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NEWS AND NOTICES

SIXTY YEARS OF JAROSLAV KURZWEIL

Jiří Jarník, Štefan Schwabik, Milan Tvrdý, Ivo Vrkoč, Praha

A prominent Czechoslovak scientist, Jaroslav Kurzweil, chief research worker of the Mathematical Institute of the Czechoslovak Academy of Sciences, Professor of Mathematics at Charles University, corresponding member of the Czechoslovak Academy of Sciences, reaches sixty years of age on May 7, 1986.

Before proceeding to describing the scientific activities of Professor Kurzweil, let us give a brief survey of the main milestones of his life:

1926 – born in Prague on May 7
1945 – secondary school leaving examination
1949 – graduation from Faculty of Science, Charles University, Prague

– Assistent Professor at Department of Mathematics and Descriptive Geometry, Czech Technical University, Prague

1950 – receives his RNDr. (doctor of natural science) degree

1951 – since July 1 research student (aspirant) at Central Mathematical Institute, later Mathematical Institute of the Czechoslovak Academy of Sciences

1953 – research stay in Poland

1954 – since January 1 employed in the Mathematical Institute, Czechoslovak Academy of Sciences, Prague

1955 – receives the degree of Candidate of Science (CSc.) and is appointed Head of Department of Ordinary Differential Equations of the Mathematical Institute

1957 – research stay in the USSR

1958 – receives the degree of Doctor of Science (DrSc.)

1964 – appointed member of the Scientific Board for Mathematics of the Academy;
– awarded Klement Gottwald State Prize

1966 – appointed Full Professor of Mathematics

1968 – elected corresponding member of the Czechoslovak Academy of Sciences;
– in the academic year 1968–1969 visiting professor at Dynamic Centre, Warwick, UK

1978 – elected honorary foreign member of the Royal Society of Edinburgh

1981 – awarded the Bernard Bolzano silver medal of the Czechoslovak Academy of Sciences "For achievements in mathematical sciences";
– elected member of merit of the Union of Czechoslovak Mathematicians and Physicists

1984 – appointed Head of Division of Mathematical Analysis in the Mathematical Institute and Head of Department for Didactics of Mathematics.

J. Kurzweil started his scientific career as a student of Professor V. Jarník in the metrical theory of diophantine approximations. The influence of V. Jarník can still be seen in Kurzweil's rigorous style and his feeling for fine and ingenious estimates. The very first Kurzweil's paper deals with the properties of Hausdorff measure of the set of real numbers x that admit no $g(q)$ approximation, that is, there are only a finite number of integers $p, q > 0$ such that $|x - p(q)| < q^{-2}\, g(q)$, where $g(q)$ is a positive function defined for positive values of q.

The next paper concerning this topic [5] is of great importance. It solves the Steinhaus problem: if $a < b$ are real numbers, denote

$$I(a, b) = \{(\xi_1, \xi_2) \in \mathbb{R}^2\,, \quad \xi_1 = \cos 2\pi x\,, \quad \xi_2 = \sin 2\pi x\,, \quad x \in [a, b]\}\,,$$

and let B be the set of all nonincreasing sequences (b_k), $k = 1, 2, \ldots$, with positive members satisfying $\sum b_k = +\infty$. Let

$$K = \{(\xi_1, \xi_2) \in \mathbb{R}^2,\ \xi_1^2 + \xi_2^2 = 1\}\,,$$

let μ be the Lebesgue measure on the circumference K and $\alpha(B)$ the set of real numbers

$x \in [0, 1)$ with the property that for every sequence $(b_k) \in B$, μ-almost all points $y \in K$ belong to infinitely many sets of the form $I \cap (kx - b_k, kx + b_k)$, $k = 1, 2, \ldots$. The set $\alpha(B)$ does not contain rational numbers and H. Steinhaus put forward the question whether $\alpha(B)$ contains all irrational numbers from the interval $(0, 1)$. Kurzweil characterized the set $\alpha(B)$ by means of the notion of approximability and his considerations implied among other that $\alpha(B) \neq \emptyset$ and that its Lebesgue measure is zero. In this way he answered Steinhaus' question in negative. The paper [5] includes further results, in particular, the problem is modified and solved in the moredimensional case.

In 1953 Kurzweil spent three months in Poznań (Poland) with Prof. Wl. Orlicz. This contact brought new impulses to his work, concerning uniform approximation of a continuous operation by an analytic one.[1])

The paper [3] was directly inspired by Wl. Orlicz. It contains a generalization of the well known theorem of S. N. Bernstein on characterization of real analyticity of a function. Kurzweil proved an assertion of this type for analytic operations defined in a Banach space X with values in a Banach space Y. In the next paper [4] he formulated the following problem: is it possible to uniformly approximate continuous operations from a Banach space X into a real Banach space Y by means of analytic operations?

The answer is given by the following assertion: Let X be a separable real Banach space satisfying the condition

(A) there exists a real polynomial q^* defined on X such that $q^*(0) = 0$ and

$$\inf_{x \in B, \|x\| = 1} q^*(x) > 0.$$

Let F be a continuous operation defined on an open set $G \subset X$ with values in an arbitrary Banach space Y. Let φ be a positive continuous functional on G. Then there exists an operation H with values in Y which is analytic in G and satisfies

$$\|F(x) - H(x)\| < \varphi(x).$$

Counterexamples of continuous functionals in $C(0, 1)$, l^p and L^p (p odd) which are not uniform limits of analytic functions were presented in the same paper.

