Czechoslovak Mathematical Journal

Masami Sekizawa

On complete lifts of reductive homogeneous spaces and generalized symmetric spaces

Czechoslovak Mathematical Journal, Vol. 36 (1986), No. 4, 516–534

Persistent URL: http://dml.cz/dmlcz/102113

Czechoslovak Mathematical Journal, 36 (111) 1986, Praha

ON COMPLETE LIFTS OF REDUCTIVE HOMOGENEOUS SPACES AND GENERALIZED SYMMETRIC SPACES

MASAMI SEKIZAWA, Tokyo

(Received March 27, 1985)

Let $M = K/H$ be a homogeneous manifold. Then the transitive action of the group K on M induces a transitive action of the tangent group TK on the tangent bundle TM. This lifting of action is compatible with the complete lifts of tensor fields and connections from M to TM (see K. Yano and S. Kobayashi [14]). So we can expect some nice liftings of other structures on homogeneous manifolds to tangent bundles.

This paper has two purposes. Firstly, we describe the group of transvections of the complete lift of an affine reductive space. Secondly, we construct a pseudo-Riemannian regular s-structure on the tangent bundle of a generalized symmetric pseudo-Riemannian space (in the sense of O. Kowalski [5]). The s-structure obtained coincides with that constructed by M. Toomanian in [11]. We use a simple method based on a lemma due to O. Kowalski, and we can avoid rather complicated calculus which was developed in [11]. Thus, we prove in a short way that the complete lift of a generalized symmetric space is a generalized symmetric space as well (Theorems 4.2 and 4.3). (In the previous paper [9], we proved the same in the special case of a simply connected manifold.)

Section 1 is a summary of results about lifting operations from manifolds to their tangent bundles. Section 2 gives basic information about actions of tangent groups on tangent bundles. In Section 3 we lift some basic structures on reductive homogeneous spaces to tangent bundles, such as the group of transvections. In Section 4 we find the lifts of regular s-structures from generalized symmetric pseudo-Riemannian spaces to their tangent bundles.

I would like to express my sincere gratitude to Professor O. Kowalski for suggesting these problems to me, for various helpful discussions and for his encouragement. I would like to thank also to the Department of Mathematical Analysis, Charles University in Prague which provided convenience for my research stay in Prague.

1. TANGENT BUNDLES

In this section we give a brief survey on prolongations of tensor fields and connections from a manifold to its tangent bundle. For more details, we refer to Yano-Ishihara [13] and Yano-Kobayashi [14].

Let M be a smooth manifold of dimension n. Let $\mathscr{X}(M)$ be the Lie algebra of all smooth vector fields on M and $\mathscr{I}(M)$ the tensor algebra of all smooth tensor fields on M. For any smooth mapping φ of M into a smooth manifold N, let φ_* denote the differential of φ, and φ^* its dual mapping.

Further, let M_x be the tangent space of M at a point x in M and $TM = \bigcup_{x \in M} M_x$ the tangent bundle over M with the canonical projection p.

First we define the vertical lifts from M to TM. For a function f on M, the function p^*f on TM induced by the projection p is denoted by f^v and is called the *vertical lift of the function* f from M to TM. Any 1-form ω on M may be regarded, in a natural way, as a function on TM. We denote this function by $\iota\omega$. The value of the function $\iota\omega$ at a point (x, X_x) in TM is $(\iota\omega)(x, X_x) = \omega_x(X_x)$, where X_x is a tangent vector of M at a point x in M. For any vector field Y on M, we define a vector field Y^v on TM by $Y^v(\iota\omega) = (\omega(Y))^v$ for all 1-forms ω on M. We call Y^v the *vertical lift of the vector field* Y from M to TM. For any function f on M, we denote by $\mathrm{d}f$ the differential of f. We define the *vertical lift of a* 1-*form* $\mathrm{d}f$ on M by $(\mathrm{d}f)^v = \mathrm{d}(f^v)$ for all functions f on M. Now, we define the vertical lift of an arbitrary 1-form ω on M. To do this, we recall the expression $\omega = \sum \omega_i \, \mathrm{d}x^i$ on the domain of a coordinate system $(U, x^1, x^2, \ldots, x^n)$. The *vertical lift* ω^v *of* ω is a 1-form given by $\omega^v = \sum (\omega_i)^v (\mathrm{d}x^i)^v$. We extend the vertical lifts defined above to a unique linear mapping of the tensor algebra $\mathscr{I}(M)$ to the tensor algebra $\mathscr{I}(TM)$ by the rule $(T \otimes S)^v = T^v \otimes S^v$ for all $T, S \in \mathscr{I}(M)$.

Next we define the complete lifts from M to TM. For a function f on M we put $f^c = \iota\, \mathrm{d}f$ and call the function f^c on TM the *complete lift of the function* f from M to TM. For a vector field Y on M we define a vector field Y^c on TM by $Y^c f^c = (Yf)^c$ for all functions f on M. We call Y^c the *complete lift of the vector field* Y from M to TM. Given a 1-form ω on M we define a 1-form ω^c on TM by $\omega^c(Y^c) = (\omega(Y))^c$ for all vector fields Y on M. We call ω^c the *complete lift of the* 1-*form* ω from M to TM. We extend the complete lifts defined above to a unique linear mapping of the tensor algebra $\mathscr{I}(M)$ to the tensor algebra $\mathscr{I}(TM)$ by the rule $(T \otimes S)^c = T^c \otimes S^v + T^v \otimes S^c$ for all $T, S \in \mathscr{I}(M)$.

Now, let ∇ be an affine connection on M. Then there exists a unique affine connection ∇^c on TM which satisfies

$$\nabla^c_{Y^c} Z^c = (\nabla_Y Z)^c$$

for all vector fields Y, Z on M. We call the connection ∇^c the *complete lift of the connection* ∇ from M to TM.

We restrict ourselves to those properties of lifting operations which will be used later. The following lemmas 1.A – 1.E are due to Yano and Kobayashi [14].

Lemma 1.A. *For any vector field Y on M and any tensor field T on M, the Lie derivative $\mathscr{L}$ satisfies*

$$\mathscr{L}_{Y^c}T^c = (\mathscr{L}_Y T)^c\,, \quad \mathscr{L}_{Y^c}T^v = (\mathscr{L}_Y T)^v\,,$$
$$\mathscr{L}_{Y^v}T^c = (\mathscr{L}_Y T)^v\,, \quad \mathscr{L}_{Y^v}T^v = 0\,.$$

In particular, if $T = Z$ is a vector field, then

$$[Y^c, Z^c] = [Y, Z]^c\,, \quad [Y^c, Z^v] = [Y, Z]^v\,,$$
$$[Y^v, Z^c] = [Y, Z]^v\,, \quad [Y^v, Z^v] = 0\,.$$

Lemma 1.B. *Let g be a tensor field of type $(0,2)$ on M. Then*

$$g^c(Y^c, Z^c) = (g(Y, Z))^c\,, \quad g^c(Y^c, Z^v) = (g(Y, Z))^v\,,$$
$$g^c(Y^v, Z^c) = (g(Y, Z))^v\,, \quad g^c(Y^v, Z^v) = 0\,.$$

Furthermore, if g is a pseudo-Riemannian metric on M, then g^c is a pseudo-Riemannian metric on TM (with n positive and n negative signs).

