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DEPENDENCES BETWEEN DEFINITIONS OF FINITENESS

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1. INTRODUCTION

To taste the fine structure of various finiteness-like notions we deal with Zermelo-Fraenkel set theory **ZF** without the (somewhat supercilious) Axiom of Choice under which the eight historical definitions of finiteness of sets are equivalent.

Apparently, the study of finiteness dates back to R. Dedekind, A. N. Whitehead, B. Rusell, A. Tarski, W. Sierpiński, A. Mostowski and others. More details about the history can be found in [12, Ch. III § 9] and [7]. Surveys are given in [12], [8, Ch. III & V], [9-life without Choice] and [5].

In this section we give the basic definitions, provide the background information and state the main results (Theorem 1.1, Theorem 1.2). The proofs will be given in Section 2. In Section 3 we discuss various applications of our main theorems and we investigate the notion of compactness in topology and logic based on the particular definition of finiteness. In general, our notation is the standard set-theoretic one, see e.g. [8].

Let us recall almost verbally the definitions of finiteness from [7] ((ii) and (viii) are due to A. Levy, (v) to R. Dedekind and the rest is due to A. Tarski).

Definition 1. A set A is said to be

(i) ***I**-finite* if every nonvoid family of subsets of A has an $\subseteq$-maximal element,

(ii) ***I**$_a$-finite* if it is not the union of two disjoint sets neither of which is finite according to definition ***I***,

(iii) ***II**-finite* if every non-void $\subseteq$-monotone family has a $\subseteq$-maximal element,

(iv) ***III**-finite* if the power set of A is irreflexive (i.e., there is no one-to-one mapping of $\mathscr{P}(A)$ onto a proper subset of $\mathscr{P}(A)$),

(v) ***IV**-finite* if it is irreflexive,

(vi) ***V**-finite* if $|A| = 0$ or $2 . |A| > |A|$,

(vii) ***VI**-finite* if $|A| = 0$ or $|A| = 1$ or $|A|^2 > |A|$,

(viii) ***VII**-finite* if A is ***I***-finite or A is not well-orderable.

The results contained in this paper were subsequently presented at the Czechoslovak Combinatorial Conference (June 1983, Šírava), Winter School on Abstract Analysis (February 1984, Srní) and at the Student Research Competition (1982, 1983, 1984, Šafárik University Košice, prepared by the first author under the guidance of the second author).

For the reader's convenience we present some equivalent definitions. A set A is: (i) $\boldsymbol{I}$-finite iff it is an one-to-one image of some natural number iff $|A| < \aleph_0$; (ii) $\boldsymbol{I_a}$-finite iff the Frèchet filter $\{B \subseteq A: |A - B| < \aleph_0\}$ is an ultrafilter; (iii) $\boldsymbol{III}$-finite iff $2^{|A|} + 1 > 2^{|A|}$ iff $2^{|A|} \ngeq \aleph_0$; (iv) $\boldsymbol{IV}$-finite iff $|A| + 1 > |A|$ (i.e. the pidgeon hole principle holds for A) iff $|A| \ngeq \aleph_0$.

To simplify the notation, let $\boldsymbol{F}, \boldsymbol{F_i}$ vary over $\boldsymbol{I}, \boldsymbol{I_a}, \boldsymbol{II}, \boldsymbol{III}, \boldsymbol{IV}, \boldsymbol{V}, \boldsymbol{VI}, \boldsymbol{VII}$ (in § 2 also over the symbols of derived definitions of finiteness). A set A is said to be ***F-infinite*** if it is not $\boldsymbol{F}$-finite and $\boldsymbol{F_1} \to \boldsymbol{F_2}$ abbreviates the following formula in the language of **ZF**(**ZFU**): $(\forall A)$ (A is $\boldsymbol{F_1}$-finite $\to$ A is $\boldsymbol{F_2}$-finite). In literature, $\boldsymbol{I}$-finite sets are also called *Tarski-finite*, $\boldsymbol{IV}$-finite are called *Dedekind-finite*, $\boldsymbol{V}$-infinite sets are known as *idemmultiple* and $\boldsymbol{VI}$-infinite sets are called *idempotent*. Denote by $\mathbf{J}_{\boldsymbol{F}}^{\mathfrak{M}}$ the (proper) class of all $\boldsymbol{F}$-finite sets in a model $\mathfrak{M}$, i.e. $\mathbf{J}_{\boldsymbol{F}}^{\mathfrak{M}} = \{A \in \mathfrak{M}: \mathfrak{M} \models A \text{ is } \boldsymbol{F}\text{-finite}\}$; we omit $\mathfrak{M}$ if there is no danger of misunderstanding.

