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ON THE URYSOHN CONDITION AND THE CONVERGENCE OF PROBABILITY DISTRIBUTIONS

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1. Denote by $\mathscr{F}_K$ the set of all probability distribution functions (p.d.f.'s) on R^k. Suppose that $F_n \in \mathscr{F}_K$ $(n = 1, 2, \ldots)$ and $F \in \mathscr{F}_K$, and consider

$(*)$ $F_n(x) \to F(x)$,

$(**)$ $\int_{R^k} \varphi(x)\, dF_n(x) \to \int_{R^k} \varphi(x)\, dF(x)$.

It is well known that the following conditions are equivalent (see [2], § 2–3):

(I^1) $(*)$ holds for each point x of the continuity set $C(F)$ of F;

(I^2) $(*)$ holds for each point x of a dense set D;

(II_1) $(**)$ holds for each φ in $\mathscr{C}_b$, the class of all bounded continuous functions;

(II_2) $(**)$ holds for each φ in $\mathscr{C}_u$, the class of all bounded uniformly continuous functions.

Consider additionally the conditions:

(II_3) $(**)$ holds for φ in $\mathscr{E}_b$, the class of all bounded smooth functions;

(II_4) $(**)$ holds for φ in $\mathscr{C}_0$, the class of all continuous functions of bounded support;

(II_5) $(**)$ holds for φ in $\mathscr{D}$, the Sobolev-Schwartz class of all smooth functions of bounded support.

We shall show that conditions $(II_3)-(II_5)$ are equivalent to the preceding ones.

The convergence expressed by one of the above conditions is called, according to (II_1), the weak convergence of p.d.f.'s and is denoted by $F_n \to^w F$. Note that this is the distributional convergence, because condition (II_5) means that $F_n \to F$ in the sense of distributions of L. Schwartz.

In [3], § 9 (Theorem 1), it is proved that for $k = 1$ the weak convergence of p.d.f.'s is equivalent to the convergence in the P. Lévy metric:

$$d_L(F, G) = \inf\{h > 0: F(x - h) - h \leqq G(x) \leqq F(x + h) + h \quad \text{for all} \quad x \in R^1\},$$

where $F, G \in \mathscr{F}_K$. Some modifications of this metric as well as other equivalent metrics have been considered lately by various authors [14], [11], [12], [13], [15].

One can show that conditions $(I^1)-(II_5)$ are equivalent to the local convergence of the respective p.d.f.'s in $L^p(0 < p < \infty)$:

(III) $\int_I |F_n(x) - F(x)|^p \, dx \to 0$ for all rectangles $I \subset R^k$,

which, for $k = 1$, is essentially shown in [15]. In Section 2, we would like to present a simple proof of this equivalence, based on the Urysohn property:

(U) if for every subsequence $\bar{F}_n$ of a sequence F_n there is a subsequence $\bar{\bar{F}}_n$ of $\bar{F}_n$ such that $\bar{\bar{F}}_n \to F$, then $F_n \to F$.

Property (U) holds evidently if (I^1) is considered. Let us recall that (U) represents a condition which plays an essential part in characterizations of topological convergence (see [6], p. 51; [7]; [1]; [4]).

Condition (III) suggests natural metrics in $\mathscr{F}_K$ connected with the local convergence in L^p. Fix an arbitrary sequence I_m of rectangles in R^k whose union is R^k. One of possible metrics in $\mathscr{F}_K$ may be defined as

$$d_p(F, G) = \sum_{s=1}^{\infty} 2^{-s} \max(1, \|F - G\|_{L^p(I_s)}) \quad (0 < p < \infty),$$

which is a modification of M. D. Taylor's metric (see [15]), and condition (III) can be expressed in the form:

(III) $d_p(F_n, F) \to 0$.

Let us consider now conditions for convergence of p.d.f.'s inspired by the Peck topology in the set of doubly stochastic measures on $[0, \infty) \times [0, \infty)$ (see [9]).

