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ON RELATIONS

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Dedicated to Professor M. Novotný on the occassion of his 65th birthday

0. INTRODUCTION

Recently, the cyclically ordered sets have been intensively studied by V. Novák (see e.g. [2], [3]). For his purposes he had to introduce certain new concepts and prove some general statements concerning ternary relations because the general behaviour of ternary relations has not yet been described sufficiently in literature. The aim of this paper is not only to give a satisfatory description of such behaviour but even to put foundations for the study of relations in the general sense. Our considerations are based on extensions of the very well-known concepts concerning binary relations and of some aspects concerning ternary relations, to relations in the general sense. Therefore some results are analogous to the well-known ones of the theory of binary relations (see e.g. [4]) while others are generalizations of results presented in [2] and [3]. Nonetheless, many new results are also to be found here.

Let G, H be non-empty sets. By a *relation in the general sense* (briefly a *relation*) R we understand a set of mappings $R \subseteq G^H$. The sets G and H are called the *carrier* and the *index set*, respectively, *of* R. Throughout the paper, G and H denote non-empty sets and N the set of all positive integers. If $n \in N$ and $H = \{1, 2, \ldots, n\}$, then $R \subseteq G^H$ is called an *n-ary relation on* G. The relations with well ordered index sets are studied in [5].

1. OPERATIONS WITH RELATIONS

1.1. Definition. Let $\operatorname{card} H \geqq 2$ ($\operatorname{card} H \geqq 3$). Then a sequence of three (four) sets $\mathscr{K} = \{K_i\}_{i=1}^{3(4)}$ is called a *b-decomposition* (*t-decomposition*) of the set H if

(1) $\bigcup_{i=1}^{3(4)} K_i = H$,

(2) $K_i \cap K_j = \emptyset$ for all $i, j \in \{1, 2, 3\}$ ($i, j \in \{1, 2, 3, 4\}$, $i \neq j$,

(3) $0 < \operatorname{card} K_1 = \operatorname{card} K_2$ ($0 < \operatorname{card} K_1 = \operatorname{card} K_2 = \operatorname{card} K_3$).

1.2. Definition. Let $\operatorname{card} H \geqq 2$ ($\operatorname{card} H \geqq 3$) and let $\mathscr{K} = \{K_i\}_{i=1}^{3(4)}$ be a *b*-

decomposition (t-decomposition) of H. The relation $\{f \in G^H | f(K_1) = f(K_2)\}$ is called *diagonal with regard to* $\mathscr{K}$, and denoted by $E_{\mathscr{K}}$.

1.3. Definition. Let $R \subseteq G^H$ be a relation with card $H \geqq 2$ (card $H \geqq 3$) and let $\mathscr{K} = \{K_i\}_{i=1}^{3(4)}$ be a b-decomposition (t-decomposition) of H. Then we define the relation $R_{\mathscr{K}}^{-1} \subseteq G^H$ by

$$f \in R_{\mathscr{K}}^{-1} \Leftrightarrow \exists g \in R: f(K_1) = g(K_2), \quad f(K_2) = g(K_1),$$
$$f(K_i) = g(K_i) \quad \text{for} \quad i = 3 \quad (\text{for } i = 3, 4).$$

$R_{\mathscr{K}}^{-1}$ is called the *inversion of R with regard to* $\mathscr{K}$.

1.4. Definition. Let $R, S \subseteq G^H$ be relations with card $H \geqq 2$ (card $H \geqq 3$) and let $\mathscr{K} = \{K_i\}_{i=1}^{3(4)}$ be a b-decomposition (t-decomposition) of H. Then we define the relation $(RS)_{\mathscr{K}} \subseteq G^H$ by

$$f \in (RS)_{\mathscr{K}} \Leftrightarrow \exists g \in R, \quad \exists h \in S: f(K_1) = g(K_1), \quad f(K_2) = h(K_2),$$
$$f(K_i) = g(K_i) = h(K_i) \quad \text{for} \quad i = 3 \ (\text{for } i = 3, 4), \quad g(K_2) = h(K_1).$$

$(RS)_{\mathscr{K}}$ is called the *composition of R and S with regard to* $\mathscr{K}$. Further, we put $R_{\mathscr{K}}^1 = R$ and $R_{\mathscr{K}}^{n+1} = (R_{\mathscr{K}}^n R)_{\mathscr{K}}$ for every $n \in N$. $R_{\mathscr{K}}^n$ is called the *n-th power of R with regard to* $\mathscr{K}$.

1.5. Remark. Let $R, S \subseteq G^H$ be relations with card $H \geqq 3$ and let $\mathscr{K} = \{K_i\}_{i=1}^4$ be a t-decomposition of H. Let $\mathscr{K}' = \{K_i'\}_{i=1}^3$ be the b-decomposition of H defined by $K_i' = K_i$ for $i = 1, 2$ and $K_3' = K_3 \cup K_4$. Obviously, then

(a) $E_{\mathscr{K}} = E_{\mathscr{K}'}$,
(b) $R_{\mathscr{K}}^{-1} \subseteq R_{\mathscr{K}'}^{-1}$ and $(RS)_{\mathscr{K}} \subseteq (RS)_{\mathscr{K}'}$ (and the inclusions can be replaced by equalities provided $K_4 = \emptyset$).

1.6. Notation. Let card $H \geqq 2$ (card $H \geqq 3$) and let $\mathscr{K} = \{K_i\}_{i=1}^{3(4)}$ be a b-decomposition (t-decomposition) of H. Then $\mathscr{K}^*$ denotes the b-decomposition (t-decomposition) of H defined by $\mathscr{K}^* = \{K_i^*\}_{i=1}^{3(4)}$ where $K_1^* = K_2$, $K_2^* = K_1$, $K_i^* = K_i$ for $i = 3$ (for $i = 3, 4$).

1.7. Lemma. *Let $R, S \subseteq G^H$ be relations with* card $H \geqq 2$ (card $H \geqq 3$) *and let $\mathscr{K}$ be a b-decomposition (t-decomposition) of H. Then*

(a) $\mathscr{K}^{**} = \mathscr{K}$,
(b) $E_{\mathscr{K}} = E_{\mathscr{K}^*}$,
(c) $R_{\mathscr{K}}^{-1} = R_{\mathscr{K}^*}^{-1}$,
(d) $(RS)_{\mathscr{K}} = (SR)_{\mathscr{K}^*}$.

Proof is obvious.

1.8. Lemma. *Let $R, S, T, U \subseteq G^H$ be relations with* card $H \geqq 2$ (card $H \geqq 3$) *and let $\mathscr{K}$ be a b-decomposition (t-decomposition) of H. Then*

(1) $E_{\mathscr{K}} = (E_{\mathscr{K}})_{\mathscr{K}}^{-1} = (E_{\mathscr{K}})_{\mathscr{K}}^2$,
(2) $R \subseteq (R_{\mathscr{K}}^{-1})_{\mathscr{K}}^{-1} = (RE_{\mathscr{K}})_{\mathscr{K}} = (E_{\mathscr{K}}R)_{\mathscr{K}}$,

(3) $(R \cup S)_{\mathscr{K}}^{-1} = R_{\mathscr{K}}^{-1} \cup S_{\mathscr{K}}^{-1}$,
(4) $(R \cap S)_{\mathscr{K}}^{-1} \subseteq R_{\mathscr{K}}^{-1} \cap S_{\mathscr{K}}^{-1}$,
(5) $R \subseteq S \Rightarrow R_{\mathscr{K}}^{-1} \subseteq S_{\mathscr{K}}^{-1}$,
(6) $((RS)_{\mathscr{K}} T)_{\mathscr{K}} = (R(ST)_{\mathscr{K}})_{\mathscr{K}}$,
(7) $R \subseteq S,\ T \subseteq U \Rightarrow (RT)_{\mathscr{K}} \subseteq (SU)_{\mathscr{K}}$,
(8) $((RS)_{\mathscr{K}})_{\mathscr{K}}^{-1} = (S_{\mathscr{K}}^{-1} R_{\mathscr{K}}^{-1})_{\mathscr{K}}$.

Proof. The assertion follows directly from the definitions of the operations. For example, let us prove (6) and (8). Let $\mathscr{K} = \{K_i\}_{i=1}^{3(4)}$.

(6) Let $f \in ((RS)_{\mathscr{K}} T)_{\mathscr{K}}$ be a mapping. Then there exist $g \in (RS)_{\mathscr{K}}$ and $h \in T$ such that $f(K_1) = g(K_1), f(K_2) = h(K_2), f(K_i) = g(K_i) = h(K_i)$ for $i = 3$ (for $i = 3, 4$), $g(K_2) = h(K_1)$. As $g \in (RS)_{\mathscr{K}}$, there exist $p \in R$ and $q \in S$ such that $g(K_1) = p(K_1)$, $g(K_2) = q(K_2)$, $g(K_i) = p(K_i) = q(K_i)$ for $i = 3$ (for $i = 3, 4$), $p(K_2) = q(K_1)$. Thus, we have $q(K_i) = h(K_i)$ for $i = 3$ (for $i = 3, 4$), $q(K_2) = h(K_1)$. Let $r \in G^H$ be a mapping fulfilling $r(K_1) = q(K_1)$, $r(K_2) = h(K_2)$, $r(K_i) = q(K_i)$ for $i = 3$ (for $i = 3, 4$). Then $r \in (ST)_{\mathscr{K}}$ and we have $f(K_1) = p(K_1)$, $f(K_2) = r(K_2)$, $f(K_i) = p(K_i) = r(K_i)$ for $i = 3$ (for $i = 3, 4$), $p(K_2) = r(K_1)$. Therefore $r \in (R(ST)_{\mathscr{K}})_{\mathscr{K}}$ and the inclusion $((RS)_{\mathscr{K}} T)_{\mathscr{K}} \subseteq (R(ST)_{\mathscr{K}})_{\mathscr{K}}$ is proved. The converse inclusion can be proved similarly.