Observe that all proofs (and hence all statements) through-out this paper are proofs (and statements) in both **ZF** and **ZFU** theories.

Theorem (A. Levy [7]). *If a set is finite according to any of the above definitions, it is finite also according to any definition which follows it.* In symbols

$$\boldsymbol{I} \to \boldsymbol{I_a} \to \boldsymbol{II} \to \boldsymbol{III} \to \boldsymbol{IV} \to \boldsymbol{V} \to \boldsymbol{VI} \to \boldsymbol{VII},$$

or

$$\mathbf{J}_{\boldsymbol{I}} \subseteq \mathbf{J}_{\boldsymbol{I_a}} \subseteq \mathbf{J}_{\boldsymbol{II}} \subseteq \mathbf{J}_{\boldsymbol{III}} \subseteq \mathbf{J}_{\boldsymbol{IV}} \subseteq \mathbf{J}_{\boldsymbol{V}} \subseteq \mathbf{J}_{\boldsymbol{VI}} \subseteq \mathbf{J}_{\boldsymbol{VII}}.$$

It is known (see [7]) that in **ZFU** all these definitions are independent (Mostowski, Lindenbaum) and in **ZFU** + **OP** (Ordering Principle) we have $\boldsymbol{I} \equiv \boldsymbol{II}$ and all others are independent (Doss, Levy). In **ZF**, the independence of all these definitions was shown in [6] (Jech, Sochor).

Easily, **AC** is equivalent to the following statement "all $\boldsymbol{I}$-infinite sets are well-orderable", i.e., to $(\boldsymbol{I} \equiv \boldsymbol{VII})$. A. Tarski in [14] has shown that the Axiom of Choice is equivalent to "all $\boldsymbol{I}$-infinite cardinals are idempotent", i.e., to $(\boldsymbol{I} \equiv \boldsymbol{VI})$. In 1924 A. Tarski asked whether **AC** is equivalent to "all $\boldsymbol{I}$-infinite cardinals are idemmultiple". The problem was known as "idemmultiple hypothesis" and has been raised on a number of occasions. It was solved in the negative by J. D. Halpern and P. E. Howard ([3]) in **ZFU** and G. Sageev ([10]) in **ZF**. Summarized in symbols:

$$\mathbf{AC} \equiv (\boldsymbol{I} \equiv \boldsymbol{VII}) \equiv (\boldsymbol{I} \equiv \boldsymbol{VI}) \rightleftarrows\!\!\!\!\!\!/\;\; (\boldsymbol{I} \equiv \boldsymbol{V}).$$

Our first theorem provides another result in this area. In particular, we prove that **AC** is equivalent to "every idemmultiple cardinal is idempotent" and equivalent to "every idempotent cardinal is well-orderable"; in symbols:

$$\mathbf{AC} \equiv (\boldsymbol{VI} \equiv \boldsymbol{VIII}) \equiv (\boldsymbol{V} \equiv \boldsymbol{VI}).$$

Note that this is a strengthening of the above mentioned result by Tarski.

Theorem 1.1. *The following assertions are equivalent*:

(i) **AC**;

(ii) $(\forall \mathfrak{a})(2\mathfrak{a} = \mathfrak{a} \to \mathfrak{a}^2 = \mathfrak{a})$;

(iii) $(\forall \mathfrak{a})(\mathfrak{a}^2 = \mathfrak{a} \to \mathfrak{a}$ *is well-orderable*$)$.

