfilled if $v_0 \geqq \delta > 0$ on $\tilde{\Gamma}_N$ for some nontrivial solution u_0, v_0 of (RD_λ), $(\mathrm{BC_0})$ with $d = d_0$, $\lambda = 0$.

2. PROPERTIES OF EIGENVALUES OF $(\mathrm{E}\tilde{\mathrm{E}})$ AND OF THE OPERATORS $B\tilde{A} - D(d)\,\tilde{I} - \lambda\tilde{A}$

Lemma 2.1. *Under the assumptions* (A), (B) *there exists the greatest critical point $d_0 > 0$ of* $(\mathrm{EE}^\sim)$ *and $\mu = 0$ is the greatest real eigenvalue of the operator $B\tilde{A} - D(d_0)\,\tilde{I}$. Further,*

(2.1) *for any $d > d_0$ all the real eigenvalues of $B\tilde{A} - D(d)\,\tilde{I}$ are negative;*

(2.2) *for any $0 < d < d_0$ there is at least one positive eigenvalue of $B\tilde{A} - D(d)\,\tilde{I}$.*

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[4]) i.e. the greatest d for which $\lambda = 0$ is an eigenvalue of (RD_λ), $(\mathrm{BC_0})$; for its existence see Lemma 2.1 (and Remark 1.2).

Moreover, if d_0 is simple) then there is $\eta > 0$ such that*

(2.3) *for any $d \in (d_0 - \eta, d_0)$ there is one simple*[5]*) positive eigenvalue of $B\tilde{A} - D(d)\tilde{I}$ and the other real eigenvalues are negative.*

Proof. The assumption (A) ensures that the eigenvalues of the operator A form a decreasing sequence $\{\varkappa_i\}$, $\varkappa_i > 0$ $(i = 1, 2, \ldots)$, $\varkappa_i \to 0$ $(i \to +\infty)$ and the corresponding eigenvectors form a complete orthonormal system $\{e_i\}$ in $\mathbb{V}$. For any $[u, v] = U$ we have

$$u = \sum_{i=1}^{\infty} \langle u, e_i \rangle e_i, \quad v = \sum_{i=1}^{\infty} \langle v, e_i \rangle e_i \tag{2.4}$$

and therefore $B\tilde{A}U - D(d)\,U = \mu U$ is equivalent to

$$\begin{aligned} &\langle u, e_i \rangle (d - b_{11}\varkappa_i + \mu) - \langle v, e_i \rangle b_{12}\varkappa_i = 0\,,\\ &\langle u, e_i \rangle b_{21}\varkappa_i \qquad\qquad - \langle v, e_i \rangle (1 - b_{22}\varkappa_i + \mu) = 0\,,\end{aligned}$$

$i = 1, 2, 3, \ldots$.

The couple $\langle u, e_i \rangle$, $\langle v, e_i \rangle$ can be nontrivial for some i only if

$$\det \begin{pmatrix} d - b_{11}\varkappa_i + \mu, & b_{12}\varkappa_i \\ b_{21}\varkappa_i, & 1 - b_{22}\varkappa_i + \mu \end{pmatrix} = 0\,, \tag{2.5}$$

i.e.

$$\begin{aligned} &\mu^2 - \mu[(b_{11} + b_{22})\,\varkappa_i - (d + 1)] + \\ &+ (d - b_{11}\varkappa_i)(1 - b_{22}\varkappa_i) - b_{12}b_{21}\varkappa_i^2 = 0\,. \end{aligned} \tag{2.5'}$$

It follows that μ is an eigenvalue of the operator $B\tilde{A} - D(d)\,\tilde{I}$ if and only if μ is a root of (2.5′) with some (at least one) i, and in this case the corresponding eigenvectors are

$$\left[e_i, \frac{d - b_{11}\varkappa_i + \mu}{b_{12}\varkappa_i}\, e_i\right]. \tag{2.6}$$

Particularly, d is a critical point of (EE$^\sim$) (i.e. $\mu = 0$ is an eigenvalue of $B\tilde{A} - D(d)\,\tilde{I}$) if and only if

$$(d - b_{11}\varkappa_i)(1 - b_{22}\varkappa_i) - b_{12}b_{21}\varkappa_i^2 = 0\,, \quad \text{i.e.}$$

$$d = d_i = \frac{b_{12}b_{21}\varkappa_i^2}{1 - b_{22}\varkappa_i} + b_{11}\varkappa_i\,,$$

for some i (remember that $1 - b_{22}\varkappa_i > 0$ by (B)).