The assumption (A) may seem rather surprising. Kurzweil resumed the study of this problem in [11], showing that for a uniformly convex Banach space in which every operation F can be uniformly approximated by analytic functions, the assumption (A) is necessarily fulfilled.

The small excursion into nonlinear functional analysis is remarkable as concerns the depth of the results and only recently has brought its fruits in an apparently distant field dealing with the geometry of Banach spaces.

Functional analysis is the topic also of [25], where Kurzweil, using elementary tools, ellegantly proved the known theorem on spectral decomposition of Hermitian

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[1]) Basic information on notions involved are found e.g. in the well known monograph *E. Hille, R. S. Phillips:* Functional Analysis and Semigroups, AMS, Providence 1957.

operator. Unlike W. P. Eberlein[2]) he started with the immediate definition of the so called spectral function.

Also the paper [32] is closely connected with the theory of Hermitian operators. It concerns estimates of eigenvalues of a system of integral equations

$$\int_\Omega K(x, t)\, u(t)\, d\mu t = \beta\, u(x)\,,$$

$$\int_\Omega K(x, t)\, v(t)\, d\nu t = \gamma\, v(x)$$

and its "attached" system

$$\int_\Omega K(x, t)\, y(t)\, d\mu t = \alpha\, z(x)\,,$$

$$\int_\Omega K(x, t)\, z(t)\, d\nu t = \alpha\, y(x)\,.$$

The result obtained by Kurzweil in this direction had been known before only in very special cases.

Theory of stability for ordinary differential equations represents an important field which has been strongly influenced by Kurzweil's research. Although the fundaments of this theory had been laid as early as in the last decades of the 19th century (H. Poincaré, A. M. Ljapunov), many problems remained open till the 50's of this century when this branch again started to flourish.

Given a system of differential equations

$$\dot{x} = f(x, t)\,, \quad x \in \mathbb{R}^n\,, \quad t \geqq 0$$

where $f(0, t) = 0$, $t \geqq 0$, then the solution $x(t) \equiv 0$ is called stable if for every $\varepsilon > 0$ there is $\delta > 0$ such that any solution $y(t)$ of the system with $\|y(0)\| < \delta$ satisfies $\|y(t)\| < \varepsilon$ for all $t \geqq 0$. A. M. Ljapunov found the following sufficient condition for stability:

If there exist functions $V(x, t)$, $U(x)$ such that $V \in C^1$, U is continuous, $U(x) > 0$ for $x \neq 0$, $V(t, 0) \equiv 0$, $V(x, t) \geqq U(x)$ for $x \in \mathbb{R}^n$, $t \geqq 0$, and if

$$W(x, t) = \frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i \leqq 0\,,$$

then the solution $x \equiv 0$ is stable.

In 1937 K. P. Persidskij showed that the conditions from this theorem are necessary as well. Persidskij also formulated a sufficient condition for uniform stability in terms of a certain Ljapunov function. The problem whether the conditions of Persidskij's theorem are also necessary was attacked by a number of mathematicians. It was answered in affirmative independently by N. N. Krasovskij and J. Kurzweil under

[2]) A note on the spectral theorem, Bull. AMS *52* (1946), 328—331.

the assumption that the components of the right hand side of the differential equation have continuous partial derivatives. Later, T. Yoshizawa proved the conversion of these theorems for continuous right hand sides. However, the Ljapunov functions constructed by Yoshizawa were not necessarily continuous. This incited the paper [10] where Kurzweil proved that stability or uniform stability can always be characterized by existence of a function satisfying the assumptions of Ljapunov's or Persidskij's theorems. He gave in this work additional (necessary and sufficient) conditions guaranteeing existence of a smooth Ljapunov function. Analogous problems for the so called second Ljapunov theorem are solved in [9]. Conversion of this theorem, which concerns asymptotic stability, was studied by J. L. Massera for periodic right hand sides of the equation. I. G. Malkin noticed that the assumptions of the second Ljapunov theorem yield results stronger than the original formulation admits. The definitive solution of the problem was given by Kurzweil in [9]. First of all, he showed that the assumptions of the second Ljapunov theorem guarantee even strong stability of the trivial solution $x \equiv 0$. Conversely, if $x \equiv 0$ is a strongly stable solution of the system, he constructed smooth functions satisfying the assumptions of the second Ljapunov theorem. In his constructions Kurzweil developed a method of approximation of Lipschitzian functions, which enabled him to prove that the desired functions are of class C^∞ even if the right hand side of the equations are merely continuous.

In the fifties, in connection with problems in mechanics, Bogoljubov's averaging method for differential equations became very popular. The method was effective in applications but it was not quite clear how to substantiate it and give it its right place in the framework of the theory of ordinary differential equations. I. I. Gichman in 1952 was the first to notice that the basis of this method is the continuous dependence on a parameter. Gichman's ideas were further developed in 1955 by M. A. Krasnoselskij and S. G. Krejn who pointed out that in order to have continuous dependence on a parameter a certain "integral continuity" of the right hand side of the differential equation is sufficient. The paper [12] in 1957 then brought the following fundamental result:

Let $f_k\colon G \times [0, T] \to \mathbb{R}^n$, $k = 0, 1, 2, \ldots$ be a sequence of functions, $G \subset \mathbb{R}^n$ an open set. Let $x_k(t)$ be a solution of the differential equation

$$\dot{x} = f_k(x, t)\,, \quad x(0) = 0$$

and let $x_0(t)$ be uniquely defined on $[0, T]$. If

$$F_k(x, t) = \int_0^t f_k(x, \tau)\, d\tau \to \int_0^t f_0(x, \tau)\, d\tau = F_0(x, t)$$

uniformly with $k \to \infty$ and if the functions $f_k(x, t)$, $k = 0, 1, 2, \ldots$ are equicontinuous in x for fixed t, then for sufficiently large k the solutions $x_k(t)$ are defined on $[0, T]$ and $x_k(t) \to x_0(t)$ uniformly on $[0, T]$ with $k \to \infty$.