To formulate our next result we introduce the following notation: $\overset{*}{=}$ means syntactically (graphically) equal, $\psi_1 \overset{*}{=} (\boldsymbol{I} \equiv \boldsymbol{I_a})$, $\psi_2 \overset{*}{=} (\boldsymbol{I_a} \equiv \boldsymbol{II})$, $\psi_3 \overset{*}{=} (\boldsymbol{II} \equiv \boldsymbol{III})$, $\psi_4 \overset{*}{=} (\boldsymbol{III} \equiv \boldsymbol{IV})$, $\psi_5 \overset{*}{=} (\boldsymbol{IV} \equiv \boldsymbol{V})$, $\psi_6 \overset{*}{=} (\boldsymbol{V} \equiv \boldsymbol{VI})$ and $\psi_7 \overset{*}{=} (\boldsymbol{VI} \equiv \boldsymbol{VII})$, For $\delta \in \{0, 1\}$ we define ψ^δ as follows: $\psi^1 \overset{*}{=} \neg\psi$, $\psi^0 \overset{*}{=} \psi$. So for each $\varepsilon = \{\delta_1, \dots, \delta_7\} \in \{0, 1\}^7$ the statement

$$\varphi_\varepsilon \overset{*}{=} \bigwedge_{i=1}^{7} \psi_i^{\delta_i}$$

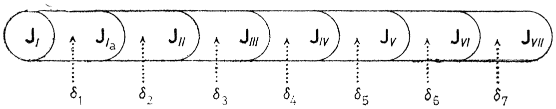
describes "the simultaneous occurrence of different types of finite sets".

Conversely to every model $\mathfrak{M}$ of **ZF**(**ZFU**) we assign a vector $\varepsilon(\mathfrak{M}) = (\delta_1, \dots, \delta_7)$ by putting

$$\delta_1 = 0 \quad \text{whenever} \quad \mathfrak{M} \models \psi_1 \quad (\text{iff } \mathbf{J}_I^{\mathfrak{M}} = \mathbf{J}_{I_a}^{\mathfrak{M}})$$

$$\delta_2 = 0 \quad \text{whenever} \quad \mathfrak{M} \models \psi_2 \dots$$

The situation can be illustrated by the following scheme:



Theorem 1.2. *For*

$$\begin{aligned}
\varepsilon_1 &= (0, 0, 0, 0, 0, 0, 0)\,,\\
\varepsilon_2 &= (0, 0, 0, 0, 0, 1, 1)\,,\\
\varepsilon_3 &= (0, 0, 0, 0, 1, 1, 1)\,,\\
\varepsilon_4 &= (0, 0, 0, 1, 1, 1, 1)\,,\\
\varepsilon_5 &= (0, 0, 1, 1, 1, 1, 1)\,,\\
\varepsilon_6 &= (0, 1, 0, 1, 1, 1, 1)\,,\\
\varepsilon_7 &= (0, 1, 1, 1, 1, 1, 1)\,,\\
\varepsilon_8 &= (1, 1, 0, 1, 1, 1, 1)\,,\\
\varepsilon_9 &= (1, 1, 1, 1, 1, 1, 1)\,,
\end{aligned}$$

the following statement is provable both in **ZF** *and* **ZFU**:

$$\varphi_{\varepsilon_1} \vee \varphi_{\varepsilon_2} \vee \varphi_{\varepsilon_3} \vee \varphi_{\varepsilon_4} \vee \varphi_{\varepsilon_5} \vee \varphi_{\varepsilon_6} \vee \varphi_{\varepsilon_7} \vee \varphi_{\varepsilon_8} \vee \varphi_{\varepsilon_9}\,.$$

Observe, that Theorem 1.2 states that for every model $\mathfrak{M}$ of **ZF** (or **ZFU**), $\varepsilon(\mathfrak{M}) \in \{\varepsilon_1, \dots, \varepsilon_9\}$, i.e., for every $\mathfrak{M}$ it suffices to consider only 9 of the possible $2^7 = 128$ statements of the form $\bigwedge_{i=1}^{7} \psi_i^{\delta_i}$; no other statement of this form can be valid. Clearly, ε_1 refers to models of **AC**. According to Theorem 1.2 the models constructed by Halpern & Howard and Sageev respectively are of type ε_2. While

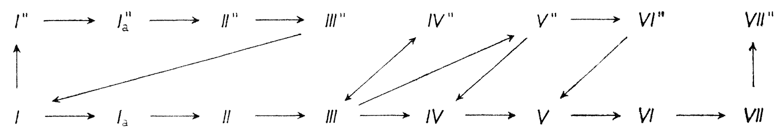
Tarski's result states that $\varepsilon(\mathfrak{M}) = (0, 0, 0, 0, 0, 0, 1)$ cannot occur, Theorem 1.1 excludes the occurrence of models for which $\min(\delta_6, \delta_7) = 0$ and simultaneously $\max\limits_{i=1,\ldots,7} (\delta_i) = 1$, i.e., it excludes further 94 possibilities. Theorem 1.2 helps us to exclude the remaining 24 possibilities.

