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ON COMPLETION OF CYCLICALLY ORDERED SETS

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In [6], a completion of linearly cyclically ordered sets (cycles) is constructed by means of cuts. In this note we give another construction of a completion, which can be applied to a larger class of monodimensional cyclically ordered sets.

1. INTRODUCTORY CONCEPTS AND ASSERTIONS

1.1. Ordered sets. Basic notions on ordered sets are assumed to be known (see e.g. [1] or [2]). *Ordinal sum* of ordered sets G, H is denoted by $G \oplus H$. If $G = (G, <)$ is an ordered set and $H \subseteq G$, then the induced order $< \cap H^2$ on H is denoted briefly by $<$. A linearly ordered set is called a *chain*. If $G = (G, <)$ is an ordered set and $H \subseteq G$ is a subset of G such that $(H, <)$ is a chain, then H is called a chain in G. A chain H in an ordered set G is *maximal* iff it is contained in no chain in G as a proper subset. As it is well known, the "Hausdorff maximal principle"

Every chain in every ordered set G is contained in a maximal chain in G

is equivalent to the Axiom of Choice.

An ordered set G is called *complete* (or a complete lattice) iff any nonvoid subset of G has the supremum and infimum in G. G is said to be *conditionally complete* iff any nonvoid bounded subset of G has the supremum and infimum in G. G is referred to as *chain complete* iff any maximal chain in G is complete.

A subset I of an ordered set G is called an *ideal* iff it has the property $x \in I$, $y \in G$, $y < x \Rightarrow y \in I$. If A is a nonvoid subset of an ordered set G, then $I(A)$ denotes the ideal in G generated by A, i.e., $I(A) = \{y \in G;$ there exists $x \in A$ such that $y \leqq x\}$. Note that $\sup A = \sup I(A)$ for any nonvoid subset A of an ordered set G, whenever one of the elements $\sup A$, $\sup I(A)$ exists. If G is a chain, then ideals in G are called *initial intervals*; the dual notion is a *final inverval* in G.

1.2. Cyclically ordered sets. A *cyclically ordered set* ([5]) is a pair (G, C) where G is a set and C is a *cyclic order* on G, i.e., C is a ternary relation on G which is

asymmetric, i.e., $(x, y, z) \in C \Rightarrow (z, y, x) \notin C$,
cyclic, i.e., $(x, y, z) \in C \Rightarrow (y, z, x) \in C$, and
transitive, i.e., $(x, y, z) \in C$, $(x, z, u) \in C \Rightarrow (x, y, u) \in C$.

If, moreover, card $G \geqq 3$ and C is

linear, i.e., $x, y, z \in G$, $x \neq y \neq z \neq x \Rightarrow$ either $(x, y, z) \in C$ or $(z, y, x) \in C$, then (G, C) is called a *linearly cyclically ordered set* or a *cycle.*

If (G, C) is a cyclically ordered set and $H \subseteq G$ is a subset of G such that $(H, C \cap H^3)$ is a cycle, then H is called a cycle in G. A cycle H in G is *maximal* iff it is contained in no cycle in G as a proper subset. In [5] (Theorem 2.5) we proved using Axiom of Choice that every cycle in every cyclically ordered set is contained in a maximal cycle; below we show that this proposition is equivalent to the Axiom of Choice.

Let $(G, <)$ be an ordered set. Let us define a ternary relation $C_<$ on G by $(x, y, z) \in C_< \Leftrightarrow$ either $x < y < z$ or $y < z < x$ or $z < x < y$. Then $(G, C_<)$ is a cyclically ordered set ([5], Theorem 3.5).

Let (G, C) be a cyclically ordered set and $x \in G$. Let us define a binary relation $<_{C,x}$ on G by

$y <_{C,x} z \Leftrightarrow$ either $(x, y, z) \in C$ or $x = y \neq z$. Then $(G, <_{C,x})$ is an ordered set with the least element x ([5], Theorem 3.1).

The proofs of the following two lemmas are trivial.

1.3. Lemma. *Let $(G, <)$ be an ordered set and* card $G \geqq 3$. *Then $(G, <)$ is a chain iff $(G, C_<)$ is a cycle.*

1.4. Lemma. *Let $(G, <)$ be an ordered set and H a chain in G with* card $H \geqq 3$. *Then H is a maximal chain in $(G, <)$ iff H is a maximal cycle in $(G, C_<)$.*

1.5. Theorem. *The proposition*

"Every cycle in every cyclically ordered set G is contained in a maximal cycle in G"

is equivalent to the Axiom of Choice.