The results of [12] discovered the very core of the assertion on continuous dependence for differential equations. When in 1975 Z. Artstein[3]) studied theorems on continuous dependence from the general viewpoint and introduced topological criteria of comparing them he found that there exist best possible theorems and that the quoted result of [12] is one of them.

However, the results of [12] brought to light also some new problems. For example, direct calculation of the solutions $x_k\colon [0, 1] \to \mathbb{R}$ of the sequence of linear differential equations

$$x' = xk^{1-\alpha} \cos kt + k^{1-\beta} \sin kt\,, \quad x(0) = 0\,, \quad k = 1, 2, \dots$$

shows that for $0 < \alpha \leqq 1$, $0 < \beta \leqq 1$, $\alpha + \beta > 1$ we have $\lim_{k\to\infty} x_k(t) = 0$ uniformly on $[0, 1]$, that is, the solutions converge to the solution of the "limit equation"

$$\dot{x} = 0\,, \quad x(0) = 0\,.$$

Theorems on continuous dependence on a paramter which could theoretically motivate and justify this convergence phenomenon were not available at the time. Even the above mentioned result from [12] gave a substantiation of the convergence effect in this case only for $\alpha = 1$ and $0 < \beta \leqq 1$.

Moreover, it was apparent that the knowledge of the function $f(x, t)$ on the right hand side of the differential equation

$$x' = f(x, t) \tag{1}$$

is in this context needed only to provide the possibility of speaking about the solution of the equation (1). Then all the essential facts can be expressed in terms of the "indefinite integral"

$$F(x, t) = \int_{t_0}^{t} f(x, \tau)\, \mathrm{d}\tau \tag{2}$$

of the right hand side $f(x, t)$ of the equation (1). A question arose how to describe the notion of a solution of the differential equation (1) in terms of the function (2). J. Kurzweil answered these questions in his work [13] where he introduced the concept of generalized differential equation. Let us briefly sketch the main points of this theory.

Given a function $F(x, t)\colon G \times [0, T] \to \mathbb{R}^n$, then a function $x\colon [a, b] \to \mathbb{R}^n$ is a solution of the generalized differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = DF(x, t) \tag{3}$$

if $(x(t), t) \in G \times [0, T]$ for every $t \in [a, b]$, and for all $s_1, s_2 \in [a, b]$ the difference $x(s_2) - x(s_1)$ is approximated with an arbitrary accuracy by the sum

$$\sum_{i=1}^{k} [F(x(\tau_i), \alpha_i) - F(x(\tau_i), \alpha_{i-1})]\,, \tag{4}$$

[3]) Continuous dependence on parameters: on the best possible results. Journal Diff. Eq. *19*, 214—225.

where $s_1 = \alpha_0 < \alpha_1 < \ldots < \alpha_k = s_2$, $\tau_i \in [\alpha_{i-1}, \alpha_i]$ is a sufficiently fine partition of the interval $[s_1, s_2]$.

In this way we express in a general form the fact that a solution of the classical equation (1) satisfies the equality

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} f(x(t), t)\, \mathrm{d}t\,, \quad s_1, s_2 \in [a, b]\,,$$

and the integral on the right hand side is approximated with an arbitrary accuracy by a sum of the form

$$\sum_{i=1}^{k} \int_{\alpha_{i-1}}^{\alpha_i} f(x(\tau_i), t)\, \mathrm{d}t\,.$$

Sums of the form (4) are the starting point of Kurzweil's concept of the generalized Perron integral developed in [13]. Here he gave the precise interpretation to the notion of arbitrarily accurate approximation of the difference $x(s_2) - x(s_1)$ by means of (4).

Let $[a, b] \subset \mathbb{R}$ be a compact interval. A finite system of real numbers

$$D = \{\alpha_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$$

will be called a partition of $[a, b]$ if

$$a = \alpha_0 < \alpha_1 < \ldots < \alpha_k = b \quad \text{and} \quad \tau_i \in [\alpha_{i-1}, \alpha_i]\,, \quad i = 1, 2, \ldots, k\,. \tag{5}$$

Given a function $\delta\colon [a, b] \to (0, +\infty)$ (a so called gauge), we say that a partition D is δ-fine if

$$[\alpha_{i-1}, \alpha_i] \subset [\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)]\,, \quad i = 1, 2, \ldots, k\,. \tag{6}$$

With a function $U\colon [a, b] \times [a, b] \to \mathbb{R}^n$ and a partition D we associate the sum

$$S(U, D) = \sum_{i=1}^{k} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})]\,.$$

Definition. We say that $I \in \mathbb{R}^n$ is the generalized Perron integral of the function U over $[a, b]$ if for every $\varepsilon > 0$ there is a gauge δ such that for every δ-fine partition D of $[a, b]$, the inequality

$$|S(U, D) - I| < \varepsilon$$

holds. The value I is denoted by the (inseparable) symbol $\int_a^b DU(\tau, t)$.