2. CLASSIFICATION OF DERIVED DEFINITIONS OF FINITENESS

Recall that Tarski's definition of ***III***-finiteness is derived from the Dedekind's definition of ***IV***-finiteness via the power set operation. In the same way (using the power set operation) for each ***I***-, $\boldsymbol{I_a}$-, ***II***-, ***III***-, ***IV***-, ***V***-, ***VI***-, ***VII***-finiteness we derive the definitions of $\boldsymbol{I''}$-, $\boldsymbol{I''_a}$-, ..., $\boldsymbol{VII''}$-finiteness. We shall investigate the relationships between all these notions (Theorem 2.1). This method, although simple, appears to be very fruitful.

Definition 2. Let $\boldsymbol{F}$ vary over $\boldsymbol{I}, \boldsymbol{I_a}, \boldsymbol{II}, \boldsymbol{III}, \boldsymbol{IV}, \boldsymbol{V}, \boldsymbol{VI}, \boldsymbol{VII}$. We say that a set A is $\boldsymbol{F''}$*-finite* if $\mathscr{P}(A)$ is $\boldsymbol{F}$-finite.

Theorem 2.1. *In* ZF (*or* ZFU) *the following implications are provable*:



Proof. $\boldsymbol{III''} \to \boldsymbol{I}$ is due to Tarski [15], who proved that if A is $\boldsymbol{I}$-infinite then $\mathscr{P}\mathscr{P}(A)$ contains a denumerable set. By definition, $\boldsymbol{III} \leftrightarrow \boldsymbol{IV''}$. As $2.2^{\mathfrak{a}} \geqq 2^{\mathfrak{a}} + 1 > 2^{\mathfrak{a}}$, we have $\boldsymbol{III} \to \boldsymbol{V''}$. We prove $\boldsymbol{V''} \to \boldsymbol{IV}$ indirectly. Assuming $\mathfrak{a} \geqq \mathfrak{a} + 1$, we have $2^{\mathfrak{a}} \geqq 2^{\mathfrak{a}+1} = 2 \,.\, 2^{\mathfrak{a}}$ and the assertion follows. Similarly, if $2\mathfrak{a} \leqq \mathfrak{a}$, then $(2^{\mathfrak{a}})^2 = 2^{2\mathfrak{a}} \leqq 2^{\mathfrak{a}}$ proves $\boldsymbol{VI''} \to \boldsymbol{V}$. Implication $\boldsymbol{VII} \to \boldsymbol{VII''}$ follows from the following observation. Each set A is isomorphic to $\{x \in \mathscr{P}(A): |x| = 1\}$ which is a definable part of $\mathscr{P}(A)$. So, if $\mathscr{P}(A)$ is well-ordered then A is well-ordered too.

Problem 2.1. (i) *We know that* $\boldsymbol{III} \to \boldsymbol{V''} \to \boldsymbol{IV}$ *and* $\boldsymbol{IV} \nrightarrow \boldsymbol{III}$. *Is it true that* $\boldsymbol{V''} \to \boldsymbol{III}$ (*i.e.* $\boldsymbol{III} \leftrightarrow \boldsymbol{V''}$) *or* $\boldsymbol{IV} \to \boldsymbol{V''}$ (*i.e.* $\boldsymbol{IV} \leftrightarrow \boldsymbol{V''}$)?

(ii) *Similarly, we know that* $\boldsymbol{III} \to \boldsymbol{VI''} \to \boldsymbol{V}$ *and we ask which one, if any, of the following three statements* $\boldsymbol{VI''} \to \boldsymbol{III}$, $\boldsymbol{VI''} \to \boldsymbol{IV}$, $\boldsymbol{VI''} \leftarrow \boldsymbol{V}$ *holds true.*

Now we prove several lemmas of the type "if $\boldsymbol{F_1} \to \boldsymbol{F_2}$ then $\boldsymbol{F_3} \to \boldsymbol{F_4}$", or, using "$\delta$-notation" from introduction, of the type "$\delta_i = 0 \to \delta_j = 0$". Proofs can be often better visualized if looking and "walking" through the diagram from Theorem 2.1 just using the fact that if $\boldsymbol{F_1} \to \boldsymbol{F_2}$ then also $\boldsymbol{F''_1} \to \boldsymbol{F''_2}$. The next lemma is a slight generalization of the nontrivial implication in Tarski's Theorem ([14]; see [8, Proposition V. 1.14]) which states $(\boldsymbol{VI} \to \boldsymbol{I}) \to \mathbf{AC}$ (i.e. if $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \delta_5 = \delta_6 = 0$ then $\delta_7 = 0$).