(8) Let $f \in ((RS)_{\mathscr{K}})_{\mathscr{K}}^{-1}$. Then there exists $g \in (RS)_{\mathscr{K}}$ such that $f(K_1) = g(K_2)$, $f(K_2) = g(K_1)$, $f(K_i) = g(K_i)$ for $i = 3$ (for $i = 3, 4$). As $g \in (RS)_{\mathscr{K}}$, there exist $h \in R$ and $p \in S$ such that $g(K_1) = h(K_1)$, $g(K_2) = p(K_2)$, $g(K_i) = h(K_i) = p(K_i)$ for $i = 3$ (for $i = 3, 4$), $h(K_2) = p(K_1)$. Let $q, r \in G^H$ be mappings such that $q(K_1) = h(K_2)$, $q(K_2) = h(K_1)$, $q(K_i) = h(K_i)$ for $i = 3$ (for $i = 3, 4$), and $r(K_1) = p(K_2)$, $r(K_2) = p(K_1)$, $r(K_i) = p(K_i)$ for $i = 3$ (for $i = 3, 4$). Then $q \in R_{\mathscr{K}}^{-1}$, $r \in S_{\mathscr{K}}^{-1}$ and we have $f(K_1) = r(K_1)$, $f(K_2) = q(K_2)$, $f(K_i) = r(K_i) = q(K_i)$ for $i = 3$ (for $i = 3, 4$), $r(K_2) = q(K_1)$. Hence $f \in (S_{\mathscr{K}}^{-1} R_{\mathscr{K}}^{-1})_{\mathscr{K}}$ and the inclusion $((RS)_{\mathscr{K}})_{\mathscr{K}}^{-1} \subseteq (S_{\mathscr{K}}^{-1} R_{\mathscr{K}}^{-1})_{\mathscr{K}}$ holds. The converse inclusion can be proved similarly.

1.9. Remark. Clearly, for binary relations the above defined operations – with regard to the b-decomposition $\mathscr{K} = \{\{1\}, \{2\}, \emptyset\}$ of the index set $H = \{1, 2\}$ – coincide with the very well-known operations (see e.g. [4]). However, the inclusions in (2) and (4) of 1.8 cannot in general be replaced by equalities. In other words, some properties of binary relations are not preserved for relations in the general sense. In the following paragraph we define such relations (called regular) for which all properties of binary relations are preserved.

1.10. Definition. Let $R \subseteq G^H$ be a relation with card $H \geqq 3$ and let $\mathscr{K} = \{K_i\}_{i=1}^4$ be a t-decomposition of H. Then we define the relation ${}^1R_{\mathscr{K}} \subseteq G^H$ by

$$f \in {}^1R_{\mathscr{K}} \Leftrightarrow \exists g \in R\colon f(K_1) = g(K_2)\,,\quad f(K_2) = g(K_3)\,,$$
$$f(K_3) = g(K_1)\,,\quad f(K_4) = g(K_4)\,.$$

Further, we put ${}^{n+1}R_{\mathscr{K}} = {}^1({}^nR_{\mathscr{K}})_{\mathscr{K}}$ for every $n \in N$. ${}^nR_{\mathscr{K}}$ is called *the n-th cyclic transposition of R with regard to* $\mathscr{K}$.

1.11. Notation. Let card $H \geqq 3$ and let $\mathscr{K} = \{K_i\}_{i=1}^4$ be a t-decomposition of H. Then ${}^*\mathscr{K}$ denotes the t-decomposition of H defined by ${}^*\mathscr{K} = \{{}^*K_i\}_{i=1}^4$ where ${}^*K_1 = K_2$, ${}^*K_2 = K_3$, ${}^*K_3 = K_1$, ${}^*K_4 = K_4$.

1.12. Lemma. *Let* $R \subseteq G^H$ *be a relation with* card $H \geqq 3$ *and let* $\mathscr{K}$ *be a t-decomposition of H. Then*

(a) ${}^{***}\mathscr{K} = \mathscr{K}$,
(b) ${}^1R_{\mathscr{K}} = {}^1R_{*\mathscr{K}}$,
(c) $E_{\mathscr{K}} = {}^1(E_{*\mathscr{K}})_{\mathscr{K}}$,
(d) $R_{\mathscr{K}}^{-1} = ({}^1R_{\mathscr{K}})_{*\mathscr{K}}^{-1}$,
(e) $(RS)_{\mathscr{K}} = {}^2[({}^1R_{\mathscr{K}}{}^1S_{\mathscr{K}})_{**\mathscr{K}}]_{\mathscr{K}}$.

Proof. The equalities (a)–(d) are evident. (e) Let $\mathscr{K} = \{K_i\}_{i=1}^4$ and let $f \in {}^2[({}^1R_{\mathscr{K}}{}^1S_{\mathscr{K}})_{**\mathscr{K}}]_{\mathscr{K}}$. Then there exists $g \in ({}^1R_{\mathscr{K}}{}^1S_{\mathscr{K}})_{**\mathscr{K}}$ fulfilling $f(K_1) = g(K_3)$, $f(K_2) = g(K_1)$, $f(K_3) = g(K_2)$, $f(K_4) = g(K_4)$. But $g \in ({}^1R_{\mathscr{K}}{}^1S_{\mathscr{K}})_{**\mathscr{K}}$ implies that there exist $h \in {}^1R_{\mathscr{K}}$ and $p \in {}^1S_{\mathscr{K}}$ such that $g(K_3) = h(K_3)$, $g(K_1) = p(K_1)$, $g(K_i) = h(K_i) = p(K_i)$ for $i = 2, 4$, $h(K_1) = p(K_3)$, because ${}^{**}K_1 = K_3$, ${}^{**}K_2 = K_1$, ${}^{**}K_3 = K_2$, ${}^{**}K_4 = K_4$. Next, $h \in {}^1R_{\mathscr{K}}$ and $p \in {}^1S_{\mathscr{K}}$ imply that there exist $q \in R$ and $r \in S$ such that $h(K_1) = q(K_2)$, $h(K_2) = q(K_3)$, $h(K_3) = q(K_1)$, $h(K_4) = q(K_4)$ and $p(K_1) = r(K_2)$, $p(K_2) = r(K_3)$, $p(K_3) = r(K_1)$, $p(K_4) = r(K_4)$. Now we have $f(K_1) = q(K_1)$, $f(K_2) = r(K_2)$, $f(K_i) = q(K_i) = r(K_i)$ for $i = 3, 4$, $q(K_2) = r(K_1)$. This yields $f \in (RS)_{\mathscr{K}}$. Hence ${}^2[({}^1R_{\mathscr{K}}{}^1S_{\mathscr{K}})_{**\mathscr{K}}]_{\mathscr{K}} \subseteq (RS)_{\mathscr{K}}$. The converse inclusion can be proved similarly.

1.13. Lemma. *Let* $R, S \subseteq G^H$ *be relations with* card $H \geqq 3$ *and let* $\mathscr{K}$ *be a t-decomposition of H. Then*

(1) $R \subseteq {}^3R_{\mathscr{K}}$,
(2) ${}^1(R \cup S)_{\mathscr{K}} = {}^1K_{\mathscr{K}} \cup {}^1S_{\mathscr{K}}$,
(3) ${}^1(R \cap S)_{\mathscr{K}} \subseteq {}^1R_{\mathscr{K}} \cap {}^1S_{\mathscr{K}}$,
(4) $R \subseteq S \Rightarrow {}^1R_{\mathscr{K}} \subseteq {}^1S_{\mathscr{K}}$,
(5) $({}^1R_{\mathscr{K}})_{\mathscr{K}}^{-1} = {}^2(R_{\mathscr{K}}^{-1})_{\mathscr{K}}$.

Proof. It is easy to see that (1), (2), (3) and (4) are valid. (5) Let $\mathscr{K} = \{K_i\}_{i=1}^4$ and let $f \in ({}^1R_{\mathscr{K}})_{\mathscr{K}}^{-1}$. Then there exists $g \in {}^1R_{\mathscr{K}}$ such that $f(K_1) = g(K_2)$, $f(K_2) = g(K_1)$, $f(K_i) = g(K_i)$ for $i = 3, 4$. As $g \in {}^1R_{\mathscr{K}}$, there exists $h \in R$ such that $g(K_1) = h(K_2)$, $g(K_2) = h(K_3)$, $g(K_3) = h(K_1)$, $g(K_4) = h(K_4)$. Let $p \in G^H$ be a mapping fulfilling $p(K_1) = h(K_2)$, $p(K_2) = h(K_1)$, $p(K_i) = h(K_i)$ for $i = 3, 4$. Then $p \in R_{\mathscr{K}}^{-1}$ and we have $f(K_1) = p(K_3)$, $f(K_2) = p(K_1)$, $f(K_3) = p(K_2)$, $f(K_4) = p(K_4)$. Consequently $f \in {}^2(R_{\mathscr{K}}^{-1})_{\mathscr{K}}$ and hence $({}^1R_{\mathscr{K}})_{\mathscr{K}}^{-1} \subseteq {}^2(R_{\mathscr{K}}^{-1})_{\mathscr{K}}$. The converse inclusion can be proved similarly.

2.1. Definition. Let $R \subseteq G^H$ be a relation with card $H \geqq 2$ (card $H \geqq 3$) and let $\mathscr{K} = \{K_i\}_{i=1}^{3(4)}$ be a b-decomposition (t-decomposition) of H. Then R is called

(1) *reflexive* [*areflexive*] *with regard to* $\mathscr{K}$ if $E_{\mathscr{K}} \subseteq R$ [$R \cap E_{\mathscr{K}} = \emptyset$],

(2) *symmetric* [*asymmetric, antisymmetric*] *with regard to* $\mathscr{K}$ if $R_{\mathscr{K}}^{-1} \subseteq R$ [$R \cap R_{\mathscr{K}}^{-1} = \emptyset$, $R \cap R_{\mathscr{K}}^{-1} \subseteq E_{\mathscr{K}}$],

(3) *transitive* [*atransitive*] *with regard to* $\mathscr{K}$ if $R_{\mathscr{K}}^2 \subseteq R$ [$R \cap R_{\mathscr{K}}^n = \emptyset$ for every $n \in N$, $n \geqq 2$],

(4) *complete with regard to* $\mathscr{K}$ if $f \in G^H$, $f(K_1) \neq f(K_2)$ ($f(K_1) \neq f(K_2) \neq f(K_3) \neq f(K_1)$) $\Rightarrow$ there exist a permutation φ of the set $\{1, 2\}$ (of the set $\{1, 2, 3\}$) and a mapping $g \in R$ such that $f(K_i) = g(K_{\varphi(i)})$ for $i = 1, 2$ (for $i = 1, 2, 3$) and $f(K_i) = g(K_i)$ for $i = 3$ (for $i = 4$),

(5) *regular with regard to* $\mathscr{K}$ if $f \in R$, $g \in G^H$, $f(K_i) = g(K_i)$ for $i = 1, 2, 3$ (for $i = 1, 2, 3, 4$) $\Rightarrow g \in R$.