This definition enables us to give a precise meaning to the notion of solution of the generalized differential equation (3): a function $x\colon [a, b] \to \mathbb{R}^n$ is a solution of (3) if $(x(t), t) \in G \times [0, T]$ and

$$x(s_2) - x(s_1) = \int_{s_1}^{s_2} DF(x(\tau), t)$$

holds for all $s_1, s_2 \in [a, b]$.

The generalized differential equations (3) were thoroughly studied in [13], [14], [15], [17], [20], [29], [34], in which Kurzweil obtained important new results concerning continuous dependence on a parameter for differential equations and substantiated convergence phenomena that had lacked theoretical explanation, including for example convergence effects for a sequence of ordinary differential equations

$$\dot{x} = f(x, t) + g(x)\, \varphi_k(x)\,, \quad k = 1, 2, \ldots,$$

with the sequence (φ_k) converging in the usual way to the Dirac function (see [14], [16]).

In [14] Kurzweil showed that generalized differential equations admit discontinuous functions as solutions. This was a quite new phenomenon in the theory of differential equations. Of course, its occurrence was the consequence of the class of right hand sides considered.

The methods of generalized differential equations were extended by Kurzweil also to the case of differential equations in a Banach space. Here he obtained new results concerning partial differential equations and some types of boundary value problems (e.g. in [27], [28], [29], [33], [34]). His contributions in this direction inspired many mathematicians working in the theory of partial differential equations.

For the series of papers on generalized differential equations J. Kurzweil was awarded the Klement Gottwald State Prize in 1964.

Let us return to the paper [13] and in particular to the above mentioned definition of integral. Kurzweil gave there two equivalent definitions, one of them in terms of majorant and minorant functions analogously to the classical Perron's definition, and the other via the integral sums as we have mentioned above. If the function U is of the form $U(\tau, t) = f(\tau)\, t$ then the corresponding integral sum is $\sum_{i=1}^{k} f(\tau_i)\,.$ $.\,(\alpha_i - \alpha_{i-1})$, thus coinciding with the Riemann integral sum. In [13] Kurzweil proved that in this special case

$$\int_a^b [Df(\tau)\, t] \quad \text{exists iff the Perron integral} \quad \int_a^b f(t)\, \mathrm{d}t \quad \text{exists}\,,$$

that is, he proved that the Perron integral can be defined by means of Riemannian sums with the above mentioned modification of the "fineness" of a partition of the interval. In this period he contributed to the theory of integral also by the paper [18] devoted to the integration by parts.

Independently of Kurzweil's results and with quite different motives, the same definition of integral was later (cca 1960) introduced by R. Henstock[4]).

This theory of integral, besides its usefulness for the theory of differential equations, is of considerable interest by itself. It is an illustrative summation definition of

[4]) See e.g. the monograph *R. Henstock:* Theory of Integration, Butterworths, London 1963.

a general, nonabsolutely convergent integral, which is also of nonnegligible didactical value.[5])

Kurzweil's ideas from 1957 are still alive and fruitful. Kurzweil himself returned to his theory of integral in 1973 by papers dealing with the change of order of two integrations [57] and an interesting problem of multipliers for the Perron integral [58], and published an appendix [B6] to Jacob's monograph on measure and integral. In 1980 he published a small monograph [B5] summarizing his results and embodying his concept of integral in the framework of the theory of integral. The survey paper [74] then represents a brief exposition of Kurzweil's approach to the theory of integral based on his works from 1957.

The years 1957—1959 were the period when principal contributions to the mathematical theory of optimal control appeared. In particular, in 1959 a group of Soviet mathematicians led by L. S. Pontrjagin published the now well known monograph on this subject. J. Kurzweil reacted very soon to this situation and inspired research in this field in Czechoslovakia. In [23] and [31] Kurzweil studied the linear regulation problem and for this case obtained results concerning especially the geometric properties of accessible sets. The paper [26] is devoted to the linear autonomous problem with a quadratic functional. He proved the existence theorem for the optimal solution approaching zero when $t \to \infty$, and solved also the so called converse problem.

The problems of the optimal control theory form the background of later Kurzweil's papers concerning differential relations (inclusions).

The averaging method did not cease to attract the attention of Prof. Kurzweil. He focused his interest to the application of this method in the case of more general spaces. In [27] he proved a theorem on averaging for differential equations in a Banach space and applied the result to the case of oscillations of a weakly nonlinear string. In particular, he discussed the weak nonlinearity of van der Pol's type. Problems of this type were much more extensively studied in [34]—[44] and [49], in which Kurzweil dealt also with problems concerning integral manifolds for systems of differential equations in a Banach space. He took much care to establish results applicable to the theory of partial differential equations and functional differential equations.