Lemma 2.2. *If every* ***VI****-finite set is* ***V****-finite, then every* ***VII****-finite set is* ***VI****-finite* (i.e. $\delta_6 = 0 \rightarrow \delta_7 = 0$).

Proof. We follow Tarski's original idea. Take an idemmultiple cardinal $\mathfrak{a}$. Since $\mathfrak{a}^+$ is a cardinal number greater then $\aleph_0$ (recall that $\mathfrak{a}^+$ denotes Hartog's number, i.e., the least ordinal β such that $\beta \nleqq \mathfrak{a}$), $\mathfrak{a} + \mathfrak{a}^+$ is also idemmultiple. Thus (without **AC**) $2\mathfrak{a}^+ = \mathfrak{a}^+$ and also $2(\mathfrak{a} + \mathfrak{a}^+) = 2\mathfrak{a} + 2\mathfrak{a}^+ = \mathfrak{a} + \mathfrak{a}^+$. Now we literally follow Tarski (see [8]). By the assumption, $\mathfrak{a} + \mathfrak{a}^+$ is idempotent, i.e.,

$$(\mathfrak{a} + \mathfrak{a}^+)^2 = \mathfrak{a}^2 + 2 \,.\, \mathfrak{a} \,.\, \mathfrak{a}^+ + (\mathfrak{a}^+)^2 = \mathfrak{a} + \mathfrak{a}^+ \,,$$

so $\mathfrak{a} \,.\, \mathfrak{a}^+ \leqq \mathfrak{a} + \mathfrak{a}^+$. This is known ([8, Lemma V. 1.13]) to imply $\mathfrak{a} \leqq \mathfrak{a}^+$, i.e., $\mathfrak{a}$ is a well-ordered infinite cardinal.

We show that the converse implication (to Lemma 2.2) holds. Namely, Lemma 2.3 yields a stronger statement.

Lemma 2.3. *If every* ***VII****-finite set is* ***VI****-finite, then every* ***VII****-finite set is* ***III****-finite* (*i.e.* $\delta_7 = 0 \rightarrow \delta_6 = \delta_5 = \delta_4 = 0$).

Proof. We use our "diagram". First, we show that $\delta_7 = 0 \rightarrow \delta_6 = 0$. Take an arbitrary ***VI***-finite set A. Then A is ***VII***-finite and so also $\mathscr{P}(A)$ is ***VII***-finite. By the assumption, $\mathscr{P}(A)$ is ***VI***-finite and, using ***VI***$'' \rightarrow$ ***V***, we have that A is ***V***-finite.

Second, we prove $\delta_7 = 0 \rightarrow \delta_5 = 0$. Take a ***V***-finite set A. It is ***VII***-finite, and so is $\mathscr{P}(A)$. From $\delta_7 = 0$ and $\delta_6 = 0$ we know that $\mathscr{P}(A)$ is ***V***-finite. By Theorem 2.1, this implies that A is ***IV***-finite.

Finally, take a ***IV***-finite set A. By the same argument as in previous steps, $A \in \mathbf{J}_{VII} \rightarrow \mathscr{P}(A) \in \mathbf{J}_{VII} \rightarrow \mathscr{P}(A) \in \mathbf{J}_{IV} \leftrightarrow A \in \mathbf{J}_{III}$, i.e., $\delta_4 = 0$.

The next lemma is a simple consequence of another Tarski's result ([15], see [8, Proposition III. 1.28]); it states that if A is ***I***-infinite then $\mathscr{P}(A)$ is ***IV***-infinite.

Lemma 2.4. *If every* ***IV****-finite set is* ***III****-finite, then every* ***IV****-finite set is* ***I****-finite* (*i.e.* $\delta_4 = 0 \rightarrow \delta_3 = \delta_2 = \delta_1 = 0$).