2.2 Remark. Let $R \subseteq G^H$ be a relation with card $H \geqq 3$ and let $\mathscr{K} = \{K_i\}_{i=1}^4$ be a t-decomposition of H. Let $\mathscr{K}'$ be the b-decomposition of H defined in the same way as in 1.5. Obviously, we then have

(a) R is reflexive (areflexive) with regard to $\mathscr{K}$ iff R is reflexive (areflexive) with regard to $\mathscr{K}'$.

(b) If R is symmetric (asymmetric, antisymmetric, transitive, atransitive, regular) with regard to $\mathscr{K}'$, then R has the same property with regard to $\mathscr{K}$ [and vice-versa provided $K_4 = \emptyset$].

2.3. Lemma. *Let* $R, S \subseteq G^H$ *be relations with* card $H \geqq 2$ (card $H \geqq 3$) *and let* $\mathscr{K}$ *be a b-decomposition (t-decomposition) of H. Then*

(a) $E_{\mathscr{K}}, R_{\mathscr{K}}^{-1}, (RS)_{\mathscr{K}}$ *are regular with regard to* $\mathscr{K}$.

(b) *If R is symmetric with regard to* $\mathscr{K}$, *then R is regular with regard to* $\mathscr{K}$.

Proof. The assertion (a) is evident. From 1.8 (2) and (5) we obtain that $R_{\mathscr{K}}^{-1} \subseteq R \Leftrightarrow R_{\mathscr{K}}^{-1} = R$. This fact and (a) yield (b).

2.4. Lemma. *Let* $R, S \subseteq G^H$ *be relations with* card $H \geqq 2$ (card $H \geqq 3$) *and let* $\mathscr{K}$ *be a b-decomposition (t-decomposition) of H. If R is regular with regard to* $\mathscr{K}$, *then*

(1) $R = (R_{\mathscr{K}}^{-1})_{\mathscr{K}}^{-1} = (RE_{\mathscr{K}})_{\mathscr{K}} = (E_{\mathscr{K}}R)_{\mathscr{K}}$,

(2) $(R \cap S)_{\mathscr{K}}^{-1} = R_{\mathscr{K}}^{-1} \cap S_{\mathscr{K}}^{-1}$.

Proof. The inclusions $\subseteq$ result from 1.8. Using regularity one can easily prove the converse ones.

2.5. Remark. It follows from 2.4 that the regular relations are exactly those relations the operations with which have the same algebraic properties as the operations with binary relations. Thus, the theory of regular relations could be constructed analogously to the theory of binary relations.

2.6. Theorem. *Let $R, S \subseteq G^H$ be relations with* card $H \geqq 2$ (card $H \geqq 3$) *and let $\mathscr{K}$ be a b-decomposition (t-decomposition) of H. Then*

(a) *If R, S are reflexive [regular] with regard to $\mathscr{K}$, then $R \cup S$, $R \cap S$, $R_{\mathscr{K}}^{-1}$, $(RS)_{\mathscr{K}}$ have the same property.*

(b) *If R, S are areflexive [symmetric] with regard to $\mathscr{K}$, then $R \cup S$, $R \cap S$, $R_{\mathscr{K}}^{-1}$ have the same property.*

(c) *If R, S are transitive with regard to $\mathscr{K}$, then $R \cap S$ and $R_{\mathscr{K}}^{-1}$ have the same property.*

(d) *If R is asymmetric [antisymmetric, atransitive] with regard to $\mathscr{K}$, then $R \cap S$ and $R_{\mathscr{K}}^{-1}$ have the same property.*

(e) *If R is complete with regard to $\mathscr{K}$, then $R \cup S$ and $R_{\mathscr{K}}^{-1}$ have the same property.*

Proof. The assertions (a)–(d) follow from 1.8, 2.3, 2.4, and (e) is evident.

2.7. Theorem. *Let $R, S \subseteq G^H$ be relations with* card $H \geqq 2$ (card $H \geqq 3$) *and let $\mathscr{K}$ be a b-decomposition (t-decomposition) of H. If R, S are symmetric with regard to $\mathscr{K}$, then $(RS)_{\mathscr{K}}$ is symmetric with regard to $\mathscr{K}$ iff $(RS)_{\mathscr{K}} = (SR)_{\mathscr{K}}$.*

Proof. The statement results from 1.8.

2.8. Lemma. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 2$ (card $H \geqq 3$) *and let $\mathscr{K}$ be a b-decomposition (t-decomposition) of H. If R has any one of the properties defined in* 2.1 *with regard to $\mathscr{K}$, then R has the same property with regard to $\mathscr{K}^*$.*

Proof. For completeness and regularity the assertion is obvious, and for the other properties it follows from 1.7.

2.9. Definition. Let $R \subseteq G^H$ be a relation with card $H \geqq 3$ and let $\mathscr{K} = \{K_i\}_{i=1}^4$ be a t-decomposition of H. Then R is called

(1) *cyclic (acylic, anticyclic) with regard to $\mathscr{K}$* if ${}^1R_{\mathscr{K}} \subseteq R$ ($R \cap {}^1R_{\mathscr{K}} = \emptyset$, $f \in R \cap {}^1R_{\mathscr{K}} \Rightarrow f(K_1) = f(K_2) = f(K_3)$).

(2) *strongly symmetric (strongly asymmetric, strongly antisymmetric) with regard to $\mathscr{K}$* if for any permutation (any odd premutation) φ of the set $\{1, 2, 3\}$ and any mapping $f \in G^H$ for which there exists a mapping $g \in R$ such that $f(K_i) = g(K_{\varphi(i)})$ for $i = 1, 2, 3$ and $f(K_4) = g(K_4)$, we have $f \in R$ ($f \notin R$, $f \in R \Rightarrow f(K_1) = f(K_2) = f(K_3)$).

2.10. Remark. a) Consequently, R is cyclic (acyclic, anticyclic) with regard to $\mathscr{K}$ iff for any even permutation φ of the set $\{1, 2, 3\}$ and any mapping $f \in G^H$ for which there exists a mapping $g \in R$ such that $f(K_i) = g(K_{\varphi(i)})$ for $i = 1, 2, 3$ and $f(K_4) = g(K_4)$ we have $f \in R$ ($f \notin R$, $f \in R \Rightarrow f(K_1) = f(K_2) = f(K_3)$).

b) Clearly, if R is cyclic with regard to $\mathscr{K}$, then R is strongly antisymmetric with regard to $\mathscr{K}$ iff for any any odd permutation φ of the set $\{1, 2, 3\}$ and any mapping $f \in G^H$ for which there exists a mapping $g \in R$ such that $f(K_i) = g(K_{\varphi(i)})$ for $i = 1, 2, 3$ and $f(K_4) = g(K_4)$, we have $f \in R \Rightarrow f(K_1) = f(K_2)$.

2.11. Lemma. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 3$ *and let $\mathscr{K}$ be a t-decomposition of H. Then*

(a) *${}^1R_{\mathscr{K}}$ is regular with regard to $\mathscr{K}$.*

(b) *If R is cyclic or strongly symmetric with regard to $\mathscr{K}$, then R is regular with regard to $\mathscr{K}$.*

Proof. The assertion (a) is clear. By 1.13 (1) and (4) we have ${}^1R_{\mathscr{K}} \subseteq R \Leftrightarrow {}^1R_{\mathscr{K}} = R$. This fact together with (a) implies the assertion (b) for cyclicity. For strong symmetry it is obvious.

2.12. Lemma. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 3$ *and let $\mathscr{K}$ be a t-decomposition of H. If R is regular with regard to $\mathscr{K}$, then*

(1) $R = {}^3R_{\mathscr{K}}$,

(2) ${}^1(R \cap S)_{\mathscr{K}} = {}^1R_{\mathscr{K}} \cap {}^1S_{\mathscr{K}}$.

Proof. The inclusions $\subseteq$ follow from 1.13. Using the regularity of R one can easily prove the converse ones.

2.13. Remark. Let us note that by virtue of 2.11 and 2.12 we have ${}^1R_{\mathscr{K}} = {}^4R_{\mathscr{K}}$ for each relation $R \subseteq G^H$ with card $H \geqq 3$ and each t-decomposition $\mathscr{K}$ of H.

2.14. Theorem. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 3$ *and let $\mathscr{K}$ be a t-decomposition of H. Then*

(1) *R is strongly symmetric with regard to $\mathscr{K}$ iff it is both symmetric and cyclic with regard to $\mathscr{K}$.*

(2) *If R is cyclic with regard to $\mathscr{K}$, then R is strongly asymmetric (strongly antisymmetric) with regard to $\mathscr{K}$ iff it is asymmetric (antisymmetri) with regard to $\mathscr{K}$.*

Proof. Let $\mathscr{K} = \{K_i\}_{i=1}^4$. (1) I. Let R be strongly symmetric with regard to $\mathscr{K}$. Then R is obviously symmetric with regard to $\mathscr{K}$. Let $f \in {}^1R_{\mathscr{K}}$ and let φ be the permutation of the set $\{1, 2, 3\}$ defined by $\varphi(1) = 2$, $\varphi(2) = 3$, $\varphi(3) = 1$. Then there exists a mapping $g \in R$ with $f(K_i) = g(K_{\varphi(i)})$ for $i = 1, 2, 3$ and $f(K_4) = g(K_4)$. Hence, $f \in R$ and we have ${}^1R_{\mathscr{K}} \subseteq R$, i.e. R is cyclic with regard to $\mathscr{K}$.

II. Let R be symmetric and cyclic with regard to $\mathscr{K}$. Let φ be a permutation of the set $\{1, 2, 3\}$ and let $f \in G^H$, $g \in R$ be mappings such that $f(K_i) = g(K_{\varphi(i)})$ for $i = 1, 2, 3$ and $f(K_4) = g(K_4)$. If φ is even, then $f \in R$. Let φ be odd and let ψ be the odd permutation of the set $\{1, 2, 3\}$ defined by $\psi(1) = 2$, $\psi(2) = 1$, $\psi(3) = 3$. Then there exists an even permutation χ of $\{1, 2, 3\}$ such that $\varphi = \chi\psi$. Let $h, p \in G^H$ be mappings such that $h(K_i) = g(K_{\chi(i)})$ and $p(K_i) = h(K_{\psi(i)})$ for $i = 1, 2, 3$, $h(K_4) = g(K_4) = p(K_4)$. Then $h \in R$ because R is cyclic with regard to $\mathscr{K}$, and hence $p \in R$ because R is symmetric with regard to $\mathscr{K}$. Now we have $p(K_i) = g(K_{\chi[\psi(i)]}) = g(K_{\varphi(i)}) = f(K_i)$ for $i = 1, 2, 3$ and $p(K_4) = f(K_4)$. By 2.3, R is regular with regard to $\mathscr{K}$, and thus $f \in R$. Therefore R is strongly asymmetric with regard to $\mathscr{K}$.