Proof. One implication is shown in [5]; we shall prove the other one. Thus, let every cycle in every cyclically ordered set be contained in a maximal cycle and let $(G, <)$ be an ordered set. Assume that H is a chain in G with card $H \geqq 3$. By Lemma 1.3 $(H, C_< \cap H^3)$ is a cycle in $(G, C_<)$ and hence there exists a maximal cycle $(K, C_< \cap K^3)$ in $(G, C_<)$ such that $K \supseteq H$. By Lemma 1.4 $(K, <)$ is a maximal chain in $(G, <)$. If H is a chain in $(G, <)$ with card $H \leqq 2$ and if there exists no 3-element chain in G containing H, then the existence of a maximal chain in G containing H is trivial. Thus the Hausdorff maximal principle and also the Axiom of Choice is true.

We shall need the following assertion; its proof can be found in [6] (Theorem 3.6 and Corollary 3.9).

1.6. Theorem. *Let G be a set,* card $G \geqq 3$ *and let $<_1, <_2$ be linear orders on G. Then the following statements are equivalent:*

(A) $C_{<_1} = C_{<_2}$

(B) *There exist disjoint subsets A, B of G such that $A \cup B = G$, $<_1 \cap A^2 = <_2 \cap A^2$, $<_1 \cap B^2 = <_2 \cap B^2$ and $(G, <_1) = A \oplus B$, $(G, <_2) = B \oplus A$.*

Let G be a set, let $<_1$, $<_2$ be orders on G. Put $<_1 \sim <_2$ iff $C_{<_1} = C_{<_2}$. Trivially, it holds

1.7. Lemma. *Let G be a set. The binary relation $\sim$ is an equivalence relation on the set of all orders on G.*

1.8. Theorem. *Let G be a set, let $<_1$, $<_2$ be orders on G. Then the following statements are equivalent:*

(A) $C_{<_1} \subseteq C_{<_2}$

(B) *If H is a maximal chain in $(G, <_1)$ with* card $H \geqq 3$, *then H is a chain in $(G, <_2)$ and there exist disjoint subsets A, B of H with $A \cup B = H$ such that $<_1 \cap A^2 = <_2 \cap A^2$, $<_1 \cap B^2 = <_2 \cap B^2$ and $(H, <_1) = A \oplus B$, $(H, <_2) = B \oplus A$.*

Proof. 1. Let (A) hold and let $(H, <_1)$ be a maximal chain in $(G, <_1)$ with card $H \geqq 3$. By Lemma 1.3 $(H, C_{<_1} \cap H^3)$ is a cycle and as $C_{<_1} \subseteq C_{<_2}$, $(H, C_{<_2} \cap H^3)$ is also a cycle, thus $C_{<_1} \cap H^3 = C_{<_2} \cap H^3$. From this it follows, by Lemma 1.3, that $(H, <_2)$ is also a chain. Applying Theorem 1.6 on the set H and linear orders $<_1 \cap H^2$, $<_2 \cap H^2$, we obtain the validity of (B).

2. Let (B) hold and suppose $(x, y, z) \in C_{<_1}$. Then either $x <_1 y <_1 z$ or $y <_1 z <_1 x$ or $z <_1 x <_1 y$ and there exists a maximal chain H in $(G, <_1)$ containing $\{x, y, z\}$. Thus card $H \geqq 3$ and, by (B), $(H, <_2)$ is a chain and there exist subsets A, B of H with the desired properties. Assume that $x <_1 y <_1 z$ holds (in other two cases the proof is analogical). We have four possibilities:

(1) $x, y, z \in A$. Then $x <_2 y <_2 z$, for $(A, <_1) = (A, <_2)$, and thus $(x, y, z) \in C_{<_2}$;

(2) $x, y \in A$, $z \in B$. Then $x <_2 y$ and $z <_2 x$, for $(H, <_2) = B \oplus A$, thus $z <_2 x <_2 y$ and $(x, y, z) \in C_{<_2}$;

(3) $x \in A$, $y, z \in B$. Then $y <_2 z$ and $z <_2 x$, hence $y <_2 z <_2 x$ and $(x, y, z) \in C_{<_2}$;

(4) $x, y, z \in B$. Then $x <_2 y <_2 z$ and $(x, y, z) \in C_{<_2}$.

We have shown that $(x, y, z) \in C_{<_1} \Rightarrow (x, y, z) \in C_{<_2}$, i.e., $C_{<_1} \subseteq C_{<_2}$ and (A) holds.