It is easy to see that the properties of the function

$$h(t) = \frac{b_{12}b_{21}t^2}{1 - b_{22}t} + b_{11}t$$

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[5]) We mean always the geometrical simplicity, i.e. the corresponding null-space is one-dimensional (see Remark 2.2, cf. Definition 1.2).

(which follow from (B)) and $\varkappa_i \to 0_+$ (for $i \to \infty$) ensure the existence of i_0 satisfying $d_{i_0} = \max_{i=1,2,\ldots} d_i > 0$. Hence, $d_0 = d_{i_0}$ is the greatest critical point of (EE$^{\sim}$). Further, under the assumption (B) it is not hard to see from the formula for the roots of (2.5′) that for $0 < d < d_i$[6]) the roots of (2.5′) are both real, the greater one being positive and for $d > 0$, $d > d_i$[6]) there is no positive root (either they are both negative or they are complex for d sufficiently large). In other words, for $d > 0$, there exists a positive eigenvalue of $B\tilde{A} - D(d)\tilde{I}$ corresponding to the eigenvector (2.6) for a given i if and only if $d < d_i$ and this implies (2.1), (2.2).

If d_0 is simple then there is a unique i_0 satisfying $d_{i_0} = d_0$. The assertion (2.3) is an easy consequence of this fact and of the previous considerations.

Remark 2.1. It is easy to see from the proof of Lemma 2.1 that for an arbitrary $d \in \mathbb{R}^+$ the operator $B\tilde{A} - D(d)\tilde{I}$ has no complex eigenvalue with a nonnegative real part. It is known that this together with (2.1), (2.2) means that the trivial solution of (AE$^{\sim}$) is stable for any $d > d_0$ and it is unstable for any $0 < d < d_0$. Using Remark 1.2 we obtain the last assertion, in particular, for the problem (RD$_L$), (BC$_0$).

Remark 2.2. If $\mu \geqq 0$ is an eigenvalue of the operator $B\tilde{A} - D(d)\tilde{I}$ [7]) (with some $d > 0$) then its geometrical simplicity is equivalent to the algebraic simplicity, i.e.

$$\dim \operatorname{Ker} (B\tilde{A} - D(d)\tilde{I} - \mu\tilde{I}) = 1$$

if and only if

$$\dim \bigcup_{k=1}^{\infty} \operatorname{Ker} (B\tilde{A} - D(d)\tilde{I} - \mu\tilde{I})^k = 1 .$$

For the proof it is sufficient to show that $\langle U, U^* \rangle_{\sim} \neq 0$, where U and U^* are the eigenvectors of $B\tilde{A} - D(d)\tilde{I}$ and of $B^*\tilde{A} - D(d)\tilde{I}$, respectively, corresponding to μ, B^* is the adjoint matrix to B (see e.g. [16]). But it follows from the proof of Lemma 2.1 that U is given by (2.6) and analogously we obtain

$$U^* = \left[e_i, \frac{d - b_{11}\varkappa_i + \mu}{b_{21}\varkappa_i} e_i \right]$$

(with the same i). This together with (2.5) and (B) implies

$$\langle U, U^* \rangle_{\sim} = 1 + \frac{(d - b_{11}\varkappa_i + \mu)^2}{b_{12}b_{21}\varkappa_i^2} = \frac{1 + d - (b_{11} + b_{22})\varkappa_i + 2\mu}{1 - b_{22}\varkappa_i + \mu} > 0 .$$

Lemma 2.2. *Let* (A), (B) *be fulfilled and let d_0 be the greatest critical point of* (EE$^{\sim}$). *Suppose that d_0 is simple. Then there exist continuous functions λ:*

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[6]) Notice that in general all d_i need not be positive.