(2) Let R be cyclic with regard to $\mathscr{K}$. I. If R is strongly asymmetric (strongly

antisymmetric) with regard to $\mathscr{K}$, then it is obviously asymmetric (antisymmetric) with regard to $\mathscr{K}$.

II. Let R be asymmetric (antisymmetric) with regard to $\mathscr{K}$. The proof of the strong asymmetry (strong antisymmetry) of R with regard to $\mathscr{K}$ is similar to part (1), II. (For the strong antisymmetry see also 2.10.b)).

2.15. Theorem. *Let $R, S \subseteq G^H$ be relations with* card $H \geqq 3$ *and let $\mathscr{K}$ be a t-decomposition of H. Then*

(a) *If R, S are cyclic (strongly symmetric) with regard to $\mathscr{K}$, then $R \cup S$, $R \cap S$, $R_{\mathscr{K}}^{-1}$, ${}^1R_{\mathscr{K}}$ have the same property.*

(b) *If R is acyclic (strongly asymmetric, strongly antisymmetric) with regard to $\mathscr{K}$, then $R \cap S$, $R_{\mathscr{K}}^{-1}$, ${}^1R_{\mathscr{K}}$ have the same property.*

(c) *If R is complete with regard to $\mathscr{K}$, then ${}^1R_{\mathscr{K}}$ has the same property.*

Proof. The assertion (a) is evident. (b) If R is acyclic (strongly asymmetric, strongly antisymmetric) with regard to $\mathscr{K}$, then clearly $R \cap S$ has the same property. Let R be acyclic with regard to $\mathscr{K}$. Then by 1.13, 2.4 and 2.12 we have ${}^1(R_{\mathscr{K}}^{-1})_{\mathscr{K}} \cap \cap R_{\mathscr{K}}^{-1} = {}^1(R_{\mathscr{K}}^{-1})_{\mathscr{K}} \cap {}^3(R_{\mathscr{K}}^{-1})_{\mathscr{K}} = {}^1(R_{\mathscr{K}}^{-1})_{\mathscr{K}} \cap {}^1[{}^2(R_{\mathscr{K}}^{-1})_{\mathscr{K}}]_{\mathscr{K}} = {}^1[R_{\mathscr{K}}^{-1} \cap {}^2(R_{\mathscr{K}}^{-1})_{\mathscr{K}}]_{\mathscr{K}} = = {}^1[R_{\mathscr{K}}^{-1}({}^1R_{\mathscr{K}})_{\mathscr{K}}^{-1}]_{\mathscr{K}} = {}^1[(R \cap {}^1R_{\mathscr{K}})_{\mathscr{K}}^{-1}]_{\mathscr{K}} = \emptyset$. Hence $R_{\mathscr{K}}^{-1}$ is acyclic with regard to $\mathscr{K}$. By 2.12 (2), ${}^2R_{\mathscr{K}} \cap {}^1R_{\mathscr{K}} = {}^1({}^1R_{\mathscr{K}} \cap R)_{\mathscr{K}} = \emptyset$ and therefore ${}^1R_{\mathscr{K}}$ is acyclic with regard to $\mathscr{K}$.

Let $\mathscr{K} = \{K_i\}_{i=1}^4$ and let R be strongly asymmetric with regard to $\mathscr{K}$. Suppose that $R_{\mathscr{K}}^{-1}$ does not have the same property. Then there exist an odd permutation φ of the set $\{1, 2, 3\}$ and mappings $f, g \in R_{\mathscr{K}}^{-1}$ such that $f(K_i) = g(K_{\varphi(i)})$ for $i = 1, 2, 3$ and $f(K_4) = g(K_4)$. Thus, there exist $h, p \in R$ such that for the odd permutation ψ of the set $\{1, 2, 3\}$ defined by $\psi(1) = 2$, $\psi(2) = 1$, $\psi(3) = 3$ we have $f(K_i) = h(K_{\psi(i)})$ and $g(K_i) = p(K_{\psi(i)})$ for $i = 1, 2, 3$, $f(K_4) = h(K_4)$ and $g(K_4) = p(K_4)$. Now we have $h(K_i) = f(K_{\psi^{-1}(i)}) = g(K_{\varphi[\psi^{-1}(i)]}) = p(K_{\psi\{\varphi[\psi^{-1}(i)]\}})$. However, this is a contradiction with the strong asymmetry of R, because $\psi\varphi\psi^{-1}$ is an odd permutation. Therefore $R_{\mathscr{K}}^{-1}$ is strongly asymmetric with regard to $\mathscr{K}$. Analogously we can prove that also ${}^1R_{\mathscr{K}}$ is strongly asymmetric with regard to $\mathscr{K}$. As for the strong antisymmetry the proof is similar. The assertion (c) is obvious.

2.16. Theorem. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 3$ *and let $\mathscr{K}$ be a t-decomposition of H. Then*

(1) *If R is complete or regular or has any one of the properties defined in* 2.9 *with regard to $\mathscr{K}$, then R has the same property with regard to ${}^*\mathscr{K}$.*

(2) *Let R be cyclic with regard to $\mathscr{K}$. If R has any one of the properties defined in* 2.1 *with regard to $\mathscr{K}$, then R has the same property with regard to ${}^*\mathscr{K}$.*

Proof. (1) For completeness and regularity the assertion is obvious. For cyclicity, acyclicity and anticyclicity it results from 1.12 (b), and for the strong symmetry from 2.14 (1) and 1.12 (d). Let $\mathscr{K} = \{K_i\}_{i=1}^4$ and let R be strongly asymmetric with regard to $\mathscr{K}$. Let ψ be the even permutation of the set $\{1, 2, 3\}$ defined by $\psi(1) = 2$ $\psi(2) = 3$, $\psi(3) = 1$. Then for ${}^*\mathscr{K} = \{{}^*K_i\}_{i=1}^4$ we have ${}^*K_i = K_{\psi(i)}$ for $i = 1, 2, 3$.

Let φ be an odd permutation of $\{1, 2, 3\}$ and let $f \in G^H$, $g \in R$ be mappings such that $f(*K_i) = g(*K_{\varphi(i)})$ for $i = 1, 2, 3$ and $f(*K_4) = g(*K_4)$. Then we have $f(K_i) = f(*K_{\psi^{-1}(i)}) = g(*K_{\varphi[\psi^{-1}(i)]}) = g(K_{\psi\{\varphi[\psi^{-1}(i)]\}})$ and $f(K_4) = g(K_4)$. Since $\psi\varphi\psi^{-1}$ is an odd permutation, we have $f \notin R$. Therefore R is strongly asymmetric with regard to $*\mathscr{K}$. For the strong antisymmetry the proof is similar.

(2) Let R be cyclic with regard to $\mathscr{K}$. If R is reflexive with regard to $\mathscr{K}$, then 1.12 (c) yields ${}^2(E_{\mathscr{K}})_{\mathscr{K}} = {}^3(E_{*\mathscr{K}})_{\mathscr{K}}$. By (1) $E_{*\mathscr{K}}$ is regular with regard to $\mathscr{K}$ and therefore ${}^3(E_{*\mathscr{K}})_{\mathscr{K}} = E_{*\mathscr{K}}$ according to 2.12 (1). Now we have $E_{*\mathscr{K}} = {}^2(E_{\mathscr{K}})_{\mathscr{K}} \subseteq {}^2R_{\mathscr{K}} \subseteq R$ and thus R is reflexive with regard to $\mathscr{K}$.

Let R be areflexive with regard to $\mathscr{K}$. By 1.12 (c) and 2.12 (2) we have $E_{\mathscr{K}} \cap R = {}^1(E_{*\mathscr{K}})_{\mathscr{K}} \cap {}^1R_{\mathscr{K}} = {}^1(E_{*\mathscr{K}} \cap R)_{\mathscr{K}} = \emptyset$. But then $E_{*\mathscr{K}} \cap R = \emptyset$ and R is reflexive with regard to $\mathscr{K}$.

According to 1.12 (d) we have $R_{\mathscr{K}}^{-1} = R_{*\mathscr{K}}^{-1}$ and hence the assertion is valid for symmetry, asymmetry and antisymmetry.

Let R be transitive with regard to $\mathscr{K}$. From 1.12 (e) it follows that $R^2_{\mathscr{K}} = {}^2(R^2_{**\mathscr{K}})_{\mathscr{K}}$. Since $R^2_{**\mathscr{K}}$ is regular with regard to $\mathscr{K}$ by 2.3 and (1), we have ${}^1(R^2_{\mathscr{K}})_{\mathscr{K}} = {}^3(R^2_{**\mathscr{K}})_{\mathscr{K}} = R^2_{**\mathscr{K}}$ according to 2.12. Further, $R^2_{\mathscr{K}} \subseteq R$ implies that ${}^1(R^2_{\mathscr{K}})_{\mathscr{K}} \subseteq {}^1R_{\mathscr{K}} = R$ by 1.13 (4). Thus $R^2_{**\mathscr{K}} \subseteq R$ and therefore R is transitive with regard to $**\mathscr{K}$. Consequently, as R is cyclic with regard to $**\mathscr{K}$ by 1.12 (b), R is transitive with regard to $****\mathscr{K} = *\mathscr{K}$.

Let R be atransitive with regard to $\mathscr{K}$. Then 1.12 (e) and 2.12 (2) yield $R^2_{*\mathscr{K}} \cap R = {}^2(R^2_{\mathscr{K}})_{\mathscr{K}} \cap {}^2R_{\mathscr{K}} = {}^2(R^2_{\mathscr{K}} \cap R)_{\mathscr{K}} = \emptyset$. Hence R is atransitive with regard to $*\mathscr{K}$.

As for completeness and regularity, the assertion follows from (1). The proof is complete.

3. HULLS OF RELATIONS

Analogously to [2] 1.6 we define:

3.1. Definition. Let $R \subseteq G^H$ be a relation with card $H \geqq 2$ (card $H \geqq 3$) and let $\mathscr{K}$ be a b-decomposition (t-decomposition) of H. Let (p) be any one of the properties with regard to $\mathscr{K}$ defined in 2.1. A relation $Q \subseteq G^H$ is called the *(p)-hull of R with regard to $\mathscr{K}$*, if

(1) $R \subseteq Q$,

(2) Q has the property (p),

(3) if $S \subseteq G^H$ is any relation having the property (p) and such that $R \subseteq S$, then $Q \subseteq S$.