Fix m, $0 \leqq m \leqq k$. Given $x = (\xi_1, \ldots, \xi_k) \in R^k$ let $x^m = (\xi_1, \ldots, \xi_m)$ and given $t = (\tau_1, \ldots, \tau_{k-m}) \in R^{k-m}$ let

$$(-\infty, t) = (-\infty, \tau_1) \times \ldots \times (-\infty, \tau_{k-m})$$

and $H_t = R^m \times (-\infty, t)$. The convergence

(IV) $\int_{H_t} \varphi(x^m) \, dF_n(x) \to \int_{H_t} \varphi(x^m) \, dF(x)$

represents, as a matter of fact, a variety of conditions, depending on various assumptions about the points t and the functions φ. Denote by (IV_j^1) condition (IV) assumed for all functions φ on R^m of the respective class described in condition (II_j) ($j = 1, \ldots, 5$) and for all $t = (\tau_1, \ldots, \tau_{k-m}) \in C_{k-m} \subset R^{k-m}$ such that F is continuous at points of the form $(s, t) = (\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_{k-m}) \in R^k$, where $s = (\sigma_1, \ldots, \sigma_m) \in R^m$. By ($IV_j^2$), denote condition (IV) satisfied for all functions φ on R^m of the class defined in (II_j) ($j = 1, \ldots, 5$) and for all points of a dense set D_{k-m} in R^{k-m}.

It appears that each of conditions (IV_j^i) ($i = 1, 2; j = 1, \ldots, 5$) is equivalent to the weak convergence of the respective p.d.f.'s. This leads to some consequences for the convergence of copulas of doubly stochastic measures (see section 3).

Finally note that analogous results can be obtained in the Rényi theory of conditional probabilities (see [10], p. 70; [4]), where p.d.f.'s are unbounded. In particular, the space of Rényi p.d.f.'s gains a natural metric while the Lévy metric cannot be modified in this case (see section 4).

2. Given a p.d.f. F and a rectangle

$$I = [a, b) = [\alpha_1, \beta_1) \times \ldots \times [\alpha_k, \beta_k) \subset R^k$$

we use the notation

$$\Delta_I F = \sum_e (-1)^{\varepsilon_1 + \ldots + \varepsilon_k} F(\beta_1 + \varepsilon_1(\alpha_1 - \beta_1), \ldots, \beta_k + \varepsilon_k(\alpha_k - \beta_k)),$$

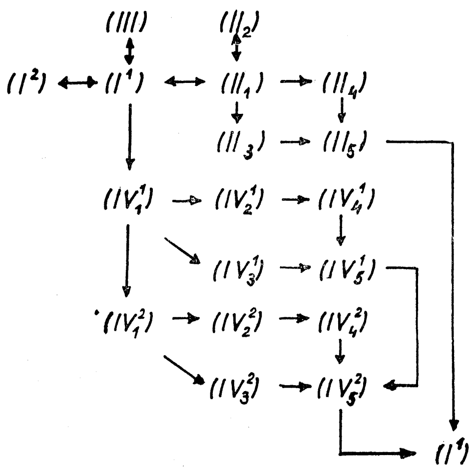
where the sum is taken over all systems $e = (\varepsilon_1, \ldots, \varepsilon_k)$ with $\varepsilon_i = 0$ or 1 for $i = 1, \ldots, k$.

Probability distributions corresponding to the p.d.f.'s F_n, F will be denoted by P_n, P, respectively.

We have

Theorem 1. *Let F_n $(n = 1, 2, \ldots)$ and F be p.d.f.'s on R^k. Then all conditions:* (I^i), (II_j), (III) *and* (IV^i_j) $(i = 1, 2; j = 1, \ldots, 5)$ *are equivalent.*

Proof. The proof will be organized according to the following diagram:



The equivalences in the chain

$$(\mathrm{I}^2) \leftrightarrow (\mathrm{I}^1) \leftrightarrow (\mathrm{II}_1) \leftrightarrow (\mathrm{II}_2)$$

are proved in [2], § 2 (Theorem 2.1) and § 3 (see also Problem 1 in § 3).