Let us roughly sketch Kurzweil's assertion on existence of an integral manifold (cf. [41]) for a system of ordinary differential equations in a Banach space $X = X_1 \times X_2$, where X_1, X_2 are also Banach spaces. Let $f = (f_1, f_2)\colon G \times \mathbb{R} \to X_1 \times X_2 = X$, where, for instance, $G = \{(x_1, x_2) \in X;\ x_1 \in X_1,\ |x_1| < 2,\ x_2 \in X_2\}$. For $x = (x_1, x_2) \in X$ put $|x| = |x_1| + |x_2|$, where $|x|, |x_1|, |x_2|$ are norms

[5]) This fact was exploited for example by the Belgian mathematician *J. Mawhin* in his lecture notes Introduction à l'Analyse, Louvain 1979. Other recent monographs devoted to Kurzweil's integral are *R. M. McLeod:* The Generalized Riemann Integral, Carus Math. Monographs 20, MAA, 1980 and *E. J. McShane:* Unified Integration, Academic Press, 1983.

of the elements x, x_1, x_2 in the spaces X, X_1, X_2, respectively. Consider the system

(7) $$\dot{x} = f(x, t), \quad \text{i.e.} \quad \dot{x}_1 = f_1(x_1, x_2, t), \quad \dot{x}_2 = f_2(x_1, x_2, t)$$

provided the function $f: G \times \mathbb{R} \to X$ is continuous, bounded and has a bounded differential $\partial f/\partial x$ uniformly continuous with respect to x and t.

Let $f_1(0, x_2, t) = 0$ for $x_2 \in X_2$, $t \in \mathbb{R}$, that is, the function $x_1(t) = 0$ is a solution of the first equation in (7) on the whole $\mathbb{R}$ and the set

$$M = \{(0, x_2, t): x_2 \in X_2, \; t \in \mathbb{R}\} \subset X \times \mathbb{R}$$

is an integral manifold of the system (7). Further, for $\tilde{x}_1 \in X_1$, $|\tilde{x}_1| \leqq \sigma$, $\tilde{x}_2 \in X_2$, $\tilde{t} \in \mathbb{R}$ let there exist such a solution (x_1, x_2) of the system (7) defined on $(\tilde{t}, +\infty)$ that $x_1(\tilde{t}) = \tilde{x}_1$, $x_2(\tilde{t}) = \tilde{x}_2$ and

$$|x_1(t)| \leqq \varkappa \mathrm{e}^{-\nu(t-\tilde{t})} |x_1|$$

for $t \geqq \tilde{t}$.

If (x_1, x_2), (y_1, y_2) are solutions of the system (7) defined for $t \in \mathbb{R}$ and lying in M, then let

(8) $$|x_2(t_2) - y_2(t_2)| \geqq \varkappa^{-1} \mathrm{e}^{-\mu(t_2 - t_1)} |x_2(t_1) - y_2(t_1)|$$

hold for $t_2 \geqq t_1$ with $\mu < \nu$.

If for $x \in C$, $t \in \mathbb{R}$ and $0 \leqq \lambda \leqq 1$ the integral

$$\left| \int_t^{t+\lambda} f(x, s) - g(x, s) \, \mathrm{d}s \right|$$

is sufficiently small, then there exists such a mapping $p: X_2 \times \mathbb{R}_1 \to X_1$ that the set

$$\tilde{M} = \{(x_1, x_2, t): x_1 = p(x_2, t), \; x_2 \in X_2, \; t \in \mathbb{R}\} \subset X \times \mathbb{R}$$

is an integral manifold for the system

(9) $$\dot{x} = g(x, t).$$

In other words: if $\tilde{x}_2 \in X$, $\tilde{t} \in \mathbb{R}$, $\tilde{x}_1 = p(\tilde{x}_2, \tilde{t})$ then there exists such a solution (x_1, x_2) of the system (9) defined for $t \in \mathbb{R}$ that $x_1(\tilde{t}) = \tilde{x}_1$, $x_2(\tilde{t}) = \tilde{x}_2$ and $x_1(t) = p(x_2(t), t)$ for $t \in \mathbb{R}$.

Moreover, the integral manifold $\tilde{M}$ of the system (9) maintains some properties of the manifold M of the system (7). For example, the mapping p is bounded and Lipschitzian in the variable x_2. Any solution of (9) starting in a neighbourhood of the manifold $\tilde{M}$ exponentially tends for $t \to \infty$ to a solution of (9) which lies in $\tilde{M}$, and every couple of solutions of (9) lying in $\tilde{M}$ satisfies an estimate of the same type as (8).

In order not to complicate the situation too much we do not give a detailed formulation of results, in which an important role is played by the interrelations of constants characterizing the systems (7) and (9) and their solutions. Of course, these are essential for the result and carry important information as well.

We have already mentioned Kurzweil's efforts to make his results widely applicable. These led him to general formulations as well as to the use of general methods of elaboration. In connection with his investigation of integral manifolds he used the notion of a flow as the basis of his conception. A flow is a certain family of mappings satisfying some conditions of axiomatic character, which are motivated by the essential properties possessed by the whole system of solutions of a differential equation. The axioms cover all features of the differential equation which are crucial for the proof of existence of the integral manifold. Kurzweil chose this approach already in the paper [34], and continued in this way in [35]. The whole set of 122 printed pages of these two essays contains numerous applications of the abstract results with proper illustration by pertinent examples. The abstract approach to the problems of existence of invariant manifolds reached its top in Kurzweil's paper [42] where the results are formulated for flows in a metric space. One section of [42] is devoted to functional differential equations in a Banach space. Kurzweil proved that if a functional differential equation is close enough to an ordinary differential equation satisfying certain boundedness conditions, then all solutions defined on the whole $\mathbb{R}$ (the so called global solutions) generate an exponentially stable integral manifold. However, the boundedness condition excluded linear equations from the class for which the result was valid. Therefore Kurzweil published in two notes [45] and [48] analogous results for equations on manifolds, which already covered the case of linear functional differential equations.