Let $R \subseteq G^H$ be a relation with card $H \geqq 3$ and let $\mathscr{K}$ be a t-decomposition of H. Let (p) be any one of the properties with regard to $\mathscr{K}$ defined in 2.9. A relation $Q \subseteq G^H$ is called the *(p)-hull of R with regard to $\mathscr{K}$* if (1), (2), (3) hold.

3.2. Remark. Thus, R has the property (p) with regard to $\mathscr{K}$ iff there exists the (p)-hull Q of R with regard to $\mathscr{K}$ and $R = Q$.

3.3 Theorem. *Let* $R \subseteq G^H$ *be a relation with* card $H \geqq 2$ (card $H \geqq 3$) *and let* $\mathscr{K}$ *be a b-decomposition* (*t-decomposition*) *of H. Then the following relations exist:*

(1) *the reflexive hull of R with regard to* $\mathscr{K}$, *and denoting it by* $R_{\mathscr{K}}^{(r)}$ *we have* $R_{\mathscr{K}}^{(r)} = R \cup E_{\mathscr{K}}$,

(2) *the symmetric hull of R with regard to* $\mathscr{K}$, *and denoting it by* $R_{\mathscr{K}}^{(s)}$ *we have* $R_{\mathscr{K}}^{(s)} = R_{\mathscr{K}}^{-1} \cup (R_{\mathscr{K}}^{-1})_{\mathscr{K}}^{-1}$,

(3) *the transitive hull of R with regard to* $\mathscr{K}$, *and denoting it by* $R_{\mathscr{K}}^{(t)}$ *we have* $R_{\mathscr{K}}^{(t)} = \bigcup_{n=1}^{\infty} R_{\mathscr{K}}^n$,

(4) *the regular hull of R with regard to* $\mathscr{K}$, *and denoting it by* $R_{\mathscr{K}}^{(g)}$ *we have* $R_{\mathscr{K}}^{(g)} = (R_{\mathscr{K}}^{-1})_{\mathscr{K}}^{-1}$.

Proof. As for (1) the statement is obvious. (2) Put $Q = R_{\mathscr{K}}^{-1} \cup (R_{\mathscr{K}}^{-1})_{\mathscr{K}}^{-1}$. By 1.8 (2) we have $R \subseteq Q$. Further, by 1.8 (3), 2.3 (a) and 2.4 (1) we have $Q_{\mathscr{K}}^{-1} = (R_{\mathscr{K}}^{-1})_{\mathscr{K}}^{-1} \cup [(R_{\mathscr{K}}^{-1})_{\mathscr{K}}^{-1}]_{\mathscr{K}}^{-1} = (R_{\mathscr{K}}^{-1})_{\mathscr{K}}^{-1} \cup R_{\mathscr{K}}^{-1} = Q$. Thus, Q is symmetric with regard to $\mathscr{K}$. Let $S \subseteq G^H$ be a relation which is symmetric with regard to $\mathscr{K}$ and such that $R \subseteq S$. Then, according to 1.5 (5), $Q = R_{\mathscr{K}}^{-1} \cup (R_{\mathscr{K}}^{-1})_{\mathscr{K}}^{-1} \subseteq S_{\mathscr{K}}^{-1} \cup (S_{\mathscr{K}}^{-1})_{\mathscr{K}}^{-1} = S$. Therefore Q is the symmetric hull of R with regard to $\mathscr{K}$.

(3) Let $\mathscr{K} = \{K_i\}_{i=1}^{3(4)}$ and put $Q = \bigcup_{n=1}^{\infty} R_{\mathscr{K}}^n$. Clearly, $R \subseteq Q$. Let $f \in Q_{\mathscr{K}}^2$ be a mapping. Then there exist $g, h \in Q$ such that $f(K_1) = g(K_1)$, $f(K_2) = h(K_2)$, $f(K_i) = g(K_i) = h(K_i)$ for $i = 3$ (for $i = 3, 4$), $g(K_2) = h(K_1)$. From $g, h \in Q$ it follows that there exist numbers $m, n \in N$ such that $g \in R_{\mathscr{K}}^m$, $h \in R_{\mathscr{K}}^n$. Consequently, $f \in (R_{\mathscr{K}}^m R_{\mathscr{K}}^n)_{\mathscr{K}} = R_{\mathscr{K}}^{m+n} \subseteq \bigcup_{n=1}^{\infty} R_{\mathscr{K}}^n = Q$. Hence $Q_{\mathscr{K}}^2 \subseteq Q$, i.e. Q is transitive with regard to $\mathscr{K}$. Let $S \subseteq G^H$ be a mapping which is transitive with regard to $\mathscr{K}$ and such that $R \subseteq S$. Then we can easily prove by induction that $R_{\mathscr{K}}^n \subseteq S$ for any $n \in N$. Therefore $Q \subseteq S$ and thus Q is the transitive hull of R with regard to $\mathscr{K}$.

As for (4), the statement results from 1.8 (2) and (5), 2.3 (a) and 2.4 (1).

3.4. Remark. Let $R \subseteq G^H$ be a relation with card $H \geqq 3$ and let $\mathscr{K} = \{K_2\}_{i=1}^4$ be a t-decomposition of H. Let $\mathscr{K}'$ be the b-decomposition of H defined in the same way as in 1.5. Then (see 2.2)

(a) $R_{\mathscr{K}}^{(r)} = R_{\mathscr{K}'}^{(r)}$,

(b) $R_{\mathscr{K}}^{(s)} \subseteq R_{\mathscr{K}'}^{(s)}$, $R_{\mathscr{K}}^{(t)} \subseteq R_{\mathscr{K}'}^{(t)}$, $R_{\mathscr{K}}^{(g)} \subseteq R_{\mathscr{K}'}^{(g)}$ (and the inclusions can be replaced by equalities provided $K_4 = \emptyset$).

3.5. Theorem. *Let* $R \subseteq G^H$ *be a relation with* card $H \geqq 2$ (card $H \geqq 3$) *and let* $\mathscr{K}$ *be a b-decomposition* (*t-decomposition*) *of H. Then*

(a) *If R is symmetric* [*antisymmetric, complete, regular, both transitive and regular*] *with regard to* $\mathscr{K}$, *then* $R_{\mathscr{K}}^{(r)}$ *has the same property.*

(b) *If R is reflexive* [*areflexive, complete, regular*] *with regard to* $\mathscr{K}$, *then* $R_{\mathscr{K}}^{(s)}$ *has the same property.*

(c) *If R is reflexive [symmetric, complete, regular] with regard to $\mathscr{K}$, then $R^{(t)}_{\mathscr{K}}$ has the same property.*

(d) *If R has any one of the properties defined in* 2.1, *then then $R^{(g)}_{\mathscr{K}}$ has the same property.*

Proof. The proofs of all assertions contained in 3.5 are very easy because of the previous results. We present only three of them. Let $\mathscr{K} = \{K_i\}_{i=1}^{3(4)}$.

1. Let R be both transitive and regular with regard to $\mathscr{K}$. Clearly, then $R^{(r)}_{\mathscr{K}}$ is regular with regard to $\mathscr{K}$. Let $f \in (R^{(r)}_{\mathscr{K}})^2_{\mathscr{K}}$. Then there exist $g, h \in R^{(r)}_{\mathscr{K}}$ such that $f(K_1) = g(K_1)$, $f(K_2) = h(K_2)$, $f(K_i) = g(K_i) = h(K_i)$ for $i = 3$ (for $i = 3, 4$) and $g(K_2) = h(K_1)$. By 3.3 (1), $g, h \in R \cup E_{\mathscr{K}}$. Since R and $E_{\mathscr{K}}$ are transitive with regard to $\mathscr{K}$, we have $g, h \in R \Rightarrow f \in R \subseteq R^{(r)}_{\mathscr{K}}$ and $g, h \in E_{\mathscr{K}} \Rightarrow f \in E_{\mathscr{K}} \subseteq R^{(r)}_{\mathscr{K}}$. If $g \in R$, $h \in E_{\mathscr{K}}$, then $f(K_i) = g(K_i)$ for $i = 1, 2, 3$ (for $i = 1, 2, 3, 4$) and hence $f \in R \subseteq R^{(r)}_{\mathscr{K}}$. Similarly, if $h \in R$, $g \in E_{\mathscr{K}}$, then $f(K_i) = h(K_i)$ for $i = 1, 2, 3$ (for $i = 1, 2, 3, 4$) and thus $f \in R \subseteq R^{(r)}_{\mathscr{K}}$. Therefore $(R^{(r)}_{\mathscr{K}})^2_{\mathscr{K}} \subseteq R^{(r)}_{\mathscr{K}}$, i.e., $R^{(r)}_{\mathscr{K}}$ is transitive with regard to $\mathscr{K}$.

2. Let R be symmetric with regard to $\mathscr{K}$. Then 1.8 (6) and 2.7 imply that $R^n_{\mathscr{K}}$ is symmetric with regard to $\mathscr{K}$ for every $n \in N$. Now, 3.3 (3) and 2.6 (b) imply that $R^{(t)}_{\mathscr{K}}$ is symmetric with regard to $\mathscr{K}$.

3. Let R be transitive with regard to $\mathscr{K}$ and let $f \in (R^{(g)}_{\mathscr{K}})^2_{\mathscr{K}}$. There exist $g, h \in R^{(g)}_{\mathscr{K}}$ such that $f(K_1) = g(K_1)$, $f(K_2) = h(K_2)$, $f(K_i) = g(K_i) = h(K_i)$ for $i = 3$ (for $i = 3, 4$) and $g(K_2) = h(K_1)$. By 3.3 (4) there exist $p, q \in R$ such that $g(K_i) = p(K_i)$ and $q(K_i) = h(K_i)$ for $i = 1, 2, 3$ (for $i = 1, 2, 3, 4$). Hence $f \in R^2_{\mathscr{K}} \subseteq R \subseteq R^{(g)}_{\mathscr{K}}$. Thus, we have $(R^{(g)}_{\mathscr{K}})^2_{\mathscr{K}} \subseteq R^{(g)}_{\mathscr{K}}$, i.e. $R^{(g)}_{\mathscr{K}}$ is transitive with regard to $\mathscr{K}$.