The implications:

$$(\mathrm{II}_1) \to (\mathrm{II}_4) \to (\mathrm{II}_5)\,; \quad (\mathrm{II}_1) \to (\mathrm{II}_3) \to (\mathrm{II}_5)\,;$$
$$(\mathrm{IV}^1_1) \to (\mathrm{IV}^1_2) \to (\mathrm{IV}^1_4) \to (\mathrm{IV}^1_5) \to (\mathrm{IV}^2_5)\,;$$
$$(\mathrm{IV}^1_1) \to (\mathrm{IV}^2_1) \to (\mathrm{IV}^2_2) \to (\mathrm{IV}^2_4) \to (\mathrm{IV}^2_5)\,;$$
$$(\mathrm{IV}^1_1) \to (\mathrm{IV}^1_3) \to (\mathrm{IV}^1_5)\,; \quad (\mathrm{IV}^2_1) \to (\mathrm{IV}^2_3) \to (\mathrm{IV}^2_5)$$

follow from the inclusions $\mathscr{C}_b \supset \mathscr{C}_u \supset \mathscr{C}_0 \supset \mathscr{D}$ and $\mathscr{C}_b \supset \mathscr{E}_b \supset \mathscr{D}$, and from the fact that the set C_{k-m} is dense in R^{k-m}.

$(\mathrm{I}^1) \to (\mathrm{III})$

Condition (I^1) implies that $F_n \to F$ almost everywhere. By the Lebesgue dominated convergence, we see that (III) holds for every p, $0 < p < \infty$.

$(\mathrm{III}) \to (\mathrm{I}^1)$

If $F_n \to F$ in the sense of (III), then $\bar{F}_n \to F$ in this sense for every subsequence $\bar{F}_n$ of F_n. By the F. Riesz theorem there is a subsequence $\bar{\bar{F}}_n$ of $\bar{F}_n$ such that $\bar{\bar{F}}_n \to F$ almost everywhere and thus on a dense set $D \subset R^k$. In view of the equivalence $(\mathrm{I}^1) \leftrightarrow \leftrightarrow (\mathrm{I}^2)$, we have $\bar{F}_n \to F$ on $C(F)$ and it remains to use the Urysohn condition.

$(\mathrm{I}^1) \to (\mathrm{IV}_1^1)$

First consider a rectangle $I = J \times K$, where $J \subset R^m, K \subset R^{k-m}$, such that F is continuous at each point of its boundary ∂I, i.e., $P(\partial I) = 0$. Given $\varepsilon > 0$ find a partition $J_1, \ldots, J_s$ of J and points x_i in $I_i = J_i \times K$ such that $\partial(I_i) = 0$ and

$$|\varphi(x^m) - (x_i^m)| < \tfrac{1}{4}\varepsilon \quad (x \in I_i)$$

for $i = 1, \ldots, s$. Since, by (I^1), $F_n(x) \to F(x)$ at each vertex of rectangle I_i, we have

$$\left|\int_I \varphi(x^m)\,\mathrm{d}F_n(x) - \int_I \varphi(x^m)\,\mathrm{d}F(x)\right| < \tfrac{1}{2}\varepsilon + \sum_{i=1}^{s} \varphi(x_i^m)\,|\Delta_{I_i}F_n - \Delta_{I_i}F| < \varepsilon$$

for sufficiently large n. Hence

$$(1) \qquad \int_U \varphi(x^m)\,\mathrm{d}F_n(x) \to \int_U \varphi(x^m)\,\mathrm{d}F(x)$$

for every finite union U of rectangles I in R^k such that $P(\partial I) = 0$.

Take a point t in R^{k-m} such that F is continuous at (s, t) for each $s \in R^m$ and denote $M = \sup\{|\phi(y)| : y \in R^m\}$. There exists U as above such that

$$(2) \qquad |P(H_t) - P(U)| < \frac{\varepsilon}{4M} \quad \text{and} \quad |P_n(H_t) - P_n(U)| < \frac{\varepsilon}{4M}$$

for sufficiently large n. The second inequality follows from the first one, because $P_n(A) \to P(A)$ for $A = H_t$ and $A = U$. In view of (1) and (2), we have

$$\left|\int_{H_t} \varphi(x^m)\,\mathrm{d}F_n(x) - \int_{H_t} \varphi(x^m)\,\mathrm{d}F(x)\right| \leqq \int_{H_t \setminus U} |\varphi(x^m)|\,\mathrm{d}F_n(x) + \int_{H_t \setminus U} |\varphi(x^m)|\,\mathrm{d}F(x) + $$
$$+ \left|\int_U \varphi(x^m)\,\mathrm{d}F_n(x) - \int_U \varphi(x^m)\,\mathrm{d}F(x)\right| < \varepsilon$$

for sufficiently large n, which proves the implication.