Proof. Take an ***IV***-finite set A. By the assumption it is ***III***-finite, i.e. $\mathscr{P}(A)$ is ***IV***-finite. So again, $\mathscr{P}(A)$ is ***III***-finite. This means precisely that $\mathscr{P}\mathscr{P}(A)$ is ***IV***-finite. By Tarski's observation ([8, III, 1.28]), this means that A is ***I***-finite.

Now we are ready to prove Theorem 1.1 and Theorem 1.2, already formulated in the introduction.

Proof of Theorem 1.1. It is obvious that **AC** implies both (i) and (ii). Lemma 2.2 says that (i) implies (ii). Lemma 2.3 says that $\delta_7 = 0$ implies $\delta_6 = \delta_5 = \delta_4 = 0$ and lemma 2.4 guarantees the rest, i.e., $\delta_4 = 0$ implies $\delta_3 = \delta_2 = \delta_1 = 0$.

Proof of Theorem 1.2. Searching for possible values of ε represented by models of **ZF** (or **ZFU**), we observe that Theorem 1.1 forbids all possibilities with $\max(\delta_i) = 1$ and $\min\{\delta_6, \delta_7\} = 0$, i.e., 95 possibilities (one of them, namely $(0, 0, 0, 0, 0, 0, 0, 1)$ is forbidden by the last Tarski's theorem). We have still 33 possible ε's. Moreover, Lemma 2.4 forbids all possibilities with $\max\{\delta_1, \delta_2, \delta_3\} =$

$= 1$, $\delta_4 = 0$, $\delta_6 = \delta_7 = 1$ and δ_5 arbitrary; this forbids another 14 ε's. We still have 19 possibilities. The next lemma takes us still further.

Lemma 2.5. *If every **II**-finite set is I_a-finite, then every I_a-finite set is **I**-finite* (*i.e.* $\delta_2 = 0 \rightarrow \delta_1 = 0$).

Proof. Contrariwise, assume $A \in \mathbf{J}_{I_a} - \mathbf{J}_I$. Now $\mathbf{J}_{I_a} \subseteq \mathbf{J}_{II}$ implies $A \in \mathbf{J}_{II}$. Then $2.A$ is also ***II***-finite (check the definition) but, obviously, $2.A$ is not I_a-finite.

Lemma 2.5 forbids the following 4 possibilities, $(1, 0, \delta_3, 1, \delta_5, 1, 1)$, $\delta_3, \delta_5 \in \in \{0, 1\}$. So, it remains to check 15 possibilities.

Our final estimate is based on Lemma 2.6; the proof appears in [13]. After having submitted [13] we learned that the same statement has been already proved by Halpern and Howard in [2].

Lemma 2.6. *If every **V**-finite set is **IV**-finite, then every **IV**-finite set is **I**-finite* (*i.e.* $\delta_5 = 0 \rightarrow \delta_4 = \delta_3 = \delta_2 = \delta_1 = 0$).

This lemma forbids (from what was still left) $\varepsilon = (\delta_1, \delta_2, \delta_3, 1, 0, 1, 1)$, where $\delta_3 \in \{0, 1\}$ and either $\delta_1 = \delta_2 = 0$, or $\delta_1 = 1 \; \& \; \delta_2 = 0$, or $\delta_1 = \delta_2 = 1$, i.e., 6 possibilities. Altogether, 9 possibilities are left – and this is at present our best achievement. It is easy to check that the remaining nine ε's are the following ones:

$$\begin{aligned}
\varepsilon_1 &= (0, 0, 0, 0, 0, 0, 0)\,, \\
\varepsilon_2 &= (0, 0, 0, 0, 0, 1, 1)\,, \\
\varepsilon_3 &= (0, 0, 0, 0, 1, 1, 1)\,, \\
\varepsilon_4 &= (0, 0, 0, 1, 1, 1, 1)\,, \\
\varepsilon_5 &= (0, 0, 1, 1, 1, 1, 1)\,, \\
\varepsilon_6 &= (0, 1, 0, 1, 1, 1, 1)\,, \\
\varepsilon_7 &= (0, 1, 1, 1, 1, 1, 1)\,, \\
\varepsilon_8 &= (1, 1, 0, 1, 1, 1, 1)\,, \\
\varepsilon_9 &= (1, 1, 1, 1, 1, 1, 1)\,.
\end{aligned}$$

In [13] it is shown that there are models of **ZF** (or **ZFU**) such that $\varepsilon(\mathfrak{M})$ attains at least 5 of the remaining 9 possible ε's. This gives an estimate from below.