Together with A. Halanay, Kurzweil in [40] studied flows in Banach spaces formed by functions defined on the whole real axis or, as the case may be, on a certain halfline. The theory from [42] was modified so that it provided an abstract basis also for functional differential systems (see e.g. [39]).

The modern theory of dynamic systems has very clearly marked connections with modern differential geometry, whose methods Kurzweil has frequently used in his investigations. As an illustration, let us present his result from [49]: let M be a submanifold of a manifold N and let $f\colon U \to N$, where f is a $C^{(1)}$ mapping from a neighbourhood U of the manifold M, such that the partial mapping $f|_M\colon M \to M$ is a diffeomorphism on M. Under certain additional assumptions, for every $g\colon U \to N$ where g is a $C^{(1)}$ mapping close to f there exists a submanifold M_g in N such that $g|_{M_g}\colon M_g \to M_g$ is a diffeomorphism on M_g.

This results is useful especially in the theory of differential equations with delayed argument.

The research in invariant manifolds was followed by a series of papers from the years 1970–1975, which dealt with global solutions of functional differential equations and, in particular, differential equations with delayed argument [45], [47], [50], [51], [52], [59].

Let us mention in more detail only the result from [59], where Kurzweil substantially deepened the results of Yu. A. Rjabov. If $x\colon [t-\tau, t] \to \mathbb{R}^n$, $\tau > 0$, then denote by $x_t\colon [-\tau, 0] \to \mathbb{R}^n$ the function defined by the relation $x_t(\sigma) = x(t+\sigma)$ for $\sigma \in [-\tau, 0]$. Consider a functional differential equation

$$\dot{x} = F(t, x_t)\,, \tag{10}$$

where F is continuous in both variables and Lipschitzian in the latter with a constant L independent of t. Rjabov had shown that if the "delay" τ is not too large (precisely if $L\tau < e^{-1}$), then through every point (t_0, x_0) there passes a unique "special" solution $\bar{x}(t) = \bar{x}(t_0, x_0; t)$ of the equation (10), which is defined on $\mathbb{R}$ and exponentially bounded for $t \to -\infty$. Strengthening further the condition on τ he had shown that for every solution x there is a (not necessarily unique) special solution $\bar{x}$ such that $x(t) - \bar{x}(t) \to 0$ for $t \to +\infty$.

In [59] Kurzweil showed that the validity of the original inequality $L\tau < e^{-1}$ is sufficient even for a substantially stronger assertion: if x is a solution of (10) and we put $x_0 = \lim\limits_{s\to\infty} \bar{x}(s, x(s); t_0)$ then

$$\sup \{\exp(t/\tau) |x(t) - \bar{x}(t_0, x_0; t)|;\ t \geqq t_0\} < \infty .$$

Obviously, $\bar{x}(t_0, x_0; t)$ is the only special solution satisfying this inequality.

Another field in which Kurzweil started to engage himself in the 70's and in which he is still interested, is the theory of differential relations (inclusions) and the problems of multifunctions connected with it.

A differential relation is a generalization of the differential equation of the form

$$\dot{x} \in F(t, x) . \tag{11}$$

The right hand side of this relation is a so called multifunction, that is, a mapping defined on $G \subset \mathbb{R} \times \mathbb{R}^n$ whose values are subsets of the space $\mathbb{R}^n$. As solutions of a differential relation we usually consider locally absolutely continuous functions u defined on an interval I, which satisfy the relation $\dot{u}(t) \in F(t, u(t))$ for almost all $t \in I$.

The beginnings of the theory of differential relations, which go back to the thirties, are connected with the names of A. Marchaud and S. Zaremba. Their development in the last 20–30 years has been caused by their relations to the optimal control theory, to the study of differential equations with discontinuous right hand sides etc. It was these relations and in particular Filippov's paper[6]) from 1960 that incited Kurzweil's still lasting interest in differential relations.

When studying differential relations, Carathéodory-type conditions are often assumed:

(i) $F(t, \cdot)$ is upper semicontinuous for almost all t;
(ii) $F(\cdot, x)$ is measurable for all x;
(iii) F satisfies an "integrable boundedness" condition.

Moreover, the sets $F(t, x)$ are usually assumed to be nonempty, compact and convex subsets of $\mathbb{R}^n$.

In this connection a question arises whether the validity of (i), (ii) suffices to guarantee "reasonable" behaviour of the multifunction F in both variables. The following condition, evidently implying (i), (ii), seems to be plausible:

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[6]) *A. F. Filippov:* Differential equations with discontinuous right hand side, Mat. sbornik *51* (*93*) (1960), 99–128 (Russian; English transl. AMS Translat. II, Ser. *42* (1964), 199–231).

(iv) for every $\varepsilon > 0$ there is a set $A_\varepsilon \subset \mathbb{R}$ such that the measure $m(\mathbb{R} \setminus A_\varepsilon) < \varepsilon$ and the restriction $F|_{(A_\varepsilon \times R^n) \cap G}$ is upper semicontinuous (with respect to the pair of variables (t, x)).

