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SOME PROPERTIES OF LATTICE ORDERED REES MATRIX SEMIGROUPS

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The purpose of this note is to give some properties of lattice ordered completely simple Rees matrix Semigroups and to construct such semigroups.

1. INTRODUCTION

As in our previous papers [3], [4], by a lattice ordered semigroup, we mean a semigroup S, on which we can define an order relation $\leqq$ such that

– $(S, \leqq)$ is distributive lattice; $\vee$ and $\wedge$ are the least upper bound and the greatest lower bound.

– $\forall a\ \forall b\ \forall c\ a(b \wedge c) = ab \wedge ac$ and $(b \wedge c)\, a = ba \wedge ca$

– $\forall a\ \forall b\ \forall c\ a(b \vee c) = ab \vee ac$ and $(b \vee c)\, a = ba \vee ca$.

In matter of Rees matrix semigroups, we keep the notations of [1]. In particular, if G is a group, I and Λ are two sets non voids, P denote a fixed matrix, a $\Lambda \times I$ matrix over G, $P = (p_{\lambda_i})$. Recall that by a Rees $I \times \Lambda$ matrix over G°, we mean an $I \times \Lambda$ matrix over G° having at most one non zero element. If $a \in G$, $i \in I$ and $\lambda \in \Lambda$, then $(a)_{i\lambda}$ will denote the Rees $I \times \Lambda$ matrix on G° having "a" in the ith row and λth column, its remaining entries being 0. The set of all Rees matrix over G° is a semigroup for the product

$$(a)_{i\lambda} \circ (b)_{j\mu} = (a p_{\lambda j} b)_{i\mu} .$$

Moreover, if P is a matrix over G, P regular, then $S = \mathscr{M}^\circ(G; I, \Lambda, P)$ is a completely 0 simple semigroup.

2. PROPERTIES OF LATTICE ORDERED COMPLETELY SIMPLE REES MATRIX SEMIGROUPS

In this paragraph, we consider $S = \mathcal{M}^\circ(G; I, \Lambda; P)$ a Rees matrix semigroup, completely 0-simple, P being a $\Lambda \times I$ matrix over G. Moreover we suppose that this semigroup is lattice ordered, for the order relation $\leqq$.

Proposition 1. *If $(a)_{i\mu} \leqq (a)_{j\mu}$, then $(x)_{i\mu} \leqq (x)_{j\mu}$ for all x of G. If $(a)_{i\lambda} \leqq (a)_{i\mu}$, then $(y)_{i\lambda} \leqq (y)_{i\mu}$ for all y of G.*

For the proof, we use the following identities

(1) $(a)_{i\mu} = (ab^{-1}p_{\lambda j}^{-1})_{i\lambda} \circ (b)_{j\mu}$ for all λ,

(2) $(a)_{i\lambda} = (b)_{i\mu} \circ (p_{\mu j}^{-1}b^{-1}a)_{j\lambda}$ for all μ.

Suppose $(a)_{i\mu} \leqq (a)_{j\mu}$: then $(a)_{i\mu} \vee (a)_{j\mu} = (a)_{j\mu}$. Using (1), we can write

$$(a)_{i\mu} = (p_{\lambda j}^{-1})_{i\lambda} \circ (a)_{j\mu}\,, \quad (a)_{j\mu} = (p_{\lambda j}^{-1})_{j\lambda} \circ (a)_{j\mu}\,,$$

and $(a)_{i\mu} \vee (a)_{j\mu} = [(p_{\lambda j}^{-1})_{i\lambda} \vee (p_j^{-1})_{j\lambda}] \circ (a)_{j\mu} = (a)_{j\mu}$. The element between the brackets is necessarly of the form $(c)_{j\nu}$ and depends only of i and j. So, we have $(c)_{j\nu} \circ (a)_{j\mu} = (a)_{j\mu}$, and $cp_{\nu j}a = a$. Consequently, $c = p_{\nu j}^{-1}$ and $(p_{\lambda j}^{-1})_{i\lambda} \vee (j_{\lambda j}^{-1})_{j\lambda} = (p_{\nu j}^{-1})_{j\nu}$. Therefore, if now, we calculate $(x)_{i\mu} \vee (x)_{j\mu}$, we find:

$$(x)_{i\mu} \vee (x)_{j\mu} = (p_{\nu j}^{-1})_{j\nu} \circ (x)_{j\mu} = (x)_{j\mu}\,. \quad \text{Q.E.D.}$$

Proposition 2. *Each $\mathcal{H}$-class of S is a lattice ordered group, subsemigroup of S and sublattice of S. Moreover, all $\mathcal{H}$-classes are isomorphic and G can be considered as a subgroup and a sublattice ordered group of $S = \mathcal{M}^\circ(G; I, \Lambda; P)$.*