[7]) We must distinguish the eigenvalues of the operator $B\tilde{A} - D(d)\tilde{I}$ (i.e. μ such that $B\tilde{A}U - D(d)U = \mu U$ has a nontrivial solution) and the eigenvalues of (EE$^{\sim}$) in the sense of Definition 1.1.

$(d_0 - r, d_0\rangle \to \mathbb{R}$, $U: (d_0 - r, d_0\rangle \to \mathbb{V}^{\sim}$ *(with some* $r > 0$*) such that* $\lambda(d)$ *is an eigenvalue of* $(\mathrm{EE}^{\sim})$ [7]) *with the corresponding eigenvector* $U(d)$, $\lambda(d) > 0$ *for all* $d \in (d_0 - r, d_0)$, $\lambda(d_0) = 0$. *Further, for any* $d \in (d_0 - r, d_0\rangle$, d *is the greatest number for which* $\lambda = \lambda(d)$ *is the eigenvalue of* $(\mathrm{EE}^{\sim})$.

Proof. We shall use similar considerations as in the proof of Lemma 2.1. The equation

$$D(d)\,U - B\tilde{A}U + \lambda\tilde{A}U = 0$$

is equivalent to

$$\begin{aligned} \langle u, e_i\rangle\,[d - (b_{11} - \lambda)\,\varkappa_i] - \langle v, e_i\rangle\, b_{12}\varkappa_i \qquad\qquad &= 0\,, \\ \langle u, e_i\rangle\, b_{21}\varkappa_i \qquad\qquad - \langle v, e_i\rangle\,[1 - (b_{22} - \lambda)\,\varkappa_i] &= 0\,, \end{aligned}$$

$i = 1, 2, 3, \ldots$, and λ is an eigenvalue of $(\mathrm{EE}^{\sim})$ (in the sense of Definition 1.1) if and only if

$$\lambda^2\varkappa_i^2 - \lambda[(b_{11} + b_{22}) - (d + 1)]\,\varkappa_i +$$
$$+ (d - b_{11}\varkappa_i)(1 - b_{22}\varkappa_i) - b_{12}b_{21}\varkappa_i^2 = 0$$

for some i. An elementary investigation of the formula for the roots of this equation with $i = i_0$ (cf. the proof of Lemma 2.1) shows that there exists a continuous function $\lambda = \lambda(d)$ on $\langle 0, d_0\rangle$ such that $\lambda(d)$ is an eigenvalue of $(\mathrm{EE}^{\sim})$, $\lambda(d) > 0$ for $d < d_0$, $\lambda(d_0) = 0$. Further, $E_B(d, \lambda(d))$ is generated by

$$\left[e_{i_0}, \frac{d - (b_{11} - \lambda(d))\,\varkappa_{i_0}}{b_{12}\varkappa_{i_0}}\, e_{i_0}\right]$$