1.9. Corollary. *Let G be a set and let $<_1$, $<_2$ be orders on G. Then $<_1 \sim <_2$ holds iff the sets of maximal, at least three element chains in $(G, <_1)$ and in $(G, <_2)$ are the same and for any such maximal chain H the condition* (B) *of Theorem* 1.8 *holds.*

2. COMPLETENESS

2.1. Cuts on cycles. Let (G, C) be a cycle. A *cut* on G ([6]) is a linear order $<$ on G such that $C = C_<$. Any cycle (G, C) contains cuts, for $<_{C,x}$ is a cut on G for any $x \in G$ ([6], Theorem 2.5). A cut $<$ on a cycle (G, C) is a *jump* iff $(G, <)$ has both the least and the greatest element ; it is a *gap*, iff $(G, <)$ has neither the least nor the greatest element; $<$ is *Dedekind* iff $(G, <)$ has just one of the boundary elements.

A cycle (G, C) is *dense* iff it contains no jumps; it is *complete* iff it contains no gaps. A cycle (G, C) is *continuous* iff it is dense and complete, i.e. iff each cut on G is Dedekind.

2.2. Definition. A cyclically ordered set (G, C) is called *cycle complete* iff each maximal cycle in G is complete.

2.3. Theorem. *Let G be a set, let $<$ be an order on G. If the ordered set $(G, <)$ is chain complete, then the cyclically ordered set $(G, C_<)$ is cycle complete.*

Proof. Let $(H, C_< \cap H^3)$ be a maximal cycle in $(G, C_<)$. By Lemmas 1.3 and 1.4, $(H, <)$ is a maximal chain in $(G, <)$. Hence $(H, <)$ is complete. Assume that the cycle $(H, C_< \cap H^3)$ is not complete. Then there exists a cut $\prec$ on $(H, C_< \cap H^3)$ which is a gap, i.e. $\prec$ is a linear order on H without the least and the greatest elements and such that $C_\prec = C_< \cap H^3$. By Theorem 1.6, there exist disjoint subsets A, B of H such that $A \cup B = H$, $< \cap A^2 = \prec \cap A^2$, $< \cap B^2 = \prec \cap B^2$ and $(H, <) = A \oplus B$, $(H, \prec) = B \oplus A$. Thus $(B, <) = (B, \prec)$ contains no least element, $(A, <)$ contains no greatest element. If $A \neq \emptyset$, then A has no supremum in $(H, <) = A \oplus B$. If $A = \emptyset$, then $B = H$ has no infimum in $(H, <)$. In either case this contradicts the assumption that $(H, <)$ is complete and thus the cycle $(H, C_< \cap H^3)$ is complete.

Theorem 2.3 cannot be reversed, i.e., if a cyclically ordered set $(G, C_<)$ is cycle complete, then the ordered set $(G, <)$ need not be chain complete as the following example shows.

2.4. Example. Let $(G, <) = [0, 1)$ with the natural ordering of reals. $(G, <)$ is not chain complete; we show that $(G, C_<)$ is a complete cycle.

Let $\prec$ be any cut on $(G, C_<)$. Then either $\prec = <$ or $(G, \prec) = B \oplus A$ where A is an initial interval, B a final interval in $[0, 1)$, and $A \neq \emptyset$, $B \neq \emptyset$. In the second case either A has the greatest element or B has the least element in $[0, 1)$. Thus $(G, \prec)$ has one of the boundary elements and $\prec$ is not a gap.

For cycles we have, however, this assertion:

2.5. Theorem. *Let $(G, <)$ be a chain with* card $G \geqq 3$. *The cycle $(G, C_<)$ is complete iff the chain $(G, <)$ is conditionally complete and has either the least or the greatest element.*

Proof. 1. Let the cycle $(G, C_<)$ be complete. The chain $(G, <)$ must contain either the least or the greatest element; otherwise the cut $<$ on $(G, C_<)$ would be a gap. Assume that $(G, <)$ is not conditionally complete. Then there exists a subset A of G, $A \neq \emptyset$ which is upper bounded and such that $\sup A$ does not exist in $(G, <)$. As $\sup A = \sup I(A)$, we may assume that A is an initial interval in $(G, <)$. Then $B = G - A$ is a final interval in $(G, <)$, $B \neq \emptyset$, we have $(G, <) = A \oplus B$ and A contains no greatest element, B contains no least element. Let $\prec$ be a linear order on G such that $(G, \prec) = B \oplus A$. By Theorem 1.6, $\prec$ is a cut on $(G, C_<)$ which is a gap. This contradicts the assumption.