3.6. Corollary. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 2$ (card $H \geqq 3$) *and let $\mathscr{K}$ be a b-decomposition (t-decomposition) of H. Then*

(a) $(R^{(r)}_{\mathscr{K}})^{(s)}_{\mathscr{K}} = (R^{(s)}_{\mathscr{K}})^{(r)}_{\mathscr{K}}$, $(R^{(r)}_{\mathscr{K}})^{(g)}_{\mathscr{K}} = (R^{(g)}_{\mathscr{K}})^{(r)}_{\mathscr{K}}$, $(R^{(s)}_{\mathscr{K}})^{(g)}_{\mathscr{K}} = (R^{(g)}_{\mathscr{K}})^{(s)}_{\mathscr{K}}$,
$(R^{(t)}_{\mathscr{K}})^{(g)}_{\mathscr{K}} = (R^{(g)}_{\mathscr{K}})^{(t)}_{\mathscr{K}}$, $(R^{(t)}_{\mathscr{K}})^{(r)}_{\mathscr{K}} \subseteq (R^{(r)}_{\mathscr{K}})^{(t)}_{\mathscr{K}}$, $(R^{(t)}_{\mathscr{K}})^{(s)}_{\mathscr{K}} \subseteq (R^{(s)}_{\mathscr{K}})^{(t)}_{\mathscr{K}}$.

(b) *If R is regular with regard to $\mathscr{K}$, then* $(R^{(t)}_{\mathscr{K}})^{(r)}_{\mathscr{K}} = (R^{(r)}_{\mathscr{K}})^{(t)}_{\mathscr{K}}$.

Proof. Since $R \subseteq R^{(s)}_{\mathscr{K}}$, we have $R^{(r)}_{\mathscr{K}} \subseteq (R^{(s)}_{\mathscr{K}})^{(r)}_{\mathscr{K}}$, and consequently $(R^{(r)}_{\mathscr{K}})^{(s)}_{\mathscr{K}} \subseteq$ $\subseteq [(R^{(s)}_{\mathscr{K}})^{(r)}_{\mathscr{K}}]^{(s)}_{\mathscr{K}}$. But by 3.5, $(R^{(s)}_{\mathscr{K}})^{(r)}_{\mathscr{K}}$ is symmetric with regard to $\mathscr{K}$, and hence $[(R^{(s)}_{\mathscr{K}})^{(r)}_{\mathscr{K}}]^{(s)}_{\mathscr{K}} = (R^{(s)}_{\mathscr{K}})^{(r)}_{\mathscr{K}}$. Thus $(R^{(r)}_{\mathscr{K}})^{(s)}_{\mathscr{K}} \subseteq (R^{(s)}_{\mathscr{K}})^{(r)}_{\mathscr{K}}$. Similarly we can prove the converse inclusion as well as the other inclusions.

3.7. Example. Here we give an example showing that none of the two inclusions in 3.6 can be replaced by equality. Let $G = \{a, b\}$ and let us be given the 4-ary relation $R = \{(a, b, a, a)\}$ on G. Consider the b-decomposition $\mathscr{K} = \{\{1, 2\}, \{3, 4\}, \emptyset\}$ of the index set $H = \{1, 2, 3, 4\}$. As R is transitive with regard to $\mathscr{K}$, we have $R^{(r)}_{\mathscr{K}} = R$.

Clearly, $(b, a, b, a) \in R^{(r)}_{\mathscr{K}}$ and since $(a, b, a, a) \in R^{(r)}_{\mathscr{K}}$, we have $(b, a, a, a) \in (R^{(r)}_{\mathscr{K}})^{(t)}_{\mathscr{K}}$. But $(b, a, a, a) \notin R^{(r)}_{\mathscr{K}} = (R^{(t)}_{\mathscr{K}})^{(r)}_{\mathscr{K}}$. Hence $(R^{(r)}_{\mathscr{K}})^{(t)}_{\mathscr{K}} \nsubseteq (R^{(t)}_{\mathscr{K}})^{(r)}_{\mathscr{K}}$.

Similarly, $(a, a, a, b) \in R^{(s)}_{\mathscr{K}}$ and since $(a, b, a, a) \in R^{(s)}_{\mathscr{K}}$, we have $(a, a, a, a) \in$

$\in (R^{(s)}_{\mathscr{K}})^{(t)}_{\mathscr{K}}$. But $(a, a, a, a) \notin R^{(s)}_{\mathscr{K}} = (R^{(t)}_{\mathscr{K}})^{(s)}_{\mathscr{K}}$ and thus $(R^{(s)}_{\mathscr{K}})^{(t)}_{\mathscr{K}} \nsubseteq (R^{(t)}_{\mathscr{K}})^{(s)}_{\mathscr{K}}$. Moreover, we have

3.8. Corollary. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 2$ (card $H \geqq 3$) *and let $\mathscr{K}$ be a b-decomposition (t-decomposition) of H. Then* $(R^{(r)}_{\mathscr{K}})^{(t)}_{\mathscr{K}} = [(R^{(t)}_{\mathscr{K}})^{(r)}_{\mathscr{K}}]^{(t)}_{\mathscr{K}}$, $(R^{(s)}_{\mathscr{K}})^{(t)}_{\mathscr{K}} = [(R^{(t)}_{\mathscr{K}})^{(s)}_{\mathscr{K}}]^{(t)}_{\mathscr{K}}$.

Proof. The inclusion $(R^{(r)}_{\mathscr{K}})^{(t)}_{\mathscr{K}} \subseteq [(R^{(t)}_{\mathscr{K}})^{(r)}_{\mathscr{K}}]^{(t)}_{\mathscr{K}}$ is clear. By 3.6 we have $(R^{(t)}_{\mathscr{K}})^{(r)}_{\mathscr{K}} \subseteq (R^{(r)}_{\mathscr{K}})^{(t)}_{\mathscr{K}}$ and consequently $[(R^{(t)}_{\mathscr{K}})^{(r)}_{\mathscr{K}}]^{(t)}_{\mathscr{K}} \subseteq (R^{(r)}_{\mathscr{K}})^{(t)}_{\mathscr{K}}$. Thus the equality is valid. The second equality can be proved analogously.

3.9. Theorem. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 3$ *and let $\mathscr{K}$ be a t-decomposition of H. Then the following relations exist:*

(1) *the cyclic hull of R with regard to $\mathscr{K}$, and denoting it by $R^{(c)}_{\mathscr{K}}$ we have* $R^{(c)}_{\mathscr{K}} = \bigcup_{n=1}^{3} {}^{n}R_{\mathscr{K}}$,

(2) *the strongly symmetric hull of R with regard to $\mathscr{K}$, and denoting it by $R^{(d)}_{\mathscr{K}}$ we have* $R^{(d)}_{\mathscr{K}} = (R^{(s)}_{\mathscr{K}})^{(c)}_{\mathscr{K}}$.

Proof. (1) Put $Q = \bigcup_{n=1}^{3} {}^{n}R_{\mathscr{K}}$. By 1.13 (1) we have $R \subseteq Q$. Further, 1.13 (2) and 2.13 yield ${}^{1}Q_{\mathscr{K}} = \bigcup_{n=2}^{4} {}^{n}R_{\mathscr{K}} = \bigcup_{n=1}^{3} {}^{n}R_{\mathscr{K}} = Q$. Let $S \subseteq G^H$ be a relation which is cyclic with regard to $\mathscr{K}$ and such that $R \subseteq S$. Then, according to 1.13 (4), $Q = \bigcup_{n=1}^{3} {}^{n}R_{\mathscr{K}} \subseteq \bigcup_{n=1}^{3} {}^{n}S_{\mathscr{K}} = S$. Therefore Q is the cyclic hull of R with regard to $\mathscr{K}$.

(2) Clearly, $R \subseteq (R^{(s)}_{\mathscr{K}})^{(c)}_{\mathscr{K}}$. If R is symmetric with regard to $\mathscr{K}$, then by 1.8 (3), 1.13 (4), (5) and 2.13 we have $[\bigcup_{n=1}^{3} {}^{n}R_{\mathscr{K}}]^{-1}_{\mathscr{K}} = ({}^{1}R_{\mathscr{K}})^{-1}_{\mathscr{K}} \cup ({}^{2}R_{\mathscr{K}})^{-1}_{\mathscr{K}} \cup ({}^{3}R_{\mathscr{K}})^{-1}_{\mathscr{K}} = {}^{2}(R^{-1}_{\mathscr{K}})_{\mathscr{K}} \cup {}^{4}(R^{-1}_{\mathscr{K}})_{\mathscr{K}} \cup {}^{6}(R^{-1}_{\mathscr{K}})_{\mathscr{K}} \subseteq {}^{2}R_{\mathscr{K}} \cup {}^{4}R_{\mathscr{K}} \cup {}^{6}R_{\mathscr{K}} = {}^{2}R_{\mathscr{K}} \cup {}^{1}R_{\mathscr{K}} \cup {}^{3}R_{\mathscr{K}} = \bigcup_{n=1}^{3} {}^{n}R_{\mathscr{K}}$, and hence $\bigcup_{n=1}^{3} {}^{n}R_{\mathscr{K}}$ is symmetric with regard to $\mathscr{K}$. Therefore $(R^{(s)}_{\mathscr{K}})^{(c)}_{\mathscr{K}} = \bigcup_{n=1}^{3} {}^{n}(R^{(s)}_{\mathscr{K}})_{\mathscr{K}}$ is symmetric with regard to $\mathscr{K}$. Thus, by 2.14 (1), $(R^{(s)}_{\mathscr{K}})^{(c)}_{\mathscr{K}}$ is strongly symmetric with regard to $\mathscr{K}$. Let $S \subseteq G^H$ be a relation which is strongly symmetric with regard to $\mathscr{K}$ and such that $R \subseteq S$. Then, clearly, $(R^{(s)}_{\mathscr{K}})^{(c)}_{\mathscr{K}} \subseteq (S^{(s)}_{\mathscr{K}})^{(c)}_{\mathscr{K}}$. But 2.14 (1) implies $(S^{(s)}_{\mathscr{K}})^{(c)}_{\mathscr{K}} = S$. Consequently, $(R^{(s)}_{\mathscr{K}})^{(c)}_{\mathscr{K}} \subseteq S$. We have proved that $(R^{(s)}_{\mathscr{K}})^{(c)}_{\mathscr{K}}$ is the strongly symmetric hull of R with regard to $\mathscr{K}$.

3.10. Remark. Of course, we have $R^{(d)}_{\mathscr{K}}$ $(R^{(c)}_{\mathscr{K}}) = \{f \in G^H |$ there exist a permutation (even permutation) φ of the set $\{1, 2, 3\}$ and a mapping $g \in R$ such that $f(K_i) = g(K_{\varphi(i)})$ for $i = 1, 2, 3$ and $f(K_4) = g(K_4)\}$.