for all $d \in (d_0 - r, d_0\rangle$ (with respect to the simplicity of d_0) with $r > 0$ sufficiently small (i_0 is uniquely determined, cf. proof of Lemma 2.1). This implies the assertion about $U(d)$. It remains to show that $r > 0$ can be chosen such that the last assertion of Lemma 2.2 holds. If this were not true there would exist sequences $\{d_n\}$, $\{\bar{d}_n\}$ such that $d_n < d_0$, $\bar{d}_n > d_n$, $d_n \to d_0$ and $\lambda(d_n)$ is an eigenvalue of $(\mathrm{EE}^{\sim})$ with $d = \bar{d}_n$ (and not only with $d = d_n$). Let $U_n \in E_B(d_n, \lambda(d_n))$, $\bar{U}_n \in E_B(\bar{d}_n, \lambda(d_n))$, $\|U_n\|_{\sim} = \|\bar{U}_n\|_{\sim} = 1$. Clearly, $\{\bar{d}_n\}$ is bounded and we can suppose $\bar{d}_n \to \bar{d} \geqq d_0$, $U_n \rightharpoonup U$, $\bar{U}_n \rightharpoonup \bar{U}$. The compactness of A implies $U_n \to U \in E_B(d_0, 0)$, $U_{..} \to \bar{U} \in E_B(\bar{d}, 0)$. The case $\bar{d} > d_0$ is impossible by the assumption that d_0 is the greatest critical point. Hence, $\bar{d} = d_0$. Set $U_n = [u_n, v_n]$, $\bar{U}_n = [\bar{u}_n, \bar{v}_n]$, $U = [u, v]$, $\bar{U} = [\bar{u}, \bar{v}]$. We have $\|u\| \neq 0$, $\|\bar{u}\| \neq 0$ because in the opposite case we would obtain from $(\mathrm{EE}^{\sim})$ (rewritten into the components) also $v = 0$ or $\bar{v} = 0$, i.e. $U = 0$ or $\bar{U} = 0$ and we know $\|U\|_{\sim} = \|\bar{U}\|_{\sim} = 1$. If we proved $\langle u, \bar{u}\rangle = 0$ then we would have a contradiction to the simplicity of d_0 and our assertion would be proved. Writing $(\mathrm{EE}^{\sim})$ for $d_n, U_n, \lambda(d_n)$ and for $\bar{d}_n, \bar{U}_n, \lambda(d_n)$ in the components, multiplying the individual equations by $\bar{u}_n, \bar{v}_n$ and u_n, v_n, respectively, and subtracting we obtain $(d_n - \bar{d}_n)\,\langle u_n, \bar{u}_n\rangle = 0$ (cf. [4], proof of Lemma 2.1). Hence $\langle u, \bar{u}\rangle = 0$ and the proof is complete.

Remark 2.3. Analogously as in Remark 2.2 it is possible to show that for any eigenvalue $\lambda \geqq 0$ of (EE$^\sim$) the algebraic and geometrical simplicity are equivalent.

Lemma 2.3. *Let* (A), (B) *be fulfilled and let the greatest critical point d_0 of* (EE$^\sim$) *be simple. Then there exists $\varrho > 0$ such that for any $d_1 \in (d_0 - \varrho, d_0\rangle$ the following assertion holds*[8]:

(2.7) *if $d > d_1$ then all the real eigenvalues of $B\tilde{A} - D(d)\tilde{I} - \lambda(d_1)\tilde{A}$ are negative;*

(2.8) *if $d \in (d_1 - \xi, d_1)$ (with some $\xi > 0$ depending on d_1) then there is one simple positive eigenvalue of $B\tilde{A} - D(d)\tilde{I} - \lambda(d_1)\tilde{A}$ and the other real eigenvalues are negative.*

Proof. Analogously as in the proof of Lemma 2.1, μ is an eigenvalue of $B\tilde{A} - D(d)\tilde{I} - \lambda(d_1)\tilde{A}$ (for a given d and d_1) if and only if

$$\mu^2 - \mu[(b_{11} + b_{22})\varkappa_i - (d + 1 + 2\lambda(d_1)\varkappa_i)] + \\ + [d - (b_{11} - \lambda(d_1))\varkappa_i][1 - (b_{22} - \lambda(d_1))\varkappa_i] - b_{12}b_{21}\varkappa_i^2 = 0\,,$$

for some i. The assertion of Lemma 2.3 can be obtained by an elementary investigation of the formula for the roots of this equation analogously as (2.1), (2.3) in the proof of Lemma 2.1. We replace only d_0 by d_1, realize that $\dim E_B(d_1, \lambda(d_1)) = 1$ if ϱ is small enough and use the last assertion of Lemma 2.2 instead of the fact that d_0 is the greatest critical point.