$(\mathrm{II}_5) \to (\mathrm{I}^1)$ and $(\mathrm{IV}_5^2) \to (\mathrm{I}^1)$

Note that the convergences (II_5) and (IV_5^2) have unique limits.

To show it for (IV_5^2) suppose that $F_n \to F$ and $F_n \to G$ in this sense and $F(x_0) \neq G(x_0)$ for some $x_0 = (y_0, z_0)$ in $C = C(F) \cap C(G)$, where $y_0 \in R^m$ and $z_0 \in R^{k-m}$. We can assume that $F(x_0) > G(x_0) + 2\varepsilon$ for some $\varepsilon > 0$. Clearly, there exist rectangles $I = [a_0, y_0) \times [b_0, z_0)$ and $J = [a_1, y_1) \times [b_0, t)$, where $a_1 < a_0$, $y_0 < y_1$, $z_0 < t$ and $t \in D_{k-m}$, having the following properties: $P(\partial I) = 0$, $\Delta_I F >$

$> \Delta_I G + \varepsilon$ and

$$\int_{J \setminus I} dG(x) < \varepsilon .$$

Let φ be a smooth function such that $0 \leqq \varphi(y) \leqq 1$ for all $y \in R^m$, $\varphi(y) = 1$ for $y \in [a_0, y_0)$ and $\varphi(y) = 0$ for $y \notin [a_1, y_1)$. Then

$$\int_{H_t} \varphi(x^m) \, dF(x) > \int_I dF(x) > \int_I dG(x) + \varepsilon > \int_{H_t} \varphi(x^m) \, dG(x) ,$$

which contradicts the supposition. Thus $F = G$ on C and, by the continuity of F and G from the left, $F = G$ on R^k. In a similar way one can show that the convergence (II_5) has unique limits.

Assume that $F_n \to F$ in the sense of (II_5) or (IV_5^2) and that $\bar{F}_n$ is an arbitrary subsequence of F_n. Since the sequence $\bar{F}_n$ is uniformly bounded, we can select its subsequence $\bar{\bar{F}}_n$ which converges to some p.d.f. G on a countable dense set D in R^k, i.e. $\bar{\bar{F}}_n \to G$ in the sense of (I^2) as well as (II_5) and (IV_5^2), by the implications already proved. Also $\bar{\bar{F}}_n \to F$ in the sense of (II_5) or (IV_5^2), according to the assumption. The uniqueness shown above yields $F = G$. Hence $\bar{\bar{F}}_n \to F$ in the sense of (I^1). Consequently, $F_n \to F$ in the sense of (I^1), by the Urysohn condition.

All the implications in the diagram are now proved, so the proof of Theorem 1 is finished.

Remark 1. We have omitted proofs of the equivalences in the chain $(I^2) \leftrightarrow (I^1) \leftrightarrow \leftrightarrow (II_1) \leftrightarrow (II_2)$ which result from general theorems on convergence of probability distributions in metric spaces given in [2], § 2, and some obvious properties of p.d.f.'s in R^k. However direct proofs can be easily obtained. Implications $(I^1) \to (I^2)$ and $(II_2) \to (II_1) \to (II_5)$ are trivial and implication $(I^2) \to (II_2)$ follows in a similar way to that demonstrated in the proof of implication $(I^1) \to (IV_1^1)$ (cf. [3], § 9). For completing the chain, it remains to add implication $(II_5) \to (I^1)$ in which we have encountered condition (U). We encounter it also if implication $(III) \to (I^1)$ is replaced by the fact that the distributional convergence in (II_5) is more general than the local convergence in L^p and again by implication $(II_5) \to (I^1)$. The method used in the proof of $(II_5) \to (I^1)$ and $(IV_5^2) \to (I^1)$ was presented in a very general setting in [8].