3.11. Theorem. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 3$ *and let $\mathscr{K}$ be a t-decomposition of H. Then*

(a) *If R is cyclic (strongly symmetric) with regard to $\mathscr{K}$, then $R^{(s)}_{\mathscr{K}}$ has the same property.*

(b) *If R has any one of the properties defined in 2.9, then $R^{(g)}_{\mathscr{K}}$ has the same property.*

(c) *If R is reflexive (symmetric, complete, regular, strongly symmetric, strongly asymmetric, strongly antisymmetric) with regard to $\mathscr{K}$, then $R^{(c)}_{\mathscr{K}}$ has the same property.*

(d) *If R is reflexive (symmetric, complete, regular, cyclic) with regard to $\mathscr{K}$, then $R^{(d)}_{\mathscr{K}}$ has the same property.*

Proof. (a) Let R be cyclic with regard to $\mathscr{K}$. Then, by virtue of 2.11, R is regular with regard to $\mathscr{K}$, and hence $(R^{-1}_{\mathscr{K}})^{-1}_{\mathscr{K}} = R$ by 2.4. Now, using 1.13, 1.8 and 3.3, we have ${}^2(R^{(s)}_{\mathscr{K}})_{\mathscr{K}} = {}^2[R \cup R^{-1}_{\mathscr{K}}]_{\mathscr{K}} = {}^2R_{\mathscr{K}} \cup {}^2(R^{-1}_{\mathscr{K}})_{\mathscr{K}} = {}^2R_{\mathscr{K}} \cup ({}^1R_{\mathscr{K}})^{-1}_{\mathscr{K}} \subseteq R \cup$ $\cup R^{-1}_{\mathscr{K}} = R^{(s)}_{\mathscr{K}}$. Thus ${}^1(R^{(s)}_{\mathscr{K}})_{\mathscr{K}} = {}^4(R^{(s)}_{\mathscr{K}})_{\mathscr{K}} \subseteq {}^2(R^{(s)}_{\mathscr{K}})_{\mathscr{K}} \subseteq R^{(s)}_{\mathscr{K}}$. Consequently, $R^{(s)}_{\mathscr{K}}$ is cyclic with regard to $\mathscr{K}$.

If R is strongly symmetric with regard to $\mathscr{K}$, then it is symmetric with regard to $\mathscr{K}$, and thus $R^{(s)}_{\mathscr{K}} = R$. Therefore $R^{(s)}_{\mathscr{K}}$ is strongly symmetric with regard to $\mathscr{K}$.

The assertion (b) is obvious.

(c) As for reflexivity, completeness and regularity, the assertion is clear. As for the strong symmetry, it follows from 2.14 (1). In the proof of 3.9 it is shown that if R is symmetric with regard to $\mathscr{K}$, then $R^{(c)}_{\mathscr{K}}$ has also this property.

Let $\mathscr{K} = \{K_i\}_{i=1}^4$ and let R be strongly asymmetric with regard to $\mathscr{K}$. Suppose that $R^{(c)}_{\mathscr{K}}$ is not asymmetric with regard to $\mathscr{K}$. Then there exist mappings $f, g \in R^{(c)}_{\mathscr{K}}$ such that $f(K_i) = g(K_{\varphi(i)})$ for $i = 1, 2, 3$ and $f(K_4) = g(K_4)$ where φ is the odd permutation of the set $\{1, 2, 3\}$ defined by $\varphi(1) = 2$, $\varphi(2) = 1$, $\varphi(3) = 3$. As $f, g \in$ $\in R^{(c)}_{\mathscr{K}}$, by 3.10 there exist even permutations ψ, χ of $\{1, 2, 3\}$ and mappings $h, p \in R$ such that $f(K_i) = h(K_{\psi(i)})$ and $g(K_i) = p(K_{\chi(i)})$ for $i = 1, 2, 3$, $f(K_4) = h(K_4)$ and $g(K_4) = p(K_4)$. Now we have $h(K_i) = f(K_{\psi^{-1}(i)}) = g(K_{\varphi[\psi^{-1}(i)]}) = p(K_{\chi\{\varphi[\psi^{-1}(i)]\}})$ for $i = 1, 2, 3$ and $h(K_4) = p(K_4)$. Since φ is odd, the composition $\chi\varphi\psi^{-1}$ is odd, too. However, this contradicts the strong asymmetry of R. Therefore R is asymmetric with regard to $\mathscr{K}$, and by 2.11 (2) it is strongly asymmetric with regard to $\mathscr{K}$. For the strong antisymmetry the proof is analogous.

The assertion (d) results from (a), (c), 3.5 (b) and 3.9.

3.12. Corollary. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 3$ *and let $\mathscr{K}$ be a t-decomposition of H. Then*

(a) $(R^{(c)}_{\mathscr{K}})^{(r)}_{\mathscr{K}} \subseteq (R^{(r)}_{\mathscr{K}})^{(c)}_{\mathscr{K}}$, $(R^{(c)}_{\mathscr{K}})^{(s)}_{\mathscr{K}} = (R^{(s)}_{\mathscr{K}})^{(c)}_{\mathscr{K}}$, $(R^{(c)}_{\mathscr{K}})^{(g)}_{\mathscr{K}} = (R^{(g)}_{\mathscr{K}})^{(c)}_{\mathscr{K}}$,

(b) $(R^{(d)}_{\mathscr{K}})^{(r)}_{\mathscr{K}} \subseteq (R^{(r)}_{\mathscr{K}})^{(d)}_{\mathscr{K}}$, $(R^{(d)}_{\mathscr{K}})^{(s)}_{\mathscr{K}} = (R^{(s)}_{\mathscr{K}})^{(d)}_{\mathscr{K}}$, $(R^{(d)}_{\mathscr{K}})^{(g)}_{\mathscr{K}} = (R^{(g)}_{\mathscr{K}})^{(d)}_{\mathscr{K}}$,

(c) $(R^{(c)}_{\mathscr{K}})^{(d)}_{\mathscr{K}} = (R^{(d)}_{\mathscr{K}})^{(c)}_{\mathscr{K}}$.

Proof. The statement follows from 3.11 analogously as 3.7 follows from 3.5.

3.13. Example. It is very easy to find an example showing that none of the two inclusions in 3.12 can be replaced by the equality: Let $G = \{a, b\}$ and let us be given the ternary relation $R = \{(a, a, a)\}$ on G. Let us consider the t-decomposition $\mathcal{K} = \{\{1\}, \{2\}, \{3\}, \emptyset\}$ of the index set $H = \{1, 2, 3\}$. Then R is cyclic with regard to $\mathcal{K}$ and hence $R^{(c)}_{\mathcal{K}} = R$. Clearly $(a, a, b) \in R^{(r)}_{\mathcal{K}}$ and consequently $(a, b, a) \in \in (R^{(r)}_{\mathcal{K}})^{(c)}_{\mathcal{K}}$. But $(a, b, a) \notin R^{(r)}_{\mathcal{K}} = (R^{(c)}_{\mathcal{K}})^{(r)}_{\mathcal{K}}$. Therefore $(R^{(r)}_{\mathcal{K}})^{(c)}_{\mathcal{K}} \nsubseteq (R^{(c)}_{\mathcal{K}})^{(r)}_{\mathcal{K}}$. Further, since $(R^{(r)}_{\mathcal{K}})^{(c)}_{\mathcal{K}} \subseteq (R^{(r)}_{\mathcal{K}})^{(d)}_{\mathcal{K}}$ clearly holds and since R is strongly symmetric with regard to $\mathcal{K}$, i.e. $R^{(d)}_{\mathcal{K}} = R$, we have $(R^{(r)}_{\mathcal{K}})^{(d)}_{\mathcal{K}} \nsubseteq (R^{(d)}_{\mathcal{K}})^{(r)}_{\mathcal{K}}$. Nevertheless, we have

3.14. Corollaty. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 3$ *and let $\mathcal{K}$ be a t-decomposition of H. Then* $[(R^{(c)}_{\mathcal{K}})^{(r)}_{\mathcal{K}}]^{(c)}_{\mathcal{K}} = (R^{(r)}_{\mathcal{K}})^{(c)}_{\mathcal{K}}$, $[(R^{(d)}_{\mathcal{K}})^{(r)}_{\mathcal{K}}]^{(d)}_{\mathcal{K}} = (R^{(r)}_{\mathcal{K}})^{(d)}_{\mathcal{K}}$.

Proof is analogous to that of 3.8.

4. PROJECTIONS OF RELATIONS

4.1. Definition. Let $R \subseteq G^H$ be a relation with card $H \geqq 3$ and let $\mathcal{K} = \{\mathcal{K}_i\}_{i=1}^4$ be a t-decomposition of H. Let $X \subseteq G$ be a subset fulfilling $0 <$ card $X \leqq$ card K_3. Then we define the relation $R_{X,\mathcal{K}} \subseteq G^{H-K_3}$ as follows:

$$g \in R_{X,\mathcal{K}} \Leftrightarrow \exists f \in R\colon f(K_3) = X\,,\quad f(K_i) = g(K_i) \quad \text{for} \quad i = 1, 2, 4\,.$$

$R_{X,\mathcal{K}}$ is called the *X-projection of R with regard to $\mathcal{K}$.*

4.2. Remark. We introduce the concept "X-projection of a relation" being motivated by [3] 1.9 where a binary relation is assigned to each ternary relation and each element of its carrier. However, only for ternary relations which are cyclic (with regard to $\mathcal{K} = \{\{1\}, \{2\}, \{3\}, \emptyset\}$) this assignment coincides with the X-projection with regard to $\mathcal{K}$.

4.3. Notation. Let H be a set with card $H \geqq 3$ and let $\mathcal{K} = \{K_i\}_{i=1}^4$ be a t-decomposition of H. Then $\tilde{\mathcal{K}}$ denotes the b-decomposition of the set $H - K_3$ defined by $\tilde{\mathcal{K}} = \{\tilde{K}_i\}_{i=1}^3$ where $\tilde{K}_i = K_i$ for $i = 1, 2$ and $\tilde{K}_3 = K_4$.