3. PROOF OF THE MAIN RESULT

The following assertion will be the basis for the proof of Theorem 1.1. It is a modification of Theorem 2.1 from [4].

Theorem 3.1. *Let* (A), (B) *be fulfilled. Suppose that $\lambda_1 \in \langle 0, b_{11})$ is a simple eigenvalue of* (EE$^\sim$) *with some $d_1 > 0$ and $E_B(d_1, \lambda_1) \cap \tilde{K}^0 \neq \emptyset$. Further, let there exist $\xi > 0$ such that*

($\overline{\text{GC}}$) *for any $d > d_1$ all the real eigenvalues of the operator $B\tilde{A} - \lambda_1\tilde{A} - D(d)\tilde{I}$ are negative; for any $d \in (d_1 - \xi, d_1)$ there is one positive simple eigenvalue of $B\tilde{A} - \lambda_1\tilde{A} - D(d)\tilde{I}$ and the other real eigenvalues of this operator are negative.*

Then there is $d_I^1 > d_1$ such that λ_1 is an eigenvalue of (EI$^\sim$) *with $d = d_I^1$, $E_I(d_I^1, \lambda_1) \subset \partial\tilde{K}$, $E_B(d_I^1, \lambda_1) = \{0\}$.*

Proof of this assertion can be obtained directly from Theorem 2.1 in [4]. It is sufficient to replace d_0 by d_1, B by $B_{\lambda_1} = B - \lambda_1 E$ (E denotes the unit matrix),

[8]) $\lambda = \lambda(d)$ denotes the function from Lemma 2.2.

to set $N = 0$, $\delta = 1$ and to use the fact the B_{λ_1} satisfies (B) under the assumption $\lambda_1 \in \langle 0, b_{11})$. We must use the simplicity of λ_1 and Remark 2.3 instead of the algebraic simplicity of d_0.

Proof of Theorem 1.1. Under the assumptions of Theorem 1.1, Lemma 2.3 ensures the existence of $\varrho > 0$ such that $(\overline{\mathrm{GC}})$ is fulfilled with an arbitrary $d_1 \in \in (d_0 - \varrho, d_0\rangle$ and with $\lambda = \lambda(d_1) > 0$ from Lemma 2.2. Further, $E_B(d_1, \lambda(d_1)) \cap \cap \tilde{K}^0 \neq \emptyset$ for all $d_1 \in (d_0 - \varrho, d_0\rangle$ if $\varrho > 0$ is sufficiently small because $E_B(d_0, 0) \cap \cap \tilde{K}^0 \neq \emptyset$ by the assumption and normed vectors $W(d)$ from $E_B(d, \lambda(d))$ depend continuously on d by Lemma 2.2. Hence, Theorem 3.1 implies that for any $d_1 \in \in (d_0 - \varrho, d_0\rangle$ there exists d_I^1 such that $\lambda(d_1) > 0$ is an eigenvalue of $(\mathrm{EI}^\sim)$ with $d = d_I^1$ in the sense of Definition 1.1, $E_I(d_I^1, \lambda(d_1)) \subset \partial\tilde{K}$, $E_B(d_I^1, \lambda(d_1)) = \{0\}$. It is sufficient to show that $d_I^1 > d_0$ for any $d_1 \in (d_0 - \varrho, d_0\rangle$ if $\varrho > 0$ is small enough. If this were not true we should have sequences $\{d_n\}$, $\{d_I^n\}$ such that $d_n < d_I^n \leqq d_0$, $d_n \to d_0$ and $\lambda(d_n) > 0$ is an eigenvalue of $(\mathrm{EI}^\sim)$ with $d = d_I^n$ corresponding to some $W_n \in \partial\tilde{K} \cap E_I(d_I^n, \lambda(d_n))$, $\|W_n\|_\sim = 1$. Hence, we have

$$\langle D(d_I^n)\, W_n - B\tilde{A}W_n + \lambda(d_n)\, \tilde{A}W_n,\ \Phi - W_n\rangle_\sim \geqq 0\,, \quad \text{for all} \quad \Phi \in \tilde{K}\,. \tag{3.1}$$