However, the converse implication, that is, (i), (ii) $\Rightarrow$ (iv), does not hold. In [64] it is proved that in spite of this fact we can restrict the study of differential relations to right hand sides satisfying (iv). Namely, the following theorem holds:

Let $\mathbb{K}^n$ be the system of all nonempty compact convex subsets of $\mathbb{R}^n$. Let $F\colon G \to \mathbb{K}^n$ satisfy (i). Then there is a function $\tilde{F}\colon G \to \mathbb{K}^n \cup \{\emptyset\}$ satisfying (iv),

(v) $\tilde{F}(t, x) \subset F(t, x)$ for all $(t, x) \in G$;

(vi) every solution of (11) is also a solution of the differential relation $\dot{x} \in \tilde{F}(t, x)$.

Kurzweil gave (iv) the name of Scorza-Dragoni property, after the Italian mathematician who had studied analogous problems for ordinary differential equations.

The assertion of the above mentioned theorem makes the study of properties of solutions of differential relations easier, as is seen for example in [65]. Here the result analogous to the following well known theorem from the theory of ordinary differential equations was proved:

For a differential equation $\dot{x} = f(t, x)$ there is a set $E \subset \mathbb{R}$ of zero measure such that for every solution $x(t)$ the derivative $\dot{x}(t)$ exists and satisfies the equation for all $t \notin E$.

(For differential relations the term "derivative" must be replaced by that of "contingent derivative".)

In [68] it was proved that the set of solutions of the differential relation (11) is closed with respect to a certain limiting process, which can be roughly described as follows: Let W be the set of functions $w\colon I_w \to \mathbb{R}^n$, $I_w = \bigcup_{i=1}^{k} [\tau_{i-1}, \tau_i)$, $\tau_0 < \tau_1 < \ldots$ $\ldots < \tau_k$ for which there exist such solutions u_i of the differential relation (11) that $w(t) = u_i(t)$ for $t \in [\tau_{i-1}, \tau_i)$. Denote by J_w the jump function of w (that is, $w - J_w$ is continuous, $J_w(t) = 0$ for $t \in [\tau_0, \tau_1)$). Then every function q which is the uniform limit of a sequence of functions $w_j \in W$ satisfying $J_{w_j} \to 0$ uniformly, is a solution of (11).

Conversely, every set of "reasonable" functions closed with respect to the limiting process described is the set of (all) solutions of a certain differential relation. This makes it possible to construct, for a given set of functions, the "minimal" relation for which all the given functions are solutions.

Also further Kurzweil's papers [61], [63], [66], [67], [69] and [70] were devoted to differential relations. Let us mention just the paper [70] in which a new summation definition of the integral of a multifunction was given and a theorem on equivalence of the differential and integral relations was proved.

The last paper indicated Kurzweil's comeback to the theory of summation integrals, and he has indeed devoted much effort to this theory recently. However, the principal impulse was Mawhin's paper[7]) in which the author gave a generalization of the Perron integral in $\mathbb{R}^n$, which guarantees validity of the divergence theorem (Stokes theorem) for all differentiable vector fields without any further assumptions. Mawhin's definition is based on the above mentioned Riemann-type definition due to Kurzweil, Henstock and McShane, but restricts the class of admissible partitions (of an n-dimensional interval) taking into account only such intervals in which the ratio of the longest and shortest edges is not too big. Nevertheless, Mawhin himself pointed out that it is not clear whether his integral has some natural properties, in particular the following type of additivity: if $J, K, J \cup K$ are intervals and if f is integrable over both J and K, then it is also integrable over $J \cup K$.

In [73] an example was found that Mawhin's integral really lacks this property, an a modified version of Mawhin's definition was proposed: instead of the ratio of the longest and shortest edges, the subintervals J forming a partition of an n-dimensional interval are characterized by the quantity $\sigma(J) = \operatorname{diam} J \,.\, m(\partial J)$ (the product of the diameter of the interval and the $(n-1)$-dimensional Lebesgue measure of its boundary).

Define a P-partition of an interval $I \subset \mathbb{R}^n$ as a finite system Π of pairs (x^j, I^j), $j = 1, 2, \ldots, k$, where I^j are nonoverlapping compact intervals whose union is I, and $x^j \in I^j$. If $\delta: I \to (0, +\infty)$ (a gauge) then a given P-partition is called δ-fine if I^j, $j = 1, 2, \ldots, k$ lies in a ball with centre x^j and radius $\delta(x^j)$. For a function $f: I \to \mathbb{R}^n$ put

$$S(f, \Pi) = \sum_{j=1}^{k} f(x^j)\, m(I^j)$$

and define: a number γ is the M-integral of the function f if for every $\varepsilon > 0$ and $C > 0$ there is a gauge δ such that $|\gamma - S(f, \Pi)| < \varepsilon$ holds for every δ-fine P-partition Π satisfying

$$\sum_{j=1}^{k} \sigma(I^j) \leqq C\,. \tag{12}$$

Since the condition (12) is evidently less restrictive then Mawhin's original condition, this definition admits a wider class of partitions and hence a narrower class of integrable functions. In [73] the properties of the new notion of integral were studied in detail. It turned out that it preserves those which had led Mawhin to the new definition (in particular, the divergence theorem or the integrability of every derivative). On the other hand, the integral has the additivity property in the above sense and, moreover, the limit theorems on monotone and dominated convergence hold. A drawback of both Mawhin's and Kurzweil's n-dimensional integral is that we can integrate only over intervals. The intervals are linked with the coordinate system and do not allow even relatively simple transformations.

[7]) *J. Mawhin:* Generalized Multiple Perron Integrals and the Green-Goursat Theorem for Differentiable Vector Fields, Czechoslovak Math. Journal *31* (*106*), (1981), 614—632.