2. Let $(G, <)$ be a conditionally complete chain which has either the least or the greatest element. Assume that the cycle $(G, C_<)$ is not complete. Then there exists a cut $\prec$ on $(G, C_<)$ which is a gap. It must necessarily be $\prec \neq <$ and thus there exist nonvoid disjoint subsets A, B of G such that $< \cap A^2 = \prec \cap A^2$, $< \cap B^2 = \prec \cap B^2$ and $(G, <) = A \oplus B$, $(G, \prec) = B \oplus A$. This implies that B has no least element, A has no greatest element. Therefore sup A does not exist in $(G, <)$, which contradicts the assumption that $(G, <)$ is conditionally complete.

Let us call a cyclically ordered set (G, C) *monodimensional* iff there exists an order $<$ on the set G such that $C = C_<$. This concept agrees with [7], for by suitable definition of the dimension in the class of cyclically ordered sets, the cyclically ordered sets of form $(G, C_<)$ are just the sets with dimension 1.

2.6. Theorem. *Let (G, C) be a monodimensional cyclically ordered set. Then there exists a cycle complete cyclically ordered set (H, D) and an isomorphic embedding of (G, C) into (H, D).*

Proof. By assumption there exists an order $<$ on G such that $C = C_<$. To the ordered set $(G, <)$ we can construct a chain complete ordered set $(H, \prec)$ such that there exists an isomorphic embedding $i: G \to H$ of $(G, <)$ into $(H, \prec)$; for instance the Dedekind-Mac Neille completion of $(G, <)$ ([2] or [4]) has this property. By Theorem 2.3, the cyclically ordered set $(H, C_\prec)$ is cycle complete and, clearly, $i: G \to H$ is an isomorphic embedding of (G, C) into $(H, C_\prec)$.

3. APPLICATION TO CYCLES

3.1. Completion of cycles by cuts. Let (G, C) be a cycle. Let $\mathscr{G}$ be the set of all cuts on (G, C), $\mathscr{G}_r = \{\langle_{C,x};\ x \in G\} \cup \{\langle \in \mathscr{G};\ < \text{ is a gap}\}$; the elements of $\mathscr{G}_r$ are called *regular cuts*. Let us define a ternary relation $\mathscr{C}$ on $\mathscr{G}$ (and also on $\mathscr{G}_r$) by $(<_1, <_2, <_3) \in \mathscr{C} \Leftrightarrow$ there exist nonvoid pairwise disjoint subsets A, B, D of G such that $<_1 \cap A^2 = <_2 \cap A^2 = <_3 \cap A^2$, $<_1 \cap B^2 = <_2 \cap B^2 = <_3 \cap B^2$, $<_1 \cap D^2 = <_2 \cap D^2 = <_3 \cap D^2$ and $(G, <_1) = A \oplus B \oplus D$, $(G, <_2) = B \oplus D \oplus A$, $(G, <_3) = D \oplus A \oplus B$. In [6] (Theorem 4.2, Corollary 4.5, Theorem 5.2 and Theorem 5.6) there is proved that both $(\mathscr{G}, \mathscr{C})$ and $(\mathscr{G}_r, \mathscr{C})$ are complete cycles and that $x \to <_{C,x}$ is an isomorphic embedding of (G, C) into $(\mathscr{G}, \mathscr{C})$ and into $(\mathscr{G}_r, \mathscr{C})$. If, moreover, (G, C) is dense. then $(\mathscr{G}_r, \mathscr{C})$ is continuous ([6], Theorem 5.9).

The results of preceding paragraph give another possibility of a construction of a completion of a cycle. Directly from Theorem 2.6 and its proof we get

3.2. Theorem. *Let (G, C) be a cycle, let $<$ be a cut on (G, C). Let $(H, \prec)$ be a complete chain such that there exists an isomorphic embedding of $(G, <)$ into $(H, \prec)$. Then $(H, C_\prec)$ is a complete cycle and there exists an isomorphic embedding of (G, C) into $(H, C_\prec)$.*

Especially, we have

3.3. Corollary. *Every cycle can be embedded into a complete cycle.*

If we choose in Theorem 3.2 $(H, \prec)$ as the Dedekind-Mac Neille completion of $(G, <)$, we get further