We can suppose $W_n \rightharpoonup W$ and the usual considerations using the compactness of A yield $W_n \to W$ (more precisely see Remark 3.1 below). Hence, $W \in \partial\tilde{K}$. Lemma 2.2 implies $\lambda(d_n) \to 0_+$ and the limiting process applied to (3.1) gives

$$\langle D(d_0)\, W - B\tilde{A}W,\ \Phi - W\rangle_\sim \geqq 0\,, \quad \text{for all} \quad \Phi \in \tilde{K}\,,$$

i.e. $W \in E_I(d_0, 0)$. However, $E_B(d_0, 0) \cap \tilde{K} = E_I(d_0, 0)$ (this holds in general under the assumption $E_B(d_0, 0) \cap \tilde{K}^0 \neq \emptyset$, see Lemma 2.1 in [4]; cf. [7]). Hence $W \in \partial\tilde{K} \cap \cap E_B(d_0, 0)$ which contradicts the simplicity of d_0 and the assumption $E_B(d_0, 0) \cap \cap \tilde{K}^0 \neq \emptyset$.

Remark 3.1. It follows from (3.1) that

$$\langle W_n, W_n\rangle_\sim = \langle D^{-1}(d_I^n)\, B\tilde{A}W_n - D^{-1}(d_I^n)\, \lambda(d_n)\, \tilde{A}W_n, W_n\rangle_\sim\,,$$
$$\langle W_n, W\rangle_\sim \geqq \langle D^{-1}(d_I^n)\, B\tilde{A}W_n - D^{-1}(d_I^n)\, \lambda(d_n)\, \tilde{A}W_n, W\rangle_\sim\,,$$

where $D^{-1}(d)$ is the inverse matrix to $D(d)$. This together with the compactness of A yields $\|W\|_\sim \geqq \lim_{n\to\infty} \|W_n\|_\sim$, i.e. $W_n \to W$ under the assumption $W_n \rightharpoonup W$.

4. A HOMOTOPY JOINING CRITICAL POINTS OF THE EQUATION AND OF THE INEQUALITY

Let us denote by $\tilde{P}$ the projection onto the closed convex cone $\tilde{K}$ in $\mathbb{V}^\sim$, i.e. $\tilde{P}$ is the mapping defined in $\mathbb{V}^\sim$ by

$$\|\tilde{P}V - V\|_\sim = \min_{W \in \tilde{K}} \|W - V\|_\sim\,.$$

Remember that $\tilde{P}$ is positive homogeneous, lipschitzian and $(\tilde{I} - \tilde{P})\, U = 0$ if and only if $U \in \tilde{K}$.

Remark 4.1. For any $V \in \mathbb{V}^{\sim}$, $\tilde{P}V$ is the unique point satisfying

$$\langle V - PV,\ W - PV \rangle_{\sim} \leqq 0\,, \quad \text{for all} \quad W \in \tilde{K}$$

(see e.g. [15]). It follows that (EI$^{\sim}$) is equivalent to the operator equation

$$D(d)\, U - \tilde{P}(B\tilde{A}U - \lambda\tilde{A}U) = 0 \tag{4.1}$$

(cf. [4], Remark 2.3).

Definition 4.1. For an arbitrary fixed λ let us denote by Z_λ the closure (in $\mathbb{R} \times \times \mathbb{V}^{\sim} \times \mathbb{R}$) of the set of all $[d, U, \tau] \in \mathbb{R}^+ \times \mathbb{V}^{\sim} \times (0, 1\rangle$ such that

(a) $$\|U\|^2_{\sim} = \tau\,,$$

(b) $$D(d)\, U - B\tilde{A}U + \lambda\tilde{A}U + \tau(\tilde{I} - \tilde{P})(B\tilde{A}U - \lambda\tilde{A}U) = 0\,.$$

Remark 4.2. If we put $\tau = 0$ and $\tau = 1$ in (b) then we obtain (EE$^{\sim}$) and (4.1) (i.e. (EI$^{\sim}$)), respectively. This means that the equation (b) represents a "homotopy joining the equation and the inequality".