Lemma 1.C. *If ∇ is the Riemannian connection of M with respect to a pseudo-Riemannian metric g, then ∇^c is the Riemannian connection of TM with respect to the pseudo-Riemannian metric g^c.*

Lemma 1.D. *Let ∇ be an affine connection on M. If Y is an infinitesimal affine transformation of M, then both Y^c and Y^v are infinitesimal affine transformations of TM with respect to ∇^c.*

Lemma 1.E. *For any vector field Y on M and any tensor field T on M, we have*

$$\nabla^c_{Y^c}T^c = (\nabla_Y T)^c\,, \quad \nabla^c_{Y^c}T^v = (\nabla_Y T)^v\,,$$
$$\nabla^c_{Y^v}T^c = (\nabla_Y T)^v\,, \quad \nabla^c_{Y^v}T^v = 0\,.$$

Finally, the following result will be useful in the sequel:

Lemma 1.1. *Let x be a fixed point in M, and let Y be a vector field on M such that $Y_x = 0$. Then we have*

$$Y^c_{(x,0)} = 0 \quad \textit{and}$$
$$Y^v_{x'} = 0 \quad \textit{for any point } x' \textit{ such that } \; p(x') = x\,.$$

Proof. By the definition of the complete lifts, we have

$$Y^c_{(x,0)}f^c = (Yf)^c_{(x,0)} = (\iota\, \mathrm{d}(Yf))\,(x, 0) = 0 \quad \text{and}$$
$$(p_* Y^c_{(x,0)})\,f = Y^c_{(x,0)}(f^v) = (Yf)^v\,(x, 0) = Y_x f = 0$$

for all functions f on M. Hence we see $Y^c_{(x,0)} = 0$.

Since $Y^v_{(x,X_x)}(\iota\omega) = \omega_x(Y_x)$ for all 1-forms ω on M and for all tangent vectors X_x on M at $x \in M$, we have $Y^v_{x'} = 0$ for all points x' such that $p(x') = x$.

Remark 1. We notice that Y^c need not be zero at a general point $x' \in TM$ even if $Y_{p(x')} = 0$.

2. TANGENT GROUPS

Let K be a Lie group and $\varphi: K \times K \to K$ the mapping of the group multiplication $\varphi(a, b) = a \,.\, b$ for all $a, b \in K$. Let $\mathfrak{k}$ be the Lie algebra of K.

Every element of the tangent bundle TK over K is a couple (a, A_a) where $a \in K$ and A_a is a tangent vector of K at a. Here A_a belongs to a unique left invariant vector field $A \in \mathfrak{k}$. Hence we have a bijective correspondence between TK and the semi-direct product $K \,.\, \mathfrak{k}$, and we shall use both symbols (a, A_a) and (a, A) to denote an element of the tangent bundle TK.

Let L_a and R_a, $a \in K$, denote the left and right translations in K, respectively, C_a the conjugation mapping on K, i.e., $C_a = L_a \circ R_{a^{-1}}$, and $a \mapsto \mathrm{ad}(a)$ the adjoint representation of K on $\mathfrak{k}$. We define a group multiplication $T\varphi$ on the tangent bundle TK of K by

$$(T\varphi)\left((a, A_a), (b, B_b)\right) = \left(a \,.\, b, (L_a)_* B_b + (R_b)_* A_a\right)$$

for all (a, A_a), $(b, B_b) \in TK$, or equivalently

$$(T\varphi)\left((a, A), (b, B)\right) = \left(a \,.\, b, B + \mathrm{ad}(b^{-1}) A\right) \tag{1}$$

for all (a, A), $(b, B) \in K \,.\, \mathfrak{k}$. Here we identified $T(K \times K)$ with $TK \times TK$. The group TK, obtained in this way, is called the *tangent group* to K. The group K, imbedded in TK as the zero-section, is a closed subgroup of TK and the vector space $\mathfrak{k}$ is a closed normal subgroup of TK. Finally, $(e, 0)$ is the unit element of TK.

Suppose that K is a Lie group acting on a manifold M by the law $\phi: K \times M \to M$. For every $a \in K$ and every $x \in M$, we often denote $\phi(a, x)$ by ax. For each element a of K, we define a transformation ϕ_a on M by $\phi_a(x) = \phi(a, x) = ax$ for all $x \in M$. We denote the group $\{\phi_a \mid a \in K\}$ of transformations on M by K^*. Then, if the action of K on M is effective, K^* is isomorphic to K. Furthermore, the Lie algebra of K^*, say $\mathfrak{k}^*$, is given by

$$\mathfrak{k}^* = \{A^* \mid A \in \mathfrak{k}\}$$

where A^* is the fundamental vector field corresponding to A, that is, for any $x \in M$,

$$A^*_x = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\exp tA)(x) \,.$$

Next, we define the "tangent action" $T\phi$ of TK on TM by

$$(T\phi)\left((a, A_a), (x, X_x)\right) = \left(ax, a_* X_x + (\psi_x)_* A_a\right)$$

for all $(a, A_a) \in TK$ and for all $(x, X_x) \in TM$, where ψ_x, $x \in M$, is a mapping of K to M defined by

$$\psi_x(a) = \phi(a, x) = ax$$

for all $a \in K$. Since $\psi_{ax} = \psi_x \circ R_a$ and $A_a = (L_a)_* A_e$, we get $(\psi_x)_* A_a = (\psi_{ax})_* \operatorname{ad}(a) A$ and hence

$$(T\phi)((a, A), (x, X_x)) = (ax, a_* X_* + (\psi_{ax})_* \operatorname{ad}(a) A) \tag{2}$$

for all $(a, A) \in K \,.\, \mathfrak{k}$ and for all $(x, X_x) \in TM$.

The following lemma is a special case of a result from [13]. Here we shall give a different proof.

Lemma 2.A. *Let K be a Lie group acting on a manifold M. Then for any element $a \in K$ and any vector field Y on M we have*

$$(a, 0)_* Y^c = (a_* Y)^c, \quad (a, 0)_* Y^v = (a_* Y)^v .$$

Proof. Let (x, X_x) be any point of TM.

We have

$$((a, 0)_* Y^c_{(x,X_x)}) f^c = Y^c_{(x,X_x)}((a, 0)^* f^c) = X_x(Ya^*f) =$$
$$= X_x((a_* Y)(a^* f)) = (a_* X_x)((a_* Y) f) = (a_* Y)^c_{(a,0)(x,X_x)} f^c$$

for all real-valued smooth functions f on M. Hence the assertion is proved for the complete lift.

Next we prove it for the vertical lift. First note that $(a, 0)^* \iota\omega = \iota a^* \omega$ holds for all 1-forms ω on M. In fact, we have

$$((a, 0)^* \iota\omega)(x, X_x) = \iota\omega(ax, a_* X_x) = \omega_{ax}(a_* X_x) =$$
$$= (a^* \omega_{ax})(X_x) = (\iota a^* \omega)(x, X_x) .$$

Thus,

$$((a, 0)_* Y^v_{(x,X_x)})(\iota\omega) = Y^v_{(x,X_x)}((a, 0)^* \iota\omega) =$$
$$= Y^v_{(x,X_x)}(\iota a^* \omega) = ((a^* \omega)(Y))^v (x, X_x) = (a^* \omega)_x (Y_x) =$$
$$= \omega_{ax}(a_* Y_x) = (\omega(a_* Y))(ax, a_* X_x) = (a_* Y)^v_{(a,0)(x,X_x)} (\iota\omega)$$

for all 1-forms ω on M.

This completes the proof.

The following proposition is due to S. Kobayashi [3].

Proposition 2.B. *If a Lie group K acts as a Lie trasformation group on a manifold M, then the tangent group TK acts as a Lie transformation group on TM by means of* (2). *If K is effective on M, then TK is effective on TM.*

As noticed in Yano-Kobayashi [14, II, Remark 4 in p. 237], the Lie algebra of TK is

$$\operatorname{Lie}(TK) = \{A^c + B^v \mid A, B \in \mathfrak{k}\} ,$$

where A^c and B^v denote the complete lift of A and the vertical lift of B, respectively, from K to TK. Hence, if K acts effectively on M, we have

$$\mathrm{Lie}\left((TK)^*\right) = \{A^{c*} + B^{v*} \mid A, B \in \mathfrak{k}\} .$$

Now we discuss a relationship between two kinds of lifting operations, one from K to TK and another from M to TM.