3.4. Corollary. *Let (G, C) be a cycle, $<$ a cut on (G, C). Let $(H, \prec)$ be the Dedekind-Mac Neille completion of $(G, <)$. Then $(H, C_\prec)$ is a complete cycle and there exists an isomorphic embedding of (G, C) into $(H, C_\prec)$.*

Further, by Theorem 2.5, the stronger version of Theorem 3.2 holds:

3.5. Theorem. *Let (G, C) be a cycle, $<$ a cut on (G, C). Let $(H, \prec)$ be a conditionally complete chain containing either the least or the greatest element such that there exists an isomorphic embedding of $(G, <)$ into $(H, \prec)$. Then $(H, C_\prec)$ is a complete cycle and there exists an isomorphic embedding of (G, C) into $(H, C_\prec)$.*

There is a close connection between the both constructions (given in 3.1 and in 3.2 or 3.5); we shall describe it.

3.6. Ideal hull of a chain. Let $(G, <)$ be a chain, let $\mathscr{I}(G)$ be the set of all ideals (initial intervals) in G, which is ordered by set inclusion. The ordered set $(\mathscr{I}(G), \subset)$ is a complete chain which we call the *ideal hull* of the chain $(G, <)$. Further, let $\mathscr{I}_0(G)$ be the set of all nonvoid initial intervals in G, i.e. $\mathscr{I}_0(G) = \mathscr{I}(G) - \{\emptyset\}$. $(\mathscr{I}_0(G), \subset)$ is a conditionally complete chain with the greatest element which we shall call a *reduced ideal hull* of $(G, <)$.

By Theorem 2.5, if $(G, <)$ is a chain with card $G \geqq 3$ and $(\mathscr{I}_0(G), \subset)$ its reduced ideal hull, then $(\mathscr{I}_0(G), C_\subset)$ is a complete cycle.

In the sequel, we assume that there is given a fixed cycle (G, C) and a fixed cut $<$ on (G, C).

3.7. Notation. Let $\prec$ be a cut on (G, C), other than $<$. By Theorem 1.6, there exist (uniquely determined) disjoint nonvoid subsets A, B of G such that $A \cup B = G$, $< \cap A^2 = \prec \cap A^2$, $< \cap B^2 = \prec \cap B^2$ and $(G, <) = A \oplus B$, $(G, \prec) = B \oplus A$. Thus, A is a nonvoid initial interval in $(G, <)$, i.e. $A \in \mathscr{I}_0(G, <)$. Let us put $i(\prec) = A$; further, put $i(<) = G$. We have therefore defined a mapping $i: \mathscr{G} \to \mathscr{I}_0(G, <)$.

3.8. Lemma. *The mapping i is a bijection of $\mathscr{G}$ onto $\mathscr{I}_0(G, <)$.*

Proof. If $i(\prec_1) = i(\prec_2) = A$ and $B = G - A$, then $(G, \prec_1) = B \oplus A = (G, \prec_2)$. Thus, i is an injection. Let $A \in \mathscr{I}_0(G, <)$ be any element. If $A = G$, then $A = i(<)$. Otherwise put $B = G - A$, thus $(G, <) = A \oplus B$. Define a linear order $\prec$ on G by setting $(G, \prec) = B \oplus A$. By Theorem 1.6, $\prec$ is a cut on (G, C), i.e. $\prec \in \mathscr{G}$ and $i(\prec) = A$. The mapping i is thus a surjection and therefore a bijection of $\mathscr{G}$ onto $\mathscr{I}_0(G, <)$.

3.9. Theorem. *The mapping $i: \mathscr{G} \to \mathscr{I}_0(G, <)$ is an isomorphism of the cycle $(\mathscr{G}, \mathscr{C})$ onto the cycle $(\mathscr{I}_0(G, <), C_\subset)$.*

Proof. By Lemma 3.8, i is a bijection. Let $\prec_1, \prec_2, \prec_3 \in \mathscr{G}$, $(\prec_1, \prec_2, \prec_3) \in \mathscr{C}$.