Remark 4.3. If $[d, 0, 0] \in Z_\lambda$ then λ is an eigenvalue of (EE$^{\sim}$) (for d under consideration) in the sense of Definition 1.1. Indeed, there exist $[d_n, U_n, \tau_n] \in Z_\lambda$ such that $\tau_n \in (0, 1\rangle$, $[d_n, U_n, \tau_n] \to [d, 0, 0]$. Setting $W_n = U_n/\|U_n\|_{\sim}$ we may suppose $W_n \rightharpoonup W$ in $\mathbb{V}^{\sim}$ and (b) divided by $\|U_n\|_{\sim}$ together with the compactness of A implies $W_n \to W$ and

$$D(d)\, W - B\tilde{A}W + \lambda\tilde{A}W = 0\,.$$

Theorem 4.1. *Let the assumptions of Theorem 1.1 be fulfilled and let $\lambda(d)$ be the function from Lemma 2.2. Then there is $\varrho > 0$ such that for any $d_1 \in (d_0 - \varrho, d_0\rangle$ there exists a closed compact connected subset $Z^+_{\lambda(d_1)}$ of $Z_{\lambda(d_1)}$ containing $[d_1, 0, 0]$ and at least one point of the type $[d^1_I, W, 1]$, $d^1_I > d_0$. Moreover, the following implications are true for all $[d, U, \tau] \in Z^+_{\lambda(d_1)}$:*

(c) *if $[d, U, \tau] \neq [d_1, 0, 0]$ then $B\tilde{A}U - \lambda\tilde{A}U \notin \tilde{K}$;*

(d) *if $[d, U, \tau] \neq [d_1, 0, 0]$ then $d_1 < d \leqq d_m$*

with some $d_m > 0$ independent of d_1.

Remark 4.4. If $[d, U, 1] \in Z_{\lambda(d_1)}$ then d, U satisfy (EI$^{\sim}$) by Remark 4.1 and $E_B(d, \lambda(d_1)) = \{0\}$ for $d > d_1$ by Lemma 2.2. Hence Theorem 1.1 is a consequence of Theorem 4.1 (cf. [4], Theorems 1.1, 1.2).

Proof of Theorem 4.1. For any $d_1 \in (d_0 - \varrho, d_0\rangle$ (with $\varrho > 0$ sufficiently small) the existence of a closed compact connected set $Z^+_{\lambda(d_1)} \subset Z_{\lambda(d_1)}$ joining $[d_1, 0, 0]$ with some $[d^1_I, U, 1]$ and satisfying (c), (d) follows from Theorem 2.2 from [4] (analogously as Theorem 3.1 follows from Theorem 2.1 from [4]). It is sufficient to replace d_0 by d_1, B by $B_{\lambda(d_1)} = B - \lambda(d_1)\, E$ (E is the unit matrix), set $N = 0$, $\delta = 1$ and use the fact that $B_{\lambda(d_1)}$ satisfies (B) again for d_1 sufficiently close to d_0, i.e. for $\lambda(d_1)$ small (see Lemma 2.2); further, we must recall Remark 2.3 and use the last assertion of Lemma 2.2 instead of the assumption that d_0 is the greatest critical point;

finally, the assumption (GC) from [4] is replaced by ($\overline{\text{GC}}$) from Theorem 3.1 and Remark 2.3; ($\overline{\text{GC}}$) is fulfilled by Lemma 2.3.

Further, analogously as in the proof of Theorem 3.1 it can be proved that $d_I^1 > d_0$ for any $[d_1^I, U, 1] \in Z^+_{\lambda(d_1)}$ with $d_1 \in (d_0 - \varrho, d_0\rangle$ if $\varrho > 0$ is sufficiently small.