Remark 2. It is evident that the set H_t in all conditions (IV) can be replaced equivalently by the set

$$(-\infty, \infty) \times \ldots \times (-\infty, \tau_1) \times \ldots \times (-\infty, \tau_k) \times \ldots \times (-\infty, \infty) .$$

3. A measure μ on the σ-algebra of all Borel sets in $X \times X$, where $X = [0, \infty)$ or $X = [0, 1]$, is doubly stochastic (d.s.m.) if

$$\mu(A \times X) = \mu(X \times A) = l(A)$$

for every Borel set $A \subset X$, where l is the Lebesgue measure on X.

The function defined by the formula

$$C(x, y) = \mu(\{s, t) \in X \times X : s < x, \ t < y\}) \quad (x, y \in X)$$

is called the copula corresponding to the d.s.m. μ on $X \times X$.

The family of copulas is equicontinuous which follows from the following inequality:

$$C(x_2, y_2) - C(x_1, y_1) \leqq \mu(\{s, t): x_1 \leqq s \leqq x_2\}) + \mu(\{(s, t): y_1 \leqq t \leqq y_2\}) =$$
$$= (x_2 - x_1) + (y_2 - y_1)$$

for arbitrary points (x_1, y_1) and (x_2, y_2) of $X \times X$ such that $x_1 < x_2$ and $y_1 < y_2$.

Observe that copulas C in $X \times X$ can be identified with the following p.d.f.'s F on $X \times X$. We put $F = C$ on $X \times X$ and $F(x, y) = 0$ if $x < 0$ or $y < 0$ in both considered cases. Moreover, if $X = [0, 1]$, we put $F(x, y) = 1$ if $x > 1$, $y \geqq 0$ or $x \geqq 0$, $y > 1$.

One can derive the following corollary from Theorem 1:

Theorem 2. *For arbitrary copulas C_n $(n = 1, 2, \ldots)$ and C on $X \times X$, where $X = [0, \infty)$ or $X = [0, 1]$, the following conditions are equivalent:*

(a) $C_n(x, y) \to C(x, y)$ *pointwise;*

(b) $C_n(x, y) \to C(x, y)$ *on a dense set in $X \times X$;*

(c) $\|C_n - C\|_{L^p([0,a]\times[0,b])} \to 0$ $(0 < p \leqq \infty)$
for arbitrary intervals $[0, a]$ and $[0, b]$ in X;

(d) $\int_{X\times X} \varphi(x, y)\, dC_n(x, y) \to \int_{X\times X} \varphi(x, y)\, dC(x, y)$
for each function φ on $X \times X$ of one of the classes $\mathscr{C}_b, \mathscr{C}_u, \mathscr{E}_b, \mathscr{C}_0, \mathscr{D}$;

(e) $\int_{X\times[0,t]} \psi(x)\, dC_n(x, y) \to \int_{X\times[0,t]} \psi(x)\, dC(x, y)$
and/or

$$\int_{[0,t]\times X} \psi(x)\, dC_n(x, y) \to \int_{[0,t]\times X} \psi(x)\, dC(x, y)$$

for each function ψ on X of one of the classes $\mathscr{C}_b, \mathscr{C}_u, \mathscr{E}_b, \mathscr{C}_0, \mathscr{D}$ and for each $t \in X$ (for each t from a dense set in X).

Proof. The equivalence of conditions (a), (b), (d), (e), and condition (c) for p, $0 < p < \infty$, follows directly from Theorem 1 and Remark 2. For $p = \infty$, the equivalence of (c) and (a) is a consequence of equicontinuity of the sequence of copulas C_n.

Remark 3. If $X = [0, 1]$, we obviously have $\mathscr{C}_b = \mathscr{C}_u = \mathscr{C}_0$ and $\mathscr{E}_b = \mathscr{D}$ in conditions (d) and (e).

Remark 4. For $X = [0, \infty)$, condition (e) means the convergence of the respective d.s.m.'s in the Peck topology (cf. [9]).