For any vector field Y on M, we denote by $\exp tY$, $t \in I \subset \mathbb{R}$, the 1-parameter group of (local) transformations generated by Y. Since we have

$$(\exp tY, 0)(x, X_x) = ((\exp tY)(x), (\exp tY)_* X_x)$$

for all $(x, X_x) \in TM$, $(\exp tY, 0)$, $t \in I \subset \mathbb{R}$, is a 1-parameter group of (local) transformations on TM.

Lemma 2.C ([14]). *For any vector field Y on M, we have*

$$(\exp tY, 0) = \exp tY^c$$

for all $t \in I \subset \mathbb{R}$.

Lemma 2.1. *For any element $A \in \mathfrak{k}$, we have*

$$A^{*c} = A^{c*} .$$

Proof. Let (x, X_x) be any point in TM. Using Lemma 2.C and the fact that $(\exp tA^*)(x) = (\exp tA)(x)$ for all $x \in M$, we have

$$\begin{aligned}(\exp tA^{*c})(x, X_x) &= (\exp tA^*, 0)(x, X_x) = ((\exp tA)(x), (\exp tA)_* X_x) = \\ &= (\exp tA, 0)(x, X_x) = (\exp tA^c)(x, X_x) = (\exp tA^{c*})(x, X_x) .\end{aligned}$$

Hence, differentiating both sides at $t = 0$, we get $A^{*c} = A^{c*}$.

Next we discuss the case of the vertical lifts.

Lemma 2.2. *For any element $A \in \mathfrak{k}$, we have*

$$\exp tA^v = (e, tA)$$

for all $t \in \mathbb{R}$, where e is the unit element of K.

Proof. By Formula (1) we have

$$(e, tA)(e, sA) = (e, sA + \mathrm{ad}(e^{-1})\, tA) = (q, (t + s) A)$$

for all $t, s \in \mathbb{R}$, where the dot on the left hand side stands for the multiplication in TK. This implies that $\{(e, tA) \mid t \in \mathbb{R}\}$ is a one parameter subgroup in TK. But, since we have $(\mathrm{d}/\mathrm{d}t)|_{t=0}\,(e, tA) = A^v$, we see that $\exp tA^v = (e, tA)$.

Lemma 2.3. *For any element $A \in \mathfrak{k}$, we have*

$$A^{*v} = A^{v*} .$$

Proof. Using Lemma 2.2 and Formula (2) we get

$$(\exp tA^v)(x, X_x) = (x, X_x + t(\phi_x)_* A)$$

for all (x, X_x) in TM. But, since $(\phi_x)_* A = A_x^*$, we get

$$(\exp tA^v)(x, X_x) = (x, X_x + tA_x^*) .$$

Hence

$$(A^{v*})_{(x,X_x)} = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} (x, X_x + tA_x^*) = (A^{*v})_{(x,X_x)} .$$

Remark 2. Lemmas 2.1 and 2.3 assert that the following diagram is commutative:

$$\begin{array}{ccc} T\mathfrak{k} & \xrightarrow{*} & \mathscr{X}(TM) \\ \uparrow{\scriptstyle c,\,v} & & \uparrow{\scriptstyle c,\,v} \\ \mathfrak{k} & \xrightarrow{*} & \mathscr{X}(M) \end{array}$$

where $T\mathfrak{k}$ denotes the Lie algebra of TK.

A vector field is *complete* if it generates a global 1-parameter group of transformations. The following lemma is a consequence of Lemmas 2.C and 2.2 (see also [14, II p. 205]).

Lemma 2.D. *If a vector field Y on M is complete, then its complete lift Y^c is a complete vector field on TM. The vertical lift Y^v of any vector field Y is always complete.*

Now we prove two propositions which will play a basic role later.

Proposition 2.4. *Let K be a Lie group acting transitively on M. Then the tangent group TK acts transitively on the tangent bundle TM.*

Proof. Let $(x, X_x), (y, Y_y)$ be two points in TM. Since K acts transitively on M, there exists an element $a \in K$ such that $y = ax$. The mapping ψ_y of K to M defined by $\psi_y(b) = by$ $(b \in K)$ is surjective, and hence the linear mapping $(\psi_y)_*$ of $\mathfrak{k}$ to the tangent space M_y at y is also surjective. This together with the surjectivity of $\mathrm{ad}(a)$ on $\mathfrak{k}$ implies that there exists $A \in \mathfrak{k}$ such that

$$Y_y - a_* X_x = (\psi_y)_* \,\mathrm{ad}(a)\, A .$$

Thus we see that there exists $(a, A) \in TK$ such that

$$(a, A)(x, X_x) = (y, Y_y) .$$

This completes the proof.

Proposition 2.5. *Let K be a connected Lie group acting effectively on a manifold M. If K preserves a tensor field T on M, then TK preserves the complete lift T^c and the vertical lift T^v of T. Also, if K preserves an affine connection ∇ on M, then TK preserves the complete lift ∇^c of ∇.*

Proof. First we note that, since K acts effectively on M, we have, from Lemma 2.1 and 2.3,

(3) $$\mathrm{Lie}((TK)^*) = \{A^{*c} + B^{*v} \mid A, B \in \mathfrak{k}\} .$$

Next, since we have $\mathscr{L}_{A^*}T = 0$ for all $A \in \mathfrak{k}$ by the assumption, Lemma 1.A yields

$$\begin{aligned} (4) \qquad & \mathscr{L}_{A^{*c}}T^c = 0\,, \quad \mathscr{L}_{A^{*v}}T^c = 0\,, \\ & \mathscr{L}_{A^{*c}}T^v = 0\,, \quad \mathscr{L}_{A^{*v}}T^v = 0\,. \end{aligned}$$

From (3) and (4) we see that TK preserves T^c and T^v because TK is connected.

If A^* is an infinitesimal affine transformation of (M, ∇), then, by Lemma 1.D, both A^{*c} and A^{*v} are infinitesimal affine transformations of (TM, ∇^c). A similar argument as above shows that TK preserves ∇^c if K preserves ∇.

This completes the proof.

Remark 3. Mr. Y. Ogawa (Ochanomizu University in Tokyo) was so kind as to send me a different proof of Proposition 2.5 based on calculations with local coordinates.

Finally, we prove a result which will be used in Section 4.

Proposition 2.6. *Let K be a Lie group acting effectively on a manifold M. If $c \in K$ commutes with any element of the isotropy subgroup H of K at $o \in M$, then $(c, 0) \in TK$ commutes with any element of the tangent group TH.*

Proof. Let (a, A) be any element of TH. By (2) we have

$$(5) \qquad (c, 0)(a, A)(x, X_x) = (cax, (ca)_* X_x + c_*(\psi_{ax})_* \operatorname{ad}(a) A)$$

for all $(x, X_x) \in TM$. Now we claim that

$$c \circ \psi_{ax} \circ C_a = \psi_{acx} \circ C_a$$

holds on H. In fact, using the assumption we have

$$c \circ \psi_{ax} \circ C_a(b) = caba^{-1}ax = abcx = aba^{-1}acx = \psi_{acx}\, C_a(b)$$

for all $b \in H$. Hence we get

$$c_*(\psi_{ax})_* \operatorname{ad}(a) A = (\psi_{acx})_* \operatorname{ad}(a) A$$

for all $A \in \mathfrak{h}$. From this and (5), we have

$$\begin{aligned} (c, 0)(a, A)(x, X_x) &= (acx, (ac)_* X_x + (\psi_{acx})_* \operatorname{ad}(a) A) = \\ &= (a, A)(cx, c_* X_x) = (a, A)(c, 0)(x, X_x) \end{aligned}$$

for all $(x, X_x) \in TM$. Since TK acts effectively on TM (see Proposition 2.B), we get $(c, 0)(a, A) = (a, A)(c, 0)$ for all $(a, A) \in TH$.