We know [1], that the $\mathcal{H}$-class are of the form $H_{i,\lambda} = \{(a)_{i,\lambda};\ a \in G\}$ and that they are isomorphic to the group G. They are sublattices of G. Effectively, if we calculate $(a)_{i\lambda} \vee (b)_{i\lambda}$, we have from (1)

$$(b)_{i\lambda} = (p_{\lambda i}^{-1})_{i\lambda} \circ b_{i\lambda} \quad \text{and} \quad (a)_{i\lambda} = (ab^{-1}p_{\lambda i}^{-1})_{i\lambda} \circ (b)_{i\lambda}\,.$$

Therefore $(a)_{i\lambda} \vee (b)_{i\lambda} = (p_{\lambda i}^{-1})_{i\lambda} \vee (ab^{-1}p_{\lambda i}^{-1})_{i\lambda}] \circ (b)_{i\lambda}$ and so $(a)_{i\lambda} \vee (b)_{i\lambda}$ is equivalent mod $\mathcal{L}$ to $(b)_{i\lambda}$. Symmetrically, we can prove that $(a)_{i\lambda} \vee (b)_{i\lambda}$ is equivalent mod $\mathcal{R}$ to $(b)_{i\lambda}$ and $(a)_{i\lambda} \vee (b)_{i\lambda}$ belongs to the same $\mathcal{H}$-class, $H_{i\lambda}$. Finally, as we go from H to H' by multiplications, because $H' = u \circ H \circ v$, $H = u' \circ H' \circ v'$ and as the products keeps the inequalities, all the $\mathcal{H}$-clases are isomorphic as lattices. We can restrict the order of S to G and we can put, in G

$$a \leqq b \Leftrightarrow \exists i \in I\,, \quad \exists \lambda \in \Lambda \quad (a)_{i\lambda} \leqq (b)_{i\lambda}$$
$$\Leftrightarrow \forall i \in I\,, \quad \forall \lambda \in \Lambda \quad (a)_{i\lambda} \leqq (b)_{i\lambda}\,.$$

In the sequel, G well be ordered by this relation.

Proposition 3. *If S is lattice ordered, we have*

$$\begin{cases} (a)_{i\mu} \leqq (a)_{j\mu} \Rightarrow p_{\nu i} \leqq p_{\nu j} & \text{in } G \text{ for all } \nu \text{ of } \Lambda \\ (a)_{i\mu} \leqq (a)_{i\lambda} \Rightarrow p_{\mu j} \leqq p_{\lambda j} & \text{in } G \text{ for all } j \text{ of } I\,. \end{cases}$$

We prove only the first implication. From

$$(a)_{i\mu} \leqq (a)_{j\mu}, \quad \text{we can deduce that for all } (c)_{k,v}$$

we have $(c)_{kv} \circ (a)_{i\mu} \leqq (c)_{kv} \circ (a)_{j\mu}$ i.e.

$(cp_{vi}a)_{k\mu} \leqq (cp_{vj}a)_{k\mu}$. As the order relation on S is an extension of this on G, (Prop. 2), we have

$$cp_{vi}a \leqq cp_{vj}a \text{ in } G, \text{ and so we have}$$

$p_{vi} \leqq p_{vj}$ by simplification in G.

Proposition 4. *If S is lattice ordered, we have: for all λ of Λ, $(a)_{i\lambda} \vee (a)_{j\lambda}$ is an element of the form $(b)_{k\lambda}$, and for all i of I, $(a)_{i\mu} \vee (a)_{iv}$ is of the form $(c)_{i\varrho}$.*

We have seen, in the proof of Proposition 1, that $(a)_{i\lambda} \vee (a)_{j\lambda} = [(p_{\lambda j}^{-1})_{i\lambda} \vee \vee (p_{\lambda j}^{-1})_{j\lambda}] \circ (a)_{j\lambda}$, and necessarily this least element is of the form $(b)_{k\lambda}$. Q.E.D.

We know that all the Rees matrix semigroups can be constructed from a normalized sandwich matrix [1]. (A normalized matrix is a matrix in which all the elements of the λ_0th row and of the i_0th column are equal to 1, if 1 is the neutral element of G.

Now, we note (*) the following condition

(*) $S = \mathscr{M}^\circ(G; I, \Lambda; P)$ where P is normalized.

Proposition 5. *If S is lattice ordered, and if the condition (*) is satisfied, we have*

$$\forall i \ \forall j \ \exists k \text{ such that } \forall a \ \forall \mu \ (a)_{i\mu} \vee (a)_{j\mu} = (a)_{k\mu}$$

$$\forall \mu \ \forall v \ \exists \lambda \text{ such that } \forall a \ \forall i \ (a)_{i\mu} \vee (a)_{iv} = (a)_{i\lambda}$$

(so k depends only of i and j, and similarly λ depends only of μ and v).