Remark 4.5. The proof of Theorem 4.1 could be done also directly by the same method as that of Theorem 2.2 in [4] (using a modification of Dancer's result [1]). The proof of our Theorem 4.1 would be easier than that of Theorem 2.2 in [4] because we have $N = 0$ (i.e. our problem is positive homogeneous; we can norm the solutions and the considerations from [4] about "sufficiently small δ" are not necessary (cf. [7]).

5. NEUMANN BOUNDARY CONDITIONS

In the case of Neumann conditions

$$\text{(NC)} \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial\Omega$$

Instead of (BC) there are the same complications as in [4]. We must use the inner product

$$\langle u, \varphi\rangle = \int_\Omega \left(\sum_{i=1}^n u_{x_i}\varphi_{x_i} + \eta u\varphi\right) \mathrm{d}x$$

(with some $\eta > 0$ fixed) in the abstract formulation and replace the expressions in ($\text{EE}^\sim$) and ($\text{EI}^\sim$) by

$$du - (b_{11} + \eta d)\, Au - b_{12}Av \ldots,$$
$$v - b_{21}Au - (b_{22} + \eta)\, Av \ldots.$$

It is the variable parameter d in the coefficient at Au which causes the fundamental trouble (cf. Remark 1.7 and Section 5 in [4]). However, using the same approach as in [4], Section 5, the following assertion can be proved:

Consider the system (RD) as in Remark 1.2 but with $\Gamma_N = \partial\Omega$. Set $\mathbb{V} = W_2^1(\Omega)$ and let K be from (1.3). Then either

(i) there exists $d_I > d_0$ such that (RD_λ), (UC_0) has a positive eigenvalue for $d = d_I$ or

(ii) there is a positive eigenvalue λ of the problem

$$\text{(SI)} \qquad \begin{cases} u = \xi \ (= \text{const}), \quad v \in K, \\ \displaystyle\int_\Omega (b_{11}\xi + b_{12}v - \lambda\xi)\, \mathrm{d}x = 0, \\ \displaystyle\int_\Omega \sum_{i=1}^n v_{x_i}(\psi_{x_i} - v_{x_i}) - (b_{21}\xi + b_{22}v + \lambda v)(\psi - v)\, \mathrm{d}x \geqq 0, \\ \qquad\qquad\qquad\qquad\qquad \text{for all} \quad \psi \in K. \end{cases}$$

The system (SI) can be called the shadow inequality to ($\text{EI}^\sim$) and can be obtained from ($\text{EI}^\sim$) by the limiting process $d \to +\infty$ (cf. [14] where the shadow system for equations is studied; cf. also [4], Section 5). If follows that either the trivial solution of $(\mathrm{RD_L})$, $(\mathrm{UC_0})$ is unstable for some $d_i > d_0$ or that the trivial solution of the shadow inequality to $(\mathrm{RD_L})$, $(\mathrm{UC_0})$

$$u(x, t) = \xi(t)\,, \quad v(x, t) \in K\,,$$

$$\int_\Omega \left[\frac{d\xi(t)}{dt} - b_{11}\,\xi(t) - b_{12}v(x, t)\right] \mathrm{d}x = 0\,,$$

$$\int_\Omega \left[\left(\frac{dv(x, t)}{dt} - b_{21}u(x, t) - b_{22}v(x, t)\right)(\psi(x) - v(x, t)) + \right.$$

$$\left. + \sum_{i=1}^n v_{x_i}(x, t)\,(\psi_{x_i}(x) - v_{x_i}(x, t))\right] \mathrm{d}x \geqq 0 \quad \text{for all} \quad \psi \in K \quad \text{and a.a.} \quad t \geqq 0$$

is unstable.

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Authors' addresses: P. Drábek, 300 00 Plzeň, Nejedlého sady 14, Czechoslovakia (Vysoká škola strojní a elektrotechnická); M. Kučera, 115 67 Praha 1, Žitná 25, Czechoslovakia (Matematický ústav ČSAV).