3. TANGENT BUNDLES OVER REDUCTIVE HOMOGENEOUS SPACES

In this section, we first prove that the tangent bundles over reductive homogeneous spaces are also reductive homogeneous spaces. We prove the same for the affine reductive spaces in the sense of [5]. Then we show that the tangent group to the group of transvections of an affine reductive space (M, ∇) is the group of transvections of the affine reductive space (TM, ∇^c).

First of all we shall recall some elementary properties of the reductive homogeneous spaces. We refer, for example, to Kobayashi-Nomizu [4, II] and Kowalski [5].

Let K be a connected Lie group and H its closed subgroup. Consider the homogeneous manifold K/H. Let $k \supset \mathfrak{h}$ be the Lie algebras of K and H, respectively. Suppose that there is a subspace $\mathfrak{m} \subset \mathfrak{k}$ such that $\mathfrak{k} = \mathfrak{m} + \mathfrak{h}$ (direct sum of vector spaces) and $\operatorname{ad}(a)\,\mathfrak{m} = \mathfrak{m}$ for all $a \in H$. Then the homogeneous space K/H is said to be *reductive* with respect to the decomposition $\mathfrak{k} = \mathfrak{m} + \mathfrak{h}$. The canonical connection of the reductive homogeneous space is characterized as follows ([4, II], [5, Proposition I.10]):

Theorem 3.A. *The canonical connection of a reductive homogeneous space $M = K/H$ is the unique K-invariant affine connection on M such that, for every $U \in \mathfrak{m}$ and every vector field Y on M, we have*

$$(\nabla_{U^*}Y)_o = [U^*, Y]_o$$

where o is the origin of K/H and U^* is the fundamental vector field on M corresponding to U.

Now, we shall study a lift of a reductive homogeneous space $M = K/H$ to its tangent bundle.

Proposition 3.1. *Let $M = K/H$ be a reductive homogeneous space with respect to a decomposition $\mathfrak{k} = \mathfrak{m} + \mathfrak{h}$. Let o' be the point in the zero-section of the tangent bundle TM such that $p(o') = o = \pi(H)$ where $\pi: K \to M$ and $p: TM \to M$ denote the canonical projections. Then the isotropy subgroup of the tangent group TK at $o' = (o, 0)$ is the tangent group TH to the isotropy subgroup H of K at o.*

Proof. We denote by $TK(o')$ the isotropy subgroup of TK at $o' \in TM$. Let ψ_x, $x \in M$, be a mapping of K to M defined by $\psi_x(a) = ax$ for all $a \in K$. We notice that ψ_0 is just the canonical projection π of K to M.

Let (a, A) be any element of $TH \cong H \,.\, \mathfrak{h}$. Then, since $\operatorname{ad}(a)\,A \in \mathfrak{h}$, we have $(\psi_0)_* \operatorname{ad}(a)\,A = 0$. Hence we see that

$$(a, A)\,(o, 0) = (ao, a_*0 + (\psi_0)_* \operatorname{ad}(a)\,A) = (o, 0)\,. \tag{6}$$

This implies $TH \subset TK(o')$.

Next we prove the converse inclusion. For any $(a, A) \in TK(o')$ we get, according to (6),

$$ao = o\,, \quad (\psi_0)_* \operatorname{ad}(a)\,A = 0\,. \tag{7}$$

The first relation implies $a \in H$. Now let $A = A_1 + A_2$ ($A_1 \in \mathfrak{m}$, $A_2 \in \mathfrak{h}$) be the decomposition of A with respect to the direct sum $\mathfrak{k} = \mathfrak{m} + \mathfrak{h}$. Then (7) implies $(\psi_0)_* \operatorname{ad}(a)\,A_1 = 0$. But, since $(\psi_0)_*|_{\mathfrak{m}} = \pi_*|_{\mathfrak{m}}$ is an isomorphism of $\mathfrak{m}$ onto M_0, we obtain $\operatorname{ad}(a)\,A_1 = 0$ and hence $A_1 = 0$, which implies $A = A_2 \in \mathfrak{h}$. Thus $TK(o') \subset TH$, q.e.d.

Now, let $T\mathfrak{k}$ and $T\mathfrak{h}$ be the Lie alebras of the tangent groups TK and TH, respectively. Put

$$\mathfrak{m}^c = \{U^c | \, U \in \mathfrak{m}\}, \quad \mathfrak{m}^v = \{U^v | \, U \in \mathfrak{m}\}.$$

$\mathfrak{m}^c$ and $\mathfrak{m}^v$ are vector subspaces of $T\mathfrak{k}$. Further, put $T\mathfrak{m} = \mathfrak{m}^c + \mathfrak{m}^v$ (sum of vector spaces). Then we have

Theorem 3.2. *Let $M = K/H$ be a reductive homogeneous space with respect to the decomposition $\mathfrak{k} = \mathfrak{m} + \mathfrak{h}$. Then the tangent bundle TM over M is a reductive homogeneous space TK/TH with respect to the decomposition $T\mathfrak{k} = T\mathfrak{m} + T\mathfrak{h}$.*

Proof. First, according to Proposition 3.1, TM can be identified with the homogeneous space TK/TH. We must only prove the reductivity. Let (a, A) be any element of TK and U any element of $\mathfrak{k}$.

Using (1) and Lemma 2.C in the case of $M = K$, we have, by routine calculation,

$$\begin{aligned} C_{(a,A)}(\exp tU^c) &= (a, A)(\exp tU, 0)(a^{-1}, -\mathrm{ad}(a) A) = \\ &= (C_a(\exp tU), \mathrm{ad}(a \, . \exp(-tU)) A - \mathrm{ad}(a) A) = \\ &= (\exp t(\mathrm{ad}(a) U), 0)(e, \mathrm{ad}(a \exp(-tU)) A - \mathrm{ad}(a) A). \end{aligned}$$

Differentiating both sides of this equation at $t = 0$ and using Lemma 2.C and Lemma 2.2, we get

$$(\mathrm{ad}(a, A))(U^c) = (\mathrm{ad}(a) U)^c + (\mathrm{ad}(a)[A, U])^v.$$

Now taking $(a, A) \in TH$, $U \in \mathfrak{m}$, and using the reductivity of the decomposition $\mathfrak{k} = \mathfrak{m} + \mathfrak{h}$, we get $\mathrm{ad}(TH)\,\mathfrak{m}^c \subset T\mathfrak{m}$.

Also using (1) and Lemma 2.2, we have

$$\begin{aligned} C_{(a,A)}(\exp tU^v) &= (a, A)(e, tU)(a^{-1}, -\mathrm{ad}(a) A) = \\ &= (e, t\,\mathrm{ad}(a) U) = \exp t((\mathrm{ad}(a) U)^v). \end{aligned}$$

Differentiating both sides of this equation at $t = 0$, we get

$$(\mathrm{ad}(a, A))(U^v) = (\mathrm{ad}(a) U)^v.$$

Specializing again $(a, A) \in TH$, $U \in \mathfrak{m}$, we get $\mathrm{ad}(TH)\,\mathfrak{m}^v \subset \mathfrak{m}^v \subset T\mathfrak{m}$.

Thus $\mathrm{ad}(TH)\, T\mathfrak{m} \subset T\mathfrak{m}$, q.e.d.

The following theorem describes the canonical connection of the lifted reductive homogeneous space.

Theorem 3.3. *Let ∇ be the canonical connection of a reductive homogeneous space $M = K/H$ with respect to a decomposition $\mathfrak{k} = \mathfrak{m} + \mathfrak{h}$. Then the complete lift ∇^c of ∇ is the canonical connection of the reductive homogeneous space $TM = TK/TH$ with respect to the decomposition $T\mathfrak{k} = T\mathfrak{m} + T\mathfrak{h}$.*

Proof. Let $o = \pi(H)$ be the origin of K/H. Then the point $o' = (o, 0)$ is the origin of TK/TH. Further, let U be any element in $\mathfrak{m}$, and let Y be any vector field on M. Then, by Theorem 3.A, we see that ∇ is K-invariant and $(\nabla_{U^*}Y)_0 = [U^*, Y]_0$ holds.