Proposition 5′. *Proposition Analogous to Proposition 5, where we replace $\vee$ by $\wedge$.*

Proof of Prop. 5. First, we show that

$$(1)_{i\lambda_0} \vee (1)_{j\lambda_0} = (1)_{k\lambda_0}.$$

From the Proposition 4, we deduce that $(1)_{i\lambda_0} \vee (1)_{j\lambda_0} = (c)_{k\lambda_0}$; we multiply this last equality on the left by $(1)_{k\lambda_0}$. We obtain:

$$(1)_{k\lambda_0} \circ [(1)_{i\lambda_0} \vee (1)_{j\lambda_0}] = (1p_{\lambda_0 i}1)_{k\lambda_0} \vee (1p_{\lambda_0 j}1)_{k\lambda_0} =$$
$$= (1)_{k\lambda_0} \text{ [because } p_{\lambda_0 i} = p_{\lambda_0 j} = 1] =$$
$$= (1)k_{\lambda_0} \circ (c)_{k\lambda_0} = (1p_{\lambda_0 k}c)_{k\lambda_0} = (c)_{k\lambda_0} (p_{\lambda_0 k} = 1).$$

Therefore $c = 1$.

Now, we calculate $a_{i\mu} \vee a_{j\mu}$; we can write this element (see Proposition 1) $[(p_{\lambda i}^{-1})_{i\lambda} \vee (p_{\lambda i}^{-1})_{j\lambda}] \circ a_{i\mu}$ for all λ of Λ, and in particular for λ_0. So we have

$$(a)_{i\mu} \vee (a)_{j\mu} = [(1)_{i\lambda_0} \vee (1)_{j\lambda_0}] \circ a_{i\mu} = (1)_{k\lambda_0} \circ (a)_{i\mu} = (a)_{k\mu}.$$

We prove the second equality similarly by using the fact that $p_{\lambda_i} = 1$ for all λ of Λ.

Proposition 6. *If S is lattice ordered, and if the condition $(*)$ is satisfied, I and Λ are distributive lattice ordered sets.*

We prove this proposition only for the set I. We define on I two binary relations:

$$I \times I \to^f I$$

$$(i,j) \mapsto f(i,j) \quad \text{such that} \quad (1)_{i\lambda_0} \vee (1)_{j\lambda_0} = (1)_{f(i,j),\lambda_0}$$

and

$$I \times I \to^g I$$

$$(i,j) \mapsto g(i,j) \quad \text{such that} \quad (1)_{i\lambda_0} \wedge (1)_{j\lambda_0} = (1)_{g(i,j),\lambda_0} \,.$$

We know (Prop. 5 and 5′) that this two binary relations are well defined from $I \times I$ on I.

It is trivial to see that $f(i, j) = f(j, i), f(i, i) = i$. Now, we prove the associativity:

$$\text{if} \quad (1)_{i\lambda_0} \vee (1)_{j\lambda_0} = (1)_{f(i,j),\lambda_0} \quad \text{then}$$

$$[(1)_{i\lambda_0} \vee (1)_{j\lambda_0}] \vee (1)_{k\lambda_0} = (1)_{f(i,j),\lambda_0} \vee (1)_{k\lambda_0} = (1)_{f[f(i,j),k],\lambda_0} =$$

$$= (1)_{i\lambda_0} \vee [(1)_{j\lambda_0} \vee (1)_{k\lambda_0}] = (1)_{i\lambda_0} \vee (1)_{f(i,j),\lambda_0} = (1)_{f[i,f(j,k)],\lambda_0} \,.$$

And finally, $f[f(i, j), k] = f[i, f(j, k)]$. Evidently, we have the same properties for g.

There is absorption between f and g. By example, we show that $g(i, f(i, j)) = i$:

$$(1)_{i\lambda_0} \vee (1)_{j\lambda_0} = (1)_{f(i,j),\lambda_0}$$

$$(1)_{i\lambda_0} \wedge [(1)_{i\lambda_0} \vee (1)_{j\lambda_0}] = (1)_{i\lambda_0} \wedge (1)_{f(i,j),\lambda_0} = (1)_{g[i,f(i,j)],\lambda_0} = (1)_{i\lambda_0} \,.$$

And so we have the asked result. Similarly, $f(i, g(i, j)) = i$. From the preceding results, we can affirm that I and J are lattice ordered sets and that if we note, as usually, $\leqq$ the order relation on I and S we have