4. In the Rényi theory of conditional probability spaces, unbounded probability distributions and unbounded p.d.f.'s are introduced (see [10], p. 245–254, and [5]). As a matter of fact, a Rényi p.d.f. is not a single function but a class of all nontrivial nondecreasing functions F on R^k (i.e. $\Delta_I F \geqq 0$ for every rectangle $I \subset R^k$), continuous from the left (in each variable), which differ from each other by the sum of functions $F_i(x_i) = F_i(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_k)$ $(i = 1, \ldots, k)$ of $k - 1$ variables. In other words, F and G belong to the same class if $\Delta_I F = \Delta_I G$ for every rectangle $I \subset R^k$ (see [5]).

Denote such a class, if it contains a function F, nondecreasing and continuous from the left, by $[F]$.

We can introduce a metric on the space $\mathscr{F}_R$ of all Rényi p.d.f.'s in the following way. Fix a sequence of rectangles I_m whose union is R^k and put for any p, $0 < p < \infty$,

$$d_p([F], [G]) = \sum_{m=1}^{\infty} 2^{-m} \max(1, d_{p,m}([F], [G])),$$

where

$$d_{p,m}([F], [G]) = \inf\{\|\tilde{F} - \tilde{G}\|_{L^p(I_m)}: \tilde{F} \in [F],\ \tilde{G} \in [G]\}.$$

It is easy to see that d_p is a metric on $\mathscr{F}_R$.

Given a Rényi p.d.f. $[F]$ and a rectangle $I = [a, b) = [\alpha_1, \beta_1) \times \ldots \times [\alpha_k, \beta_k)$ in R^k, define the function F_I on R^k:

$F_I(x) = F_I(\xi_1, \ldots, \xi_k) = 0$ if $\xi_i < \alpha_i$ for some $i = 1, \ldots, k$; $F_I(x) = 1$ if $\xi_i > 1$ and $\xi_j \geqq 0$ $(j \neq i)$ for some $i = 1, \ldots, k$; and

$$F_I(x) = \Delta_{[a,x)} F / \Delta_{[a,b)} F \quad \text{for} \quad x \in I = [a, b).$$

Clearly, F_I is a usual p.d.f. on R^k and the definition does not depend on the choice of a representant of the class $[F]$.

Given Rényi p.d.f.'s $[F_n]$ and $[F]$, consider the following conditions for $r = 1, 2$ and $s = 4, 5$:

(i^r) $F_{n,I} \to^w F_I$ for every rectangle I in R^k whose boundary points are in the set considered in condition (I^r);

(ii^r) $\dfrac{\Delta_I F_n}{\Delta_J F_n} \to \dfrac{\Delta_I F}{\Delta_J F}$ for arbitrary rectangles I, J in R^k whose boundary points are in the set considered in (I^r);

(iii^r) there exist representants $\tilde{F}_n$ and $\tilde{F}$ of $[F_n]$ and $[F]$, respectively, such that $\tilde{F}_n(x) \to \tilde{F}(x)$ for each x of the set considered in (I^r);

(iv_s) there exist constants $\alpha_n > 0$ such that

$$\alpha_n \int_{R^k} \varphi(x)\, dF_n(x) \to \int_{R^k} \varphi(x)\, dF(x)$$

for each function φ of the class considered in (II_s);

(v) $d_p([F_n], [F]) \to 0$ $(0 < p < \infty)$.

Theorem 3. *Let* $[F_n]$ $(n = 1, 2, \ldots)$ *and* $[F]$ *be Rényi p.d.f.'s on* R^n. *Then all the conditions* (i^r), (ii^r), (iii^r), (iv_s) *and* (v) *for* $r = 1, 2$ *and* $s = 4, 5$ *are equivalent.*

Proof. The equivalence of conditions (i^r), (ii^r), (iii^r) and (iv_s) for $r = 1, 2$ and $s = 4, 5$ in the one-dimensional case is proved in [5]. The proof in the case $k > 1$ is analogous. Similarly as in the case of equivalence (I^1) $\leftrightarrow$ (III), we prove that (iii^2) $\leftrightarrow$ (iii^1) $\leftrightarrow$ (v).

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