Now, from Proposition 2.5 we see that ∇^c is TK-invariant. Also, by Lemmas 1.B, 1.E and Lemmas 1.1, 2.1, 2.3, we get

$$(\nabla^c_{U^{c*}} Y^c)_{o'} = [U^{c*}, Y^c]_{o'}, \quad (\nabla^c_{U^{c*}} Y^v)_{o'} = [U^{c*}, Y^v]_{o'},$$
$$(\nabla^c_{U^{v*}} Y^c)_{o'} = [U^{v*}, Y^c]_{o'}, \quad (\nabla^c_{U^{v*}} Y^v)_{o'} = [U^{v*}, Y^v]_{o'}.$$

For example, the first equation is shown as follows:

$$(\nabla^c_{U^{c*}} Y^c)_{o'} = (\nabla^c_{U^{*c}} Y^c)_{o'} = (\nabla_{U^*} Y)^c_{o'} = [U^*, Y]^c_{o'} = [U^{*c}, Y^c]_{o'} = [U^{c*}, Y^c]_{o'}.$$

The other identities are proved similarly. Hence, using Theorem 3.A once again, we see that ∇^c is the canonical connection of TK/TH with respect to the decomposition $T\mathfrak{k} = T\mathfrak{m} + T\mathfrak{h}$.

We prove another result for later use:

Proposition 3.4. *Let K/H be a reductive homogeneous space with respect to a decomposition $\mathfrak{k} = \mathfrak{m} + \mathfrak{h}$. If $\mathfrak{k} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ holds, then the Lie algebra $T\mathfrak{k} = T\mathfrak{m} + T\mathfrak{h}$ of the tangent group TK is equal to $T\mathfrak{m} + [T\mathfrak{m}, T\mathfrak{m}]$.*

Proof. For any element $A \in \mathfrak{k}$ we can write

$$A = U + \sum [V_i, W_i], \quad U, V_i, W_i \in \mathfrak{m}.$$

By the linearity of the lifting operations and Lemma 1.B, we have

$$A^c = U^c + \sum [V_i^c, W_i^c],$$
$$A^v = U^v + \sum [V_i^c, W_i^v].$$

Now, since $T\mathfrak{k} = \{A^c + B^v \mid A, B \in \mathfrak{k}\}$ and $T\mathfrak{m} = \mathfrak{m}^c + \mathfrak{m}^v$, we have $T\mathfrak{k} \subset T\mathfrak{m} + [T\mathfrak{m}, T\mathfrak{m}]$. Hence $T\mathfrak{k} = T\mathfrak{m} + [T\mathfrak{m}, T\mathfrak{m}]$, q.e.d.

Next, we shall deal with affine reductive spaces, which have been defined by O. Kowalski [5]. Let (M, ∇) be a connnected manifold with an affine connection.The group of all affine transformations of M preserving each holonomy subbundle of the frame bundle $\mathscr{F}(M)$ is called the *group of transvections* of (M, ∇). It will be denoted by $\mathrm{Tr}(M, \nabla)$. (M, ∇) is called an *affine reductive space* if the group $\mathrm{Tr}(M, \nabla)$ acts transitively on each holonomy subbundle. The following theorem was proved in [5, pp. 37–40]:

Theorem 3.B. *Let (M, ∇) be a connected manifold with an affine connection. Then the following two conditions are equivalent:*

(i) *(M, ∇) is an affine reductive space;*

(ii) *M can be expressed as a reductive homogeneous space K/H with respect to a decomposition $\mathfrak{k} = \mathfrak{m} + \mathfrak{h}$, where K is effective on M, and ∇ is the canonical connection of K/H.*

More specifically, if (i) *is satisfied, then* $\mathrm{Tr}(M, \nabla)$ *is a connected Lie group and M can be expressed in the form* (ii) *with $K =$* $\mathrm{Tr}(M, \nabla)$. *For every expression of M in the form* (ii), $\mathrm{Tr}(M, \nabla)$ *is a normal Lie subgroup of K and its Lie algebra is isomorphic to the ideal $\mathfrak{l} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ of $\mathfrak{k}$.*

We shall now give the main theorem of this section.

Theorem 3.5. *Let (M, ∇) be an affine reductive space. Then (TM, ∇^c) is also an affine reductive space. Further, the group of transvections of (TM, ∇^c) is tangent to the group of transvections of (M, ∇), that is,* $\mathrm{Tr}(TM, \nabla^c) = T(\mathrm{Tr}(M, \nabla))$ *holds.*

Proof. According to Theorem 3.B, M can be expressed as $M = K/H$ where $K = \mathrm{Tr}(M, \nabla)$ and H is the isotropy subgroup of K at a fixed point $o \in M$. Also, we have a reductive decomposition $\mathfrak{k} = \mathfrak{m} + \mathfrak{h}$, where $\mathfrak{k} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$. By Proposition 3.4, $T\mathfrak{k} = T\mathfrak{m} + [T\mathfrak{m}, T\mathfrak{m}]$.

On the other hand, since the complete lift ∇^c of ∇ is the canonical connection of $TM = TK/TH$ (Theorem 3.3), we see from Theorem 3.B that the group of transvections of (TM, ∇^c) is a Lie subgroup of TK, and its Lie algebra is isomorphic to $T\mathfrak{m} + [T\mathfrak{m}, T\mathfrak{m}]$.

Because $T\mathfrak{m} + [T\mathfrak{m}, T\mathfrak{m}] = T\mathfrak{k}$, we get $\mathrm{Tr}(TM, \nabla^c) = TK$, i.e.,

$$T(\mathrm{Tr}(M, \nabla)) = \mathrm{Tr}(TM, \nabla^c).$$

This completes the proof of Theorem 3.5.

Remark 4. For an affine reductive space (M, ∇), Theorem 3.B says that $\mathrm{Tr}(M, \nabla)$ is a connected Lie group. But even for a general connected affine manifold (M, ∇), $\mathrm{Tr}(M, \nabla)$ is still a Lie group. I am obliged to Professor S. Kobayashi (University of California) and Dr. J. Grabowski (University of Warsaw) for communicating to me two different proofs of this fact. It remains an open problem whether the equality $T(\mathrm{Tr}(M, \nabla)) = \mathrm{Tr}(TM, \nabla^c)$ is still valid in the general case.

In the rest of this section we shall study the tangent lifts of naturally reductive pseudo-Riemannian homogeneous spaces.

A homogeneous space $M = K/H$ with a K-invariant pseudo-Riemannian metric g is said to be *naturally reductive* if it admits a reductive decomposition $\mathfrak{k} = \mathfrak{m} + \mathfrak{h}$ satisfying the condition

$$B([U, V]_{\mathfrak{m}}, W) = B(U, [V, W]_{\mathfrak{m}}) \tag{8}$$

for all $U, V, W \in \mathfrak{m}$, where $[U, V]_{\mathfrak{m}}$ denotes the projection of $[U, V] \in \mathfrak{k}$ on $\mathfrak{m}$, and B denotes the inner product on $\mathfrak{m}$ induced by the metric g via the canonical identification $\pi_*: \mathfrak{m} \to M_0$, $o = \pi(H) \in M$.

Theorem 3.6. *Let $M = K/H$ be a naturally reductive homogeneous space with a K-invariant metric g. Then the tangent bundle $TM = TK/TH$ is a naturally reductive homogeneous space with the TK-invariant metric g^c.*

Proof. First, (8) is equivalent to

$$g(([U, V]_{\mathfrak{m}})^*, W^*) = g(U^*, ([V, W]_{\mathfrak{m}})^*) \tag{9}$$

for all $U, V \in \mathfrak{m}$, where we denote by stars the corresponding fundamental vector fields on M. Next, the tangent bundle TM is a reductive homogeneous space TK/TH with respect to the decomposition $T\mathfrak{k} = T\mathfrak{m} + T\mathfrak{h}$ (Theorem 3.3).