P_1 $i \leqq j \Leftrightarrow f(i,j) = j \Leftrightarrow g(i,j) = i$,
P_2 $i \leqq j \Leftrightarrow (1)_{i\lambda_0} \leqq (1)_{j\lambda_0}$,
P_3 $i \leqq j \Leftrightarrow (a)_{i\mu} \leqq (a)_{j\mu}$ for one a of G and one μ of Λ,
P'_3 $i \leqq j \Leftrightarrow (a)_{i\mu} \leqq (a)_{j\mu}$ for any a of G and any μ of Λ,
P_4 $\lambda \leqq \mu \Leftrightarrow (1)_{i_0\lambda} \leqq (1)_{i_0\mu}$,
P_5 $\lambda \leqq \mu \Leftrightarrow (a)_{i\lambda} \leqq (a)_{i\mu}$ for one a of G and one i of I,
P'_5 $\lambda \leqq \mu \Leftrightarrow (a)_{i\lambda} \leqq (a)_{i\mu}$ for any a of G and any i of I.

P_1 is true by definition of $f(i, j)$ and of $g(i, j)$ which define the least upper bound and the greatest lower bound in I.

P_2 is true by definition of $f(i, j)$ and of $g(i, j)$. From the propositions 5 and 5′ we deduce easily P_3 and P'_3; P_4, P_5, P'_5 are the analogous (for Λ) of P_2, P_3, P'_3.

Finally, it is trivial to see that under this order relation I and Λ are distributive lattices.

Theorem. *If $S = \mathscr{M}^\circ(G; I, \Lambda; P)$ is a lattice ordered completely simple semigroup, where P is a normalized sandwich matrix, then I and Λ are distributive lattice ordered sets and we can construct an isotone application φ of $\Lambda \times I$ in G, defined by $\varphi(\lambda, i) = p_{\lambda_i}$.*

$\Lambda \times I$ is a lattice ordered set under the relation $(\lambda, i) \leqq (\mu, j) \Leftrightarrow \lambda \leqq \mu$ in Λ and $i \leqq j$ in I, since Λ and I are lattice ordered by proposition 6. Therefore, if $(\lambda, i) \leqq \leqq (\mu, j)$, we have $\lambda \leqq \mu$ and $i \leqq j$. Hence $(a)_{k\lambda} \leqq (a)_{k\mu}$ for any k of I and $(a)_{i\nu} \leqq \leqq (a)_{j\nu}$ for any ν of Λ and this for any a of G. Let us now use Proposition 3; we obtain

$$p_{\nu i} \leqq p_{\nu j} \quad \text{for any } \nu \text{ of } \Lambda \text{ and } \quad p_{\lambda}k \leqq p_{\mu}k \quad \text{for any } k \text{ of } I\,.$$

Finally (putting $k = i$, $\nu = \mu$) we obtain $p_{\lambda i} \leqq p_{\mu j}$ et so φ is isotone.

CONSTRUCTION OF A LATTICE ORDERED REES MATRIX SEMIGROUPS

Let G be a lattice ordered group; let Λ and I be two distributive lattices. Then we can construct a Rees matrix semigroup $S = \mathscr{M}^{\circ}(G; I, \Lambda; P)$ where P is a regular sandwich matrix whose entries belong to G and are all equal to 1, neutral element of G: $p_{\lambda i} = 1$ for all λ of Λ and all i of J.

We define $\leqq$ an order relation on S by

$$(a)_{i\lambda} \leqq (b)_{j\mu} \Leftrightarrow \lambda \leqq \mu \text{ in } \Lambda\,, \quad i \leqq j \text{ in } I \text{ and } a \leqq b \text{ in } G$$

(product order of I, Λ and G).

Therefore, S is a lattice, a distributive lattice as I, Λ and G. (Recall that a lattice ordered group is always a distributive lattice [2]).

But, S is also a lattice ordered semigroup. We have

$$(a)_{i\lambda} \vee (b)_{j\mu} = (a \vee b)_{i \vee j, \lambda \vee \mu} \quad \text{and} \quad (a)_{i\lambda} \circ (b)_{j\mu} = (ab)_{i\mu}\,.$$

So

$$(x)_{k\nu} \circ [a_{i\lambda} \vee b_{j\mu}] = x_{k\nu} \circ [(a \vee b)_{i \vee j, \lambda \vee \mu}] = [x(a \vee b)]_{k, \lambda \vee \mu}\,,$$

$$(x)_{k\nu} \circ (a_i\lambda) \vee (a)_{k\nu} \circ (b_{j\mu}) = (xa)_{k\lambda} \vee (xb)_{k\mu} = x(a \vee b)_{k, \lambda \vee \mu}\,.$$

By an argument of symmetry, we have also

$$(a_{i\lambda} \vee b_{j\mu}) \circ (x)_{k\nu} = (a_{i\lambda} \circ x_{k\nu}) \vee (b_{j\mu} \circ x_{k\nu})$$

and also the same equalities by replacing $\vee$ by $\wedge$.

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