3. DISCUSSION, PROBLEMS AND REFLECTIONS

3.1. Finer classification of Dedekind finiteness. It is a common feeling that definitions up to ***IV*** are good approximations of finiteness; the other definitions (***V***, ***VI***, ***VII***) are technical tools only for studying the universe of non-well-orderable sets. Further, ***VII*** is an upper bound for any future reasonable generalization of finiteness. Our classification from Theorem 1.2 restricts simultaneous occurrence of various Dedekind-finite sets. Denote by $\bar{\varepsilon}$ the obvious modification of ε for Dedekind-finite sets. For 5 considered definitions ***I***, I_a, ***II***, ***III***, ***IV*** there are $2^4 = 16$ types of $\bar{\varepsilon}$'s possible, but our theorem allows at most 7 of them, namely:

$$\bar{\varepsilon}_1 = (0, 0, 0, 0)\,,$$
$$\bar{\varepsilon}_2 = (0, 0, 0, 1)\,,$$
$$\bar{\varepsilon}_3 = (0, 0, 1, 1)\,,$$
$$\bar{\varepsilon}_4 = (0, 1, 0, 1)\,,$$
$$\bar{\varepsilon}_5 = (0, 1, 1, 1)\,,$$
$$\bar{\varepsilon}_6 = (1, 1, 0, 1)\,,$$
$$\bar{\varepsilon}_7 = (1, 1, 1, 1)\,.$$

Problem 3.1. *What are further restrictions on $\bar{\varepsilon}$'s or which of them do occur in models?*

3.2. Linearly ordered Dedekind cardinals. More information we obtain while concerning the following problem. In notation we follow G. Sageev ([11]): γ_Δ denotes the statement that any two *IV*-finite cardinals are comparable. Sageev constructed a model of **ZF** + γ_Δ + $\mathbf{J}_I \neq \mathbf{J}_{IV}$. Moreover, J. Truss ([16]) proved that γ_Δ implies $\mathbf{J}_I = \mathbf{J}_{Ia}$. Observe that under γ_Δ & $\mathbf{J}_I \neq \mathbf{J}_{IV}$ only 4 types of simultaneous occurrence of various Dedekind-finite cardinals are possible, namely:

$$\bar{\varepsilon}_2 = (0, 0, 0, 1)\,,$$
$$\bar{\varepsilon}_3 = (0, 0, 1, 1)\,,$$
$$\bar{\varepsilon}_4 = (0, 1, 0, 1)\,,$$
$$\bar{\varepsilon}_5 = (0, 1, 1, 1)\,.$$

This sheds some light to a question by G. Sageev ([11]) which we reformulate as follows.

Problem 3.2. (i) *Does γ_Δ imply one of these possibilities?*
(ii) *Which $\bar{\varepsilon}$ is assigned to Sageev's model?*

3.3. CAC, AD. Note that under **CAC** (Countable Axiom of Choice) and therefore also under **AD** (Axiom of Determinateness) we have $I \equiv IV$ (for every $n \in \omega$, if $A \notin \mathbf{J}_I$ then the set $\{f\colon n \to A; f \text{ is one-to-one}\}$ is nonempty). So, under **CAC** & $\neg$**AC** (or **AD**) the universe admits only the following ε's:

$$\varepsilon_2 = (0, 0, 0, 0, 0, 1, 1)\,,$$
$$\varepsilon_3 = (0, 0, 0, 0, 1, 1, 1)\,.$$

This shows the necessity of finer classification of the universe of non-well-ordered sets, in particular, the finer classification of the realm between *VI*-finiteness and *VII*-finiteness.