From Lemma 1.A, we have

$$[U^c, V^c] = ([U, V]_{\mathfrak{m}})^c + ([U, V]_{\mathfrak{h}})^c$$

for all $U, V \in \mathfrak{m}$. Because $T\mathfrak{m} = \mathfrak{m}^c + \mathfrak{m}^v$, we get

$$(10) \qquad [U^c, V^c]_{T\mathfrak{m}} = [(U, V]_{\mathfrak{m}})^c$$

for all $U, V \in \mathfrak{m}$. Thus, by Lemmas 2.1, 2.3, Lemma 1.C and (9), (10), we get on TM:

$$g^c(([U^c, V^c]_{T\mathfrak{m}})^*, W^{c*}) = g^c(([U, V]_{\mathfrak{m}})^{c*}, W^{c*}) =$$
$$= g^c(([U, V]_{\mathfrak{m}})^{*c}, W^{*c}) = (g(([U, V]_{\mathfrak{m}})^*, W^*))^c =$$
$$= (g(U^*, ([V, W]_{\mathfrak{m}})^*))^c = g^c(U^{c*}, ([V^c, W^c]_{T\mathfrak{m}})^*)$$

for all $U, V, W \in \mathfrak{m}$. We get similar formulas for various combinations of the complete and vertical lifts of the vector fields U, V, W.

This implies that TK/TH is naturally reductive.

4. TANGENT BUNDLES OVER REGULAR s-MANIFOLDS

Let (M, g) be a smooth pseudo-Riemannian manifold. An *s-structure* on (M, g) is a family $\{s_x | x \in M\}$ of isometries of (M, g) (called *symmetries*) such that each s_x has the point x as an isolated fixed point. An s-structure $\{s_x\}$ on (M, g) is said to be *regular* if

(i) the mapping $(x, y) \mapsto s_x(y)$ of $M \times M$ into M is smooth,

(ii) for every pair of points $x, y \in M$ we have $s_x \circ s_y = s_z \circ s_x$, where $z = s_x(y)$.

If we define the tangent tensor field S of type (1,1) of $\{s_x\}$ by $S_x = (s_x)_{*x}$ for each $x \in M$, we can see that $\{s_x\}$ is regular if and only if the tensor field S is smooth and invariant with respect to all symmetries s_x, $x \in M$. An s-structure $\{s_x\}$ is said to be of *order* k ($k \geqq 2$) if k is the least integer such that $(s_x)^k$ is the identity mapping of M for all $x \in M$. We say that an s-structure is of *infinite order* if such k does not exist.

A *generalized symmetric pseudo-Riemannian space* is a connected pseudo-Riemannian manifold (M, g) admitting at least one regular s-structure. Every generalized symmetric pseudo-Riemannian space is a homogeneous pseudo-Riemmanian manifold. Let (M, g) be a generalized pseudo-Riemannian space and $\{s_x\}$ a fixed regular s-structure on (M, g). Then the triplet $(M, g, \{s_x\})$ will be called a *pseudo-Riemannian regular s-manifold.* Let Φ be an isometry of a pseudo-Riemannian s-manifold $(M, g, \{s_x\})$ onto itself. We call Φ an *automorphism* of $(M, g, \{s_x\})$ if Φ satisfies

$$\Phi(s_x(y)) = s_{\Phi(x)}(\Phi(y))$$

for all $x, y \in M$.

Now let L be the group generated by the set $\{s_x | x \in M\}$. Then it follows from the definition of the regular s-structure that L is a subgroup of the group of all automorphisms on $(M, g, \{s_x\})$. The following lemma is a consequence of the proof of Theorem 2 in Ledger-Obata [6].

Lemma 4.A. *Let $(M, g, \{s_x\})$ be a pseudo-Riemannian regular s-manifold. Then the group L generated by the set $\{s_x | x \in M\}$ acts transitively on M.*

Let $I(M, g)$ be the full isometry group of a pseudo-Riemannian regular s-manifold $(M, g, \{s_x\})$. $I(M, g)$ is a Lie transformation group with respect to the compact-open topology. Now, let G be the closure of L in $I(M, g)$. Since G is a closed subgroup of the Lie group $I(M, g)$, G is a Lie transformation group of automorphisms which acts transitively on (M, g). The following two propositions are essentially due to O. Kowalski [5, Lemmas 0.13 and 0.14].

Proposition 4.B. *Let $(M, g, \{s_x\})$ be a pseudo-Riemannian regular s-manifold and G the closure of the group generated by the set $\{s_x | x \in M\}$. Then for any point $o \in M$ the symmetry s_0 commutes with any element of the isotropy subgroup $G(o)$ of G at o.*

Proof. Because $G(o)$ is a group of automorphisms of $(M, g, \{s_x\})$, we have $a \circ s_0 = s_{ao} = s_{ao} \circ a = s_o \circ a$ for each $a \in G(o)$.

Proposition 4.C. *Let (M, g) be a connected pseudo-Riemannian manifold. Further, let G be a Lie group of isometries acting transitively on M, and let $G(o)$ denote the isotropy subgroup of G at $o \in M$. Suppose that there is a symmetry s_0 at o which commutes with any element of $G(o)$. Then there is exactly one regular s-structure $\{s_x | x \in M\}$ with the initial value s_0.*

Proof. We can suppose, without a loss of generality, that $s_0 \in G(o)$. Namely, if $s_0 \notin G(o)$, we take the closure, say $\tilde{G}$, of the group generated by G and s_0 in $I(M, g)$. Then $\tilde{G}$ is a Lie transformation group with respect to the compact-open topology, and s_0 lies in the centre of $\tilde{G}(o)$.

For each $x \in M$ define s_x by $s_x = a \circ s_0 \circ a^{-1}$, where $a \in G$ is any element such that $ao = x$. Then s_x is a symmetry at x which is independent of the choice of $a \in G$ such that $ao = x$ and thus the family $\{s_x | x \in M\}$ is well-defined. Further, for any $y \in M$ and $a \in G$ we have (using an auxiliary element $b \in G$ such that $bo = y$):

$$a \circ s_y \circ a^{-1} = (ab) \circ s_0 \circ (ab)^{-1} = s_{ay} . \tag{12}$$

Looking at the tangent map we get $a_{*y} \circ S_y = S_{ay} \circ a_{*y}$, i.e., the tensor field S is uniquely determined by its initial value S_o and thus the obtained s-structure $\{s_x\}$ is uniquely determined. Since $s_x \in G$ for each $x \in M$, the s-structure $\{s_x\}$ is regular by (12).

Now let $o \in M$ be a fixed point and $o' = (o, 0) \in TM$ the corresponding point of the zero-section in TM. Let s_0 be a symmetry of (M, g) at o. We define a transformation $s'_{o'}$ of TM by $s'_{o'} = (s_0, 0)$, that is,

$$s'_{o'}(y, Y_y) = (s_0(y), (s_0)_* Y_y) \tag{13}$$

for all points $(y, Y_y) \in TM$.

Proposition 4.1. *Let (M, g) be a pseudo-Riemannian manifold, $o \in M$ a point, and s_0 a symmetry of (M, g) at o. Then $s'_{o'}$ is a symmetry of the pseudo-Riemannian manifold (TM, g^c) at $o' = (o, 0)$.*

Proof. Since s_0 is an isometry of (M, g), we obtain from Lemmas 1.B, 2.A,

$$g^c((s'_{o'})_* Y^c, (s'_{o'})_* Z^c) = g^c(((s_o)_* Y)^c, ((s_o)_* Z)^c) =$$
$$= (g((s_o)_* Y, (s_o)_* Z))^c = (g(Y, Z))^c = g^c(Y^c, Z^c)$$

for all vector fields Y, Z on M. In the similar way, we get

$$g^c((s'_{o'})_* Y^c, (s'_{o'})_* Z^v) = g^c(Y^c, Z^v),$$
$$g^c((s'_{o'})_* Y^v, (s'_{o'})_* Z^v) = g^c(Y^v, Z^v)$$

for all vector fields Y, Z on M. Because the tangent space $(TM)_{x'}$ at any point $x' \in TM$ is spanned by $\{Y^c_{x'} + Z^v_{x'} | \, Y, Z \in \mathscr{X}(M)\}$, we see that $s'_{o'}$ is an isometry of (TM, g^c).