Therefore, the next step of Kurzweil's research was to find a definition that would remove this drawback.

This was successfully achieved in [76]. Here, instead of the above mentioned type of partitions, partitions based on partition of unity were used. If f is a function with compact suport $\operatorname{supp} f$, then any finite system Δ of pairs (x^j, ϑ_j), $j = 1, 2, \ldots, k$, is called a PU-partition provided ϑ_j are functions of class $C^{(1)}$ with compact supports, $0 \leqq \vartheta_j(x) \leqq 1$, $\operatorname{Int}\{x \in \mathbb{R}^n;\ \sum_{j=1}^{k} \vartheta_j(x) = 1\} \supset \operatorname{supp} f$. Further, let us define $S(f, \Delta) = \sum_{j=1}^{k} f(x^j) \int \vartheta_j(x)\, dx$ and instead of the condition (12) let us introduce

$$(13) \qquad \sum_{j=1}^{k} f(x^j) \int \|x - x^j\| \sum_{i=1}^{n} \left|\frac{\partial \vartheta_j}{\partial x_i}(x)\right| dx \leqq C$$

(in both cases we actually integrate only over certain compact sets). If we replace the intervals I^j in the definition of δ-fineness of a partition by the sets $\operatorname{supp} \vartheta_j$, then we can introduce the definition of the PU-integral (PU for partition of unity) formally in the same way as that of the M-integral.

For the PU-integral the usual transformation theorem and also the Stokes theorem for differentiable functions (or forms on manifolds) hold without any additional assumptions. It is easy to see that among PU-integrable functions there are also some nonabsolutely integrable ones so that the PU-integral is a proper extension of the Lebesgue integral. It is not, however, a generalization of the Perron integral (though there exist PU-integrable functions which do not possess the Perron integral).

In a forthcoming paper it is shown that a suitable modification of the condition (13) leads to an integral for which Stokes' theorem can be proved for functions for which the differentiability condition (or even the condition of continuity or boundedness) is violated at some points.

The survey of Kurzweil's results given above represents a choice which is far from being complete. Nevertheless, Kurzweil's research activity does not at all cover his contribution to the development of Czechoslovak Mathematics.

Prof. Kurzweil has for many years been teaching at Charles University in Prague. At first he delivered special lectures for advanced students in which the students got acquainted with the domains of his own research. Since 1964 he has been systematically lecturing the standard course of ordinary differential equations. He created a modern curriculum of this course and prepared the corresponding lecture notes for students.

His teaching experience was a starting point also for his book [B4] devoted to the classical theory of ordinary differential equations. It is not only a detailed and rigorous textbook in which a complete account of the analytical fundaments of the theory is given, but it also has many features of a monograph, outlining some aspects of the modern theory of differential equations. As an example let us mention the original exposition of the differential relations, which is not to be found in current texts. The book carries the sign of Kurzweil's style consisting in rigorous elaboration of all details. It leads the reader to a thorough study, which in view of the character of the text cannot be superficial.

By rearranging and amending some parts of the book [B4], Kurzweil gave rise to its English version [B7]. For instance, the account of boundary value problems in [B7] is really remarkable.

Prof. Kurzweil has founded and led the regular Thursday Seminar in Ordinary Differential Equations in the Mathematical Institute of the Czechoslovak Academy of Sciences. It started in 1952 and is far from being restricted only to the subject of ordinary differential equations, which is a consequence of Kurzweil's extraordinary scope of interest in mathematics. The seminar has received numerous speakers from all parts of the world.

The work of the Department of Ordinary Differential Equations of the Mathematical Institute led by Kurzweil from 1955 till 1984 carries the impress of his scientific personality full of original ideas. The authors of these lines can declare from their own experience that to work with J. Kurzweil is gratifying and extraordinarily stimulative, and that many results of theirs would never come into existence without his help.

Prof. Kurzweil was chief editor of Časopis pro pěstování matematiky (Journal for Cultivation of Mathematics) from 1956 till 1970. In various offices he has taken part in both the preparation and fulfilment of the National projects of basic research. He has been member of the Scientific Board for Mathematics of the Czechoslovak Academy of Sciences, chairman or member of committees for scientific degrees etc.

The survey of Kurzweil's activity in mathematics would be incomplete without mentioning his deep interest in the problems of mathematical education in our schools. In this field he has been active both in the Institute and in the Union of Czechoslovak Mathematicians and Physicists. Here he has always supported approaches based on the employment of children's natural intellect, experience and skills. Being confident that it is necessary to educate children and young people in accordance with the present state of science, he is firmly convinced that abstract concepts and schemes which have significantly contributed to the development of mathematics as a branch of science lead the pupils in many cases to formal procedures which are irrational at least to the same extent as the old system of mathematical education. Prof. Kurzweil devoted much time and energy to these questions.

The scientific activity of Jaroslav Kurzweil has been lasting for about 35 years. During this period he has created admirable work of research that has notedly influenced Czechoslovak mathematics and enriched contemporary mathematical knowledge in an exceptionally broad part of its spectrum. He is a specialist acknowledged throughout the world, with friends (both mathematical and personal) in many countries.

All those who have met Prof. Jaroslav Kurzweil know him as a good and wise man who does not lack the sense of humour, who loves people with all their assets and drawbacks and they respect and love him in return.

We extend to Jaroslav Kurzweil our best wishes of firm health and success, so that for many years to come our mathematics may enjoy the favourable scientific milieu he creates around himself.

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