3.4. Subsets of reals. The behaviour of subsets of reals without **AC** was studied, e.g., in [4]. It is known that for any linearly ordered set *I*-infinite subsets are *II*-infinite. Moreover, it goes back to Tarski that *I*-infinite subsets of reals are *III*-infinite, already. Assume the nontrivial case, namely, let R be non-well-ordered .As $2\,.\,R \approx$

$\approx R^2 \approx R$, R itself is a *VII*-finite and *VI*-infinite set. Proofs of Theorem 1.1 and Theorem 1.2 can be carried out in any structure closed under the power set operation, but this is not the case of subsets of reals. So, there are 8 possible types of simultaneous occurrence of different types of

$$\eta = (0, 0, 0, \varrho_1, \varrho_2, \varrho_3, 1)$$

of finite subsets of reals and our results yield no restriction. Such questions will be discussed in [13].

3.5. Estimating the iteration of the power set operation. Tarski ([15]) showed that that for $A \notin \mathbf{J}_I$ we have $\mathscr{P}\mathscr{P}(A) \notin \mathbf{J}_{IV}$. From our results (looking at the table of ε's) we can conclude for instance that given $A \in \mathbf{J}_{IV} - \mathbf{J}_{III}$, there are $k_1, k_2, k_3 \in \omega$ such that

$$\mathscr{P}^{k_1}(A) \in \mathbf{J}_V - \mathbf{J}_{IV},$$
$$\mathscr{P}^{k_2}(A) \in \mathbf{J}_{VI} - \mathbf{J}_V,$$
$$\mathscr{P}^{k_3}(A) \in \mathbf{J}_{VII} - \mathbf{J}_{VI}.$$

Let k_1, k_2, k_3 be minimal with respect to this property.

Problem 3.3. *Estimate the values of k_1, k_2, k_3. Do the values of k_1, k_2, k_3 depend on the choice of the set $A \in \mathbf{J}_{IV} - \mathbf{J}_{III}$ and on the particular model of* **ZF** (**ZFU**) *we are working with?*

3.6. Reflections on compactness. The notion of compactness is widely used in different mathematical theories. It is strongly connected with the notion of finiteness. It is natural to call a topological space *F*-compact if every open cover contains an *F*-finite subcover (*F*-finiteness stands for any reasonable definition of finiteness).

Example. Assume that $A \in \mathbf{J}_{III} - \mathbf{J}_{II}$. Then the discrete space $(A, \mathscr{P}(A))$ is *IV*-compact and not *II*-compact. Indeed, $\mathscr{P}(A)$ is *IV*-finite and any subset of a *IV*-finite set is *IV*-finite and moreover, the covering $\{\{a\}: a \in A\}$ is an open cover without any *II*-finite subcover.

Several natural problems arise:

Problem 3.4. (i) *Describe pairs of F_1-finiteness and F_2-finiteness such that there exists (consistently) an F_1-compact space which fails to be F_2-compact.*

(ii) *Describe topological properties of F-compact spaces and give their characterizations.*

Let $\mathscr{L}$ be a logic with its language and syntax formalized in set theory (or in a model of **ZF**, **ZFU**). Then $\mathscr{L}$ is said to be *F*-compact provided for every theory $\mathscr{T}$ in $\mathscr{L}$ there is a model if and only if every *F*-finite subtheory $\mathscr{T}_0 \subseteq \mathscr{T}$ has a model.

In an oral communication (answering our question), H. G. Woodin conjectured that in the very Fraenkel-Mostowski model the first order predicate calculus is Dedekind-compact and not *I*-compact.

3.7. Vector spaces. As an illustration of a possible application of different phenomena of finiteness in the absence of the Axiom of Choice repeat the following example.

Example. Assume that $A \in \mathbf{J}_F - \mathbf{J}_I$. Define $S = \{f: A \to Real; \{a: f(a) \neq 0\}$ is $\boldsymbol{I}$-finite$\}$ and operations coordinatewise $(f + g)(a) = f(a) + g(a)$, $(k \,.\, f)(a) = k \,.\, f(a)$. The space S has an $\boldsymbol{F}$-finite base, namely $\{e_a: a \in A\}$, where $e_a(a) = 1$ and $e_a(b) = 0$ for $b \neq a$. S has no $\boldsymbol{I}$-finite base. Suppose on the contrary $\{c_1, \ldots, c_n\} \subseteq S$ is a base. Then $B = \bigcup_{i=1}^{n} \{a: c_i(a) \neq 0\}$ is a $\boldsymbol{I}$-finite set. So, take $a \in A - B$, then e_a is not a combination of $c_1, \ldots, c_n$.

Recently A. Blass has proved that if every vector space has a base then **AC** holds.

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