As $i(\prec_1)$, $i(\prec_2)$, $i(\prec_3)$ are nonvoid pairwise distinct initial intervals in a linearly ordered set $(G, <)$, they are linearly ordered by set inclusion. Let us assume that $i(\prec_3) \subset i(\prec_2) \subset i(\prec_1)$ holds. Choose arbitrary elements $x \in i(\prec_3)$, $y \in i(\prec_2) - i(\prec_3)$, $z \in i(\prec_1) - i(\prec_2)$. Then $y \prec_3 z \prec_3 x$, $z \prec_2 x \prec_2 y$, $x \prec_1 y \prec_1 z$. This implies by Lemma 4.3 of [6] that $(\prec_3, \prec_2, \prec_1) \in \mathscr{C}$, which is a contradiction. Analogously we show that both $i(\prec_2) \subset i(\prec_1) \subset i(\prec_3)$ and $i(\prec_1) \subset i(\prec_3) \subset i(\prec_2)$ are impossible. Thus it must hold either $i(\prec_1) \subset i(\prec_2) \subset i(\prec_3)$ or $i(\prec_2) \subset i(\prec_3) \subset i(\prec_1)$ or $i(\prec_3) \subset i(\prec_1) \subset i(\prec_2)$ and this implies $(i(\prec_1), i(\prec_2), i(\prec_3)) \in C_\subset$ in all cases. Hence i is an isomorphism.

Theorem 3.9 shows that the complete hull $(\mathscr{G}, \mathscr{C})$ of a cycle (G, C) can be constructed in the following way: we choose any cut $<$ on (G, C), find the reduced ideal hull of $(G, <)$ and construct the cycle corresponding to that chain. The regular hull $(\mathscr{G}_r, \mathscr{C})$ of a cycle (G, C), however, can be obtained by a similar construction. Let us recall that the Dedekind - Mac Neille completion of a chain $(G, <)$ can be defined as the set of all initial intervals A in $(G, <)$ with the property: either $B = G - A$ has the least element or A has no greatest element and B has no least element; this set is ordered by set inclusion ([3], chaper IV, par. 5).

3.10. Reduced Dedekind - Mac Neille completion. Let $(G, <)$ be a chain. Let $\mathscr{I}_r(G)$ be the set of all nonvoid initial intervals A in $(G, <)$ with the property: either $B = G - A$ has the least element or A has no greatest element and B has no least element; further, let $G \in \mathscr{I}_r(G)$. The chain $(\mathscr{I}_r(G), \subset)$ will be called *reduced Dedekind - Mac Neille completion* of $(G, <)$.

Clearly, $\mathscr{I}_r(G) \subseteq \mathscr{I}_0(G)$ and $(\mathscr{I}_r(G), \subset)$ is a conditionally complete chain with the greatest element, so that $(\mathscr{I}_r(G), C_\subset)$ is a complete cycle whenever card $G \geqq 3$.

3.11. Theorem. *The mapping i defined in* 3.7 *is an isomorphism of $(\mathscr{G}_r, \mathscr{C})$ onto $(\mathscr{I}_r(G, <), C_\subset)$.*

Proof. We show that i maps bijectively $\mathscr{G}_r$ onto $\mathscr{I}_r(G, <)$. Let $\prec \in \mathscr{G}_r$. If $\prec = <$, then $i(\prec) = G \in \mathscr{I}_r(G, <)$. Let $\prec \neq <$. Then either $\prec = <_{C,x}$ for some $x \in G$ or $\prec$ is a gap in (G, C). In the first case we obtain $(G, <) = A \oplus B$, $(G, \prec) = B \oplus A$ and as $(G, \prec) = (G, <_{C,x})$ has the least element x, B has the least element x. Thus, $i(\prec) = A \in \mathscr{I}_r(G, <)$. In the second case we have $(G, <) = A \oplus B$, $(G, \prec) = B \oplus A$ and as $\prec$ is a gap, B has not least element and A has no greatest element. Thus, $i(\prec) = A \in \mathscr{I}_r(G, <)$. We have shown that i maps $\mathscr{G}_r$ into $\mathscr{I}_r(G, <)$. Let $A \in \mathscr{I}_r(G, <)$. Put $B = G - A$. Then $(G, \prec) = B \oplus A$ and either B has the least element x or A has no greatest element and B has no least element. In the first case $(G, \prec)$ has the least element x, $\prec = <_{C,x}$ and $\prec \in \mathscr{G}_r$, $i(\prec) = A$. In the second case $\prec$ is a gap, $\prec \in \mathscr{G}_r$ and $i(\prec) = A$. We have shown that $i: \mathscr{G}_r \to \mathscr{I}_r(G, <)$ is a surjection; by Lemma 3.8 it is a bijection. The proof that i is an isomorphism of $(\mathscr{G}_r, \mathscr{C})$ onto $(\mathscr{I}_r(G, <), C_\subset)$ is the same as that of Theorem 3.9.

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