Next, let $(y, Y_y) \in TM$ be a point which is fixed by $s'_{o'}$. Then we have

$$s_0(y) = y\,, \quad (s_o)_* Y_y = Y_y\,.$$

But o is an isolated fixed point of s_o. Thus, for (y, Y_y) sufficiently close to o', the point y is sufficiently close to o and we get $y = o$, $Y_y = 0$. Thus $o' = (o, 0)$ is an isolated fixed point of $s'_{o'}$, q.e.d.

Now we construct a regular s-structure on the tangent bundle over a pseudo-Riemannian regular s-manifold. First we prove the existence of such a regular s-structure. Next we find the explicit formula for the symmetries.

Theorem 4.2. *Let (M, g) be a connected pseudo-Riemannian manifold admitting a regular s-structure $\{s_x\}$. Further, let TM be the tangent bundle over M and g^c the complete lift of g from M to TM. Then the pseudo-Riemannian manifold (TM, g^c) admits a regular s-structure $\{s'_{x'}\}$. Here ,$\{s'_{x'}\}$ is of order k, or of infinite order, according to whether $\{s_x\}$ is of order k or of infinite order, respectively.*

In other words, the tangent bundle of a generalized symmetric pseudo-Riemannian space is a generalized symmetric pseudo-Riemannian space.

Proof. Let G be the closure in $I(M, g)$ of the group generated by the set $\{s_x | \, x \in M\}$, and $G^\bullet$ the identity component of G. Then the Lie group $G^\bullet$ acts transitively on M (see Lemma 4.A). Now we fix a point o in M. Then, by Proposition 4.B, the symmetry s_o commutes with any element of the isotropy subgroup $G^\bullet(o)$ of $G^\bullet$ at o.

To prove the assertion, we apply Proposition 4.C to the tangent group $TG^\bullet$. First, by Propositions 2.4 and 2.5, $TG^\bullet$ is a Lie subgroup of $I(TM, g^c)$ acting transitively on TM. Next, we define a transformation $s'_{o'}$: $TM \to TM$ by (13). $s'_{o'}$, is a symmetry on TM by Proposition 4.1 and commutes with any element of the isotropy subgroup $TG^\bullet(o')$ of $TG^\bullet$ at o' by Proposition 2.6. Thus, Proposition 4.C implies that there exists a regular s-structure $\{s'_{x'}\}$ on (TM, g^c).

Since $s'_{x'} = (a, A) \circ s'_{o'} \circ (a, A)^{-1}$ $(x' = (a, A)\, o', (a, A) \in TG^\bullet)$, $\{s'_{x'}\}$ is of order k, or of infinite order, according to whether $\{s_x\}$ is of order k or of infinite order, respectively.

This completes the proof of Theorem 4.2.

Remark 5. Let us remark that some related results about tangent lifts of

s-manifolds (from the group-theoretical point of view) have been obtained recently by N. A. Stepanov [10].

Remark 6. Let T^*M be the cotangent bundle over an affine manifold (M, ∇) of dimension n. Then the *Riemann extension* of (M, ∇) is the pseudo-Riemannian manifold $(T^*M, \bar{g})$ with a metric given, in terms of the system of local coordinates $(U, (x^1, \ldots, x^n, w_1, \ldots, w_n))$ of T^*M, by

$$\bar{g} = \sum g_{ij}\, dx^i\, dx^j + 2 \sum dx^i\, dw_i$$

where $g_{ij} = -2 \sum \Gamma^h_{ij} w_h, \sum \Gamma^h_{ij}\, \partial/\partial x^h = \nabla_{\partial/\partial x^i}\, \partial/\partial x^j\, (i, j = 1, 2, \ldots, n)$ (see, for example, [8]). Now let (M, g) be a pseudo-Riemannian manifold and let b: $TM \to T^*M$ be a mapping defined for each $(x, X_x) \in TM$ by

$$b(x, X_x) = (x, \omega_x)$$

where ω is the 1-form given by $\omega(Y) = g(X, Y)$ for all $Y \in \mathscr{X}(M)$. Then b is an isometry of (TM, g^c) onto $(T^*M, \bar{g})$. Hence, as was first noticed by M. Toomanian (Theorem 1.2 in [11]), the Riemann extension of a generalized symmetric pseudo-Riemannian space is a generalized symmetric pseudo-Riemannian space.

To give the explicit formula for symmetries $s'_{x'}$, $x' \in TM$, we fix some notations. Let G, as before, be the closure in $I(M, g)$ of the group generated by the set $\{s_x | \; x \in M\}$. For any point $y \in M$, let T_y be the mapping of M into G defined by $T_y(x) = s_y^{-1} \circ s_x$ for all $x \in M$. For any vector $X_x \in M_x$, let $\tilde{X}_e \in G_e$ be the vector given by $\tilde{X}_e = (T_x)_* X_x$.

Theorem 4.3. *Under the assumptions of Theorem* 4.2 *and the above notation, the symmetries $s'_{x'}$, $x' \in TM$, on the tangent bundle (TM, g^c) are given by*

$$s'_{x'}(y') = (s_x(y), (s_x)_* Y_y + (\psi_{s_x(y)})_* \operatorname{ad}(s_x) \tilde{X}_e) \tag{14}$$

for all $x' = (x, X_x)$, $y' = (y, Y_y) \in TM$.

Proof. We keep the notations from the proof of Theorem 4.2. Since $TG^\bullet$ acts transitively on TM, there exists, for any point $x' = (x, X_x) \in TM$, an element $(a, A) \in TG^\bullet$ such that $x' = (a, A)\, o'$. Here we have

$$x = ao\,, \quad X_x = (\psi_x)_* \operatorname{ad}(a) A\,. \tag{15}$$

We recall that $s_x = a \circ s_0 \circ a^{-1}$ and $s'_{x'} = (a, A) \circ s'_{o'} \circ (a, A)^{-1}$, where $(a, A)^{-1} = (a^{-1}, -\operatorname{ad}(a) A)$. Let $y' = (y, Y_y)$ be another point in TM. Then

$$\begin{aligned} s'_{x'}(y') &= (a, A)\, s'_{o'}(a, A)^{-1}\, (y. Y_y) = \\ &= (a, A) s'_{o'}(a^{-1} y, (a^{-1})_* Y_y - (\psi_{a^{-1}y})_* A) = \\ &= (a, A)\, (s_o(a^{-1} y), (s_o \circ a^{-1})_* Y_y - (s_o \circ \psi_{a^{-1}y})_* A) = \\ &= (s_x(y), (s_x)_* Y_y - (a \circ s_o \circ \psi_{a^{-1}y})_* A + (\psi_{sx(y)})_* \operatorname{ad}(a) A)\,. \end{aligned}$$

Now we put for the sake of brevity

$$\mathscr{A} = -(a \circ s_o \circ \psi_{a^{-1}y})_* A + (\psi_{s_x(y)})_* \operatorname{ad}(a) A\,.$$

Since

$$(a \circ s_o \circ \psi_{a^{-1}y})(b) = as_o(ba^{-1}y) = s_x \circ \psi_y \circ C_a(b)$$

for all $b \in K$, we get

$$(a \circ s_o \circ \psi_{a^{-1}y})_* = (s_x \circ \psi_y)_* \operatorname{ad}(a) .$$

Hence we obtain

$$\mathscr{A} = [-(s_x \circ \psi_y)_* + (\psi_{s_x(y)})_*] \operatorname{ad}(a) A .$$

Next, we claim that

$$(16) \qquad -(s_x \circ \psi_y)_* + (\psi_{s_x(y)})_* = (\psi_{s_x(y)} \circ C_{s_x} \circ T_x \circ \psi_x)_* .$$

Indeed, let B be any left invariant vector field on $G^\bullet$, and let $b_t = \exp tB$ be the corresponding 1-parameter subgroup of $G^\bullet$. Then we have (recalling that $b_0 y = y$)

$$((\psi_{s_x(y)} \circ C_{s_x} \circ T_x \circ \psi_x)_* B) f = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(b_t s_x(b_{-t}y)) =$$

$$= \lim \frac{1}{t} [f(b_t s_x(b_{-t}y)) - f(b_t s_x(y))] + \lim \frac{1}{t} [f(b_t s_x(y)) - f(b_0 s_x(y))] =$$

$$= \lim \frac{1}{t} [f \circ \psi_{b_t} \circ s_x \circ \psi_y(b_{-t}) - f \circ \psi_{b_t} \circ s_x \circ \psi_y(b_0)] +$$

$$+ \lim \frac{1}{t} [f \circ \psi_{s_x(y)}(b_t) - f \circ \psi_{s_x(y)}(b_0)] = -((s_x \circ \psi_y)_* B) f + ((\psi_{s_x(y)})_* B) f$$

for all real-valued smooth functions f on $G^\bullet$. Hence we get (16).

Finally, using (15) and (16), we have

$$\mathscr{A} = (\psi_{s_x(y)} \circ C_{s_x} \circ T_x \circ \psi_x)_* \operatorname{ad}(a) A =$$

$$= (\psi_{s_x(y)})_* \operatorname{ad}(s_x) (T_x)_* X_x = (\psi_{s_x(y)})_* \operatorname{ad}(s_x) \tilde{X}_e ,$$

and hence (14) follows.

This completes the proof of Theorem 4.3.

Remark 7. Toomanian [11] used (14) as the definition of his lifted s-structure on the tangent bundle, and he proved the regularity directly in a rather complicated way.

Remark 8. (14) can be also expressed, in accordance with Formula (2), as

$$s'_{x'}(y') = (s_x, \tilde{X}_e)(y, Y_y) = (s_x, (T_x)_* X_x)(y, Y_y) .$$

Remark 9. For each point $x' \in TM$ and $x = p(x')$, the following diagram is commutative.

$$\begin{array}{ccc} TM & \xrightarrow{s'_{x'}} & TM \\ \downarrow p & & \downarrow p \\ M & \xrightarrow{s_x} & M \end{array}$$

At the end, we shall return to the groups of transvections. Following Kowalski [5], we define the *elementary transvections* of a pseudo-Riemannian regular s-manifold $(M, g, \{s_x\})$ as the automorphisms of the form $s_x \circ s_y^{-1}$, $x, y \in M$. Further, the group generated by all elementary transvections is called the *group of transvections of* $(M, g, \{s_x\})$ and is denoted by $\mathrm{Tr}(M, \{s_x\})$ (it does not depend on g). This group is a Lie group of automorphisms acting transitively on M (Proposition II. 39 in [5]). We recall that an affine connection $\tilde{\nabla}$ on $(M, g, \{s_x\})$ is called the *canonical connection* if $\tilde{\nabla}$ is invariant with respect to all symmetries s_x and $\tilde{\nabla}S = 0$. The following theorem is part of Theorem II.4 in [5]:

Theorem 4.D. *A pseudo-Riemannian regular s-manifold* $(M, g, \{s_x\})$ *admits a unique canonical connection* $\tilde{\nabla}$. *The affine manifold* $(M, \tilde{\nabla})$ *is an affine reductive space.*

We also recall one of the nice results about the groups of transvections due to O. Kowalski (Theorem II.32 in [5]).

Theorem 4.E. *Let* $(M, g, \{s_x\})$ *be a pseudo-Riemannian regular s-manifold and* $\tilde{\nabla}$ *its canonical connection. Then the group* $\mathrm{Tr}(M, \{s_x\})$ *of transvections of* $(M, g, \{s_x\})$ *coincides with the group* $\mathrm{Tr}(M, \tilde{\nabla})$ *of transvections of the affine reductive space* $(M, \tilde{\nabla})$.

Thus, the following result is an immediate consequence of Theorems 4.D, 4.E and Theorems 3.5, 4.2.

Corollary 4.4. *Let* $(M, g, \{s_x\})$ *be a pseudo-Riemannian regular s-manifold. Then the group of transvections of the pseudo-Riemannian regular s-manifold* $(TM, g^c, \{s'_{x'}\})$ *is tangent to the group of transvections of* $(M, g, \{s_x\})$, *that is,* $\mathrm{Tr}(TM, \{s'_{x'}\}) = T(\mathrm{Tr}(M, \{s_x\}))$ *holds.*

APPENDIX

The notion of an affine regular s-manifold is introduced by replacing the term ,,isometry" in the definition of a pseudo-Riemannian regular s-manifold by the term ,,affine transformation" (see, for example, Ledger-Obata [6]). By arguments similar to those in Section 4 we get the following conclusion.

Let (M, ∇) *be an affine manifold admitting an affine regular s-structure* $\{s_x\}$. *Then* (TM, ∇^c) *admits an affine regular s-structure* $\{s'_{x'}\}$. *Further, the structure* $\{s'_{x'}\}$ *is given by the same formula* (14) *as in Theorem* 4.3. *Corollary* 4.4 *is also true in the affine case.*

References

[1] *N. Bourbaki:* Éléments de Mathématique, Groupes et algébres de Lie, Chapitre III, Groupes de Lie, Herman, Paris, 1971, 1972.

]2] *J. Černý* and *O. Kowalski:* Classification of generalized symmetric pseudo-Riemannian spaces of dimension $n \leqq 4$, Tensor, N.S. *38* (1982), 255—267.

[3] *S. Kobayashi:* Theory of connections, Annali di Mat. *43* (1957), 119—194.

[4] *S. Kobayashi* and *K. Nomizu:* Foundations of differential geometry, Interscience Publ., New York, London, I (1963); II (1969).

[5] *O. Kowalski:* Generalized symmetric spaces, Lecture Note in Mathematics 805, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[6] *A. J. Ledger* and *M. Obata:* Affine and Riemannian s-manifolds, J. Differential Geometry *2* (1968), 451—459.

[7] *K. Nomizu:* Invariant affine connections on homogeneous spaces, Amer. J. Math. *76* (1954), 33—65.

[8] *E. M. Patterson* and *A. G. Walker:* Riemann extensions, Quart. J. Math. Oxford *3* (1952), 19—28.

[9] *M. Sekizawa:* Lifts of generalized symmetric spaces to tangent bundles, to appear.

[10] *N. A. Stepanov:* The tangent bundle of φ-spaces, Soviet Math. (Iz. VUZ) *27* (1983), 94—103.

[11] *M. Toomanian:* Riemann extensions and complete lifts of s-spaces, Ph. D. Thesis, The University, Southampton, 1975.

[12] *M. Toomanian:* Killing vectorfields and infinitesimal affine transformations on a generalized Riemann extension, Tensor, N.S. *32* (1978), 335—338.

[13] *K. Yano* and *S. Ishihara:* Tangent and cotangent bundles, Marcel Dekker, Inc., New York, 1973.

[14] *K. Yano* and *S. Kobayashi:* Prolongations of tensor fields and connections to tangent bundles, J. Math. Soc. Japan, *I 18* (1966), 194—210; II *18* (1966), 236—246.

Added in proof: see also

[15] *M. Toomanian*: Regular s-structure on TM, Tensor, N. S. *42* (1985), 225—228.

Author's address: Tokyo Gakugei University, Nukui-Kita Machi 4-1-1, Koganei Shi, Tokyo, 184, Japan.