4.4. Lemma. *Let $R, S \subseteq G^H$ be relations with* card $H \geqq 3$ *and let $\mathcal{K}$ be a t-decomposition of H. Let $X \subseteq G$ be a subset fulfilling* $0 <$ card $X \leqq$ card K_3. *Then*

(1) $(E_{\mathcal{K}})_{X,\mathcal{K}} = E_{\tilde{\mathcal{K}}}$,
(2) $(R^{-1}_{\mathcal{K}})_{X,\mathcal{K}} = (R_{X,\mathcal{K}})^{-1}_{\tilde{\mathcal{K}}}$,
(3) $(R \cup S)_{X,\mathcal{K}} = R_{X,\mathcal{K}} \cup S_{X,\mathcal{K}}$,
(4) *if R or S is regular with regard to $\mathcal{K}$, then* $(R \cap S)_{X,\mathcal{K}} = R_{X,\mathcal{K}} \cap S_{X,\mathcal{K}}$,
(5) $[(RS)_{\mathcal{K}}]_{X,\mathcal{K}} = (R_{X,\mathcal{K}} S_{X,\mathcal{K}})_{\tilde{\mathcal{K}}}$,
(6) *if $R \subseteq S$, then* $R_{X,\mathcal{K}} \subseteq S_{X,\mathcal{K}}$.

Proof. The statement follows directly from the definitions. For example, let us prove (4). So, let $\mathcal{K} = \{K_i\}_{i=1}^4$ and suppose that R is regular with regard to $\mathcal{K}$. The inclusion $(R \cap S)_{X,\mathcal{K}} \subseteq R_{X,\mathcal{K}} \cap S_{X,\mathcal{K}}$ is clear. Let $f \in R_{X,\mathcal{K}} \cap S_{X,\mathcal{K}}$. Then there exist mappings $g \in R$ and $h \in S$ such that $g(K_3) = h(K_3) = X$, $g(K_i) = h(K_i) =$

$= f(K_i)$ for $i = 1, 2, 4$. Hence, by the definition 2.1 (5), we have $h \in R$. Thus $h \in R \cap S$ and therefore $f \in (R \cap S)_{X,\mathscr{K}}$.

Obviously, the assumption of regularity cannot be omitted in (4).

4.5. Theorem. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 3$ *and let $\mathscr{K}$ be a t-decomposition of H. Let $X \subseteq G$ be a subset fulfilling* $0 <$ card $X \leqq$ card K_3. *Then*

(a) *$R_{X,\mathscr{K}}$ is regular with regard to $\tilde{\mathscr{K}}$.*

(b) *If R has any one of the properties with regard to $\mathscr{K}$, defined in* 2.1 *except completeness, then $R_{X,\mathscr{K}}$ has the same property, however, with regard to $\tilde{\mathscr{K}}$.*

Proof. The assertion (a) is obvious and (b) follows from 4.4.

4.6. Lemma. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 3$ *and let $\mathscr{K} = \{K_i\}_{i=1}^4$ be a t-decomposition of H. Let R be cyclic and complete with regard to $\mathscr{K}$. Let $f \in G^H$ be a mapping such that $f(K_1) \neq f(K_2) \neq f(K_3) \neq f(K_1)$. Then $f \in R \cup R_{\mathscr{K}}^{-1}$.*

Proof As R is complete with regard to $\mathscr{K}$, there exist a permutation φ of the set $\{1, 2, 3\}$ and a mapping $g \in R$ such that $f(K_i) = g(K_{\varphi(i)})$ for $i = 1, 2, 3$ and $f(K_4) = g(K_4)$. If φ is even, then $f \in R$ because R is cyclic with regard to $\mathscr{K}$ Let φ be odd and let ψ be the odd permutation of $\{1, 2, 3\}$ defined by $\psi(1) = 2, \psi(2) = 1, \psi(3) = 3$. Then there exists an even permutation χ of $\{1, 2, 3\}$ such that $\varphi = \chi\psi$. Let $h, p \in G^H$ be mappings such that $h(K_i) = g(K_{\chi(i)})$ and $p(K_i) = h(K_{\psi(i)})$ for $i = 1, 2, 3$, and $h(K_4) = p(K_4) = g(K_4)$. Then $h \in R$ and hence $p \in R_{\mathscr{K}}^{-1}$. We have $f(K_i) = g(K_{\chi[\psi(i)]}) = h(K_{\psi(i)}) = p(K_i)$ for $i = 1, 2, 3$ and $f(K_4) = p(K_4)$. Therefore $f \in R_{\mathscr{K}}^{-1}$. Thus, $f \in R \cup R_{\mathscr{K}}^{-1}$.

4.7. Notation. Let $R \subseteq G^H$ be a relation and $F \subseteq G$ a non-empty subset. The restriction of R onto F is denoted by R/F, i.e. $R/F \subseteq F^H$ is the relation defined by $R/F = R \cap F^H$.

4.8. Theorem. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 3$ *and let $\mathscr{K} = \{K_i\}_{i=1}^4$ be a t-decomposition of H. Let $X \subseteq G$ be a subset fulfilling* $0 <$ card $X \leqq$ card K_3. *Let R be cyclic with regard to $\mathscr{K}$. If R is complete with regard to $\mathscr{K}$, then $R_{X,\mathscr{K}}/(G - \{x\})$ is complete with regard to $\tilde{\mathscr{K}}$ for each element $x \in X$.*

Proof. Let $x \in X$ be an element, let R be complete with regard to $\mathscr{K}$ and let $f \in (G - \{x\})^{H - K_3}$ be a mapping with $f(K_1) \neq f(K_2)$. We are to find a mapping $g \in R_{X,\mathscr{K}}$ and a permutation φ of the set $\{1, 2\}$ such that $f(K_i) = g(K_{\varphi(i)})$ for $i = 1, 2$ and $f(K_4) = g(K_4)$. Let $h \in G^H$ be a mapping such that $h(K_3) = X$ and $h(K_i) = f(K_i)$ for $i = 1, 2, 4$. Then $h(K_1) \neq h(K_2) \neq h(K_3) \neq h(K_1)$. Thus, by 4.6, $h \in R \cup R_{\mathscr{K}}^{-1}$. Supposing $h \in R$ we have $f \in R_{X,\mathscr{K}}$, hence $f(K_i) = g(K_{\varphi(i)})$ for $i = 1, 2$ and $f(K_4) = g(K_4)$ are valid trivially for $g = f$ and for the identity permutation φ of $\{1, 2\}$. Assume $h \in R_{\mathscr{K}}^{-1}$. Then there exists $p \in R$ such that $h(K_1) = p(K_2)$, $h(K_2) = p(K_1)$, $h(K_i) = p(K_i)$ for $i = 3, 4$. Let $q \in G^{H - K_3}$ be a mapping fulfilling $q(K_i) = p(K_i)$ for $i = 1, 2, 4$. Then $q \in R_{X,\mathscr{K}}$ and we have $f(K_1) = q(K_2)$, $f(K_2) = q(K_1)$,

$f(K_4) = q(K_4)$. Now, putting $g = q$ and $\varphi(1) = 2$, $\varphi(2) = 1$, we have $f(K_i) = g(K_{\varphi(i)})$ for $i = 1, 2$ and $f(K_4) = g(K_4)$. Therefore $R_{X,\mathscr{K}}/(G - \{x\})$ is complete with regard to $\mathscr{K}$.

4.9. Lemma. *Let $R, S \subseteq G^H$ be relations with* card $H \geqq 3$ *and let* $\mathscr{K} = \{K_i\}_{i=1}^4$ *be a t-decomposition of H. Let S be regular with regard to $\mathscr{K}$. If $R_{X,\mathscr{K}} \subseteq S_{X,\mathscr{K}}$ for any subset $X \subseteq G$ fulfilling* $0 <$ card $X \leqq$ card K_3, *then $R \subseteq S$.*

Proof. Let $f \in R$ and let $g \in G^{H-K}$ be such that $f(K_i) = g(K_i)$ for $i = 1, 2, 4$. Putting $X = f(K_3)$ we have $g \in R_{X,\mathscr{K}}$. Thus $g \in S_{X,\mathscr{K}}$ and consequently there exists a mapping $h \in S$ such that $h(K_3) = X$ and $h(K_i) = g(K_i)$ for $i = 1, 2, 4$. But now $f(K_i) = h(K_i)$ for $i = 1, 2, 3, 4$, hence $f \in S$ because S is regular with regard to $\mathscr{K}$. The inclusion $R \subseteq S$ is proved.

4.10. Theorem. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 3$ *and let* $\mathscr{K} = \{K_i\}_{i=1}^4$ *be a t-decomposition of H. Then*

(a) *If $R_{X,\mathscr{K}}$ is areflexive (asymmetric, antisymmetric, atransitive) with regard to $\tilde{\mathscr{K}}$ for any subset $X \subseteq G$ fulfilling* $0 <$ card $X \leqq$ card K_3, *then R has the same property, however, with regard to $\mathscr{K}$.*

(b) *Let R be regular with regard to $\mathscr{K}$. If $R_{X,\mathscr{K}}$ is reflexive (symmetric, transitive, complete) with regard to $\tilde{\mathscr{K}}$ for any subset $X \subseteq G$ fulfilling* $0 <$ card $X \leqq$ card K_3, *then R has the same property, however, with regard to $\mathscr{K}$.*

Proof. Except for completeness, the statement follows from 4.4 and 4.9. Thus, let $R_{X,\mathscr{K}}$ be complete with regard to $\tilde{\mathscr{K}}$. Let $f \in G^H$ be a mapping such that $f(K_1) \neq f(K_2) \neq f(K_3) \neq f(K_1)$. Put $X = f(K_3)$. Let $g \in G^{H-K}$ be a mapping fulfilling $g(K_i) = f(K_i)$ for $i = 1, 2, 4$. Then $g(K_1) \neq g(K_2)$ and therefore $g \in R_{X,\mathscr{K}}$. Consequently, there exists a mapping $h \in R$ such that $h(K_3) = X$ and $h(K_i) = g(K_i)$ for $i = 1, 2, 4$. Now we have $h(K_i) = f(K_i)$ for $i = 1, 2, 3, 4$, hence $f \in R$. Thus R is complete with regard to $\mathscr{K}$.

4.11. Theorem. *Let $R \subseteq G^H$ be a relation with* card $H \geqq 3$ *and let* $\mathscr{K} = \{K_i\}_{i=1}^4$ *be a t-decomposition of H. Let $X \subseteq G$ be a subset fulfilling* $0 <$ card $X \leqq$ card K_3. *If Q is the reflexive (symmetric, transitive, regular) hull of R with regard to $\mathscr{K}$, then $Q_{X,\mathscr{K}}$ is the same hull of $R_{X,\mathscr{K}}$, however, with regard to $\tilde{\mathscr{K}}$.*

Proof. The statement results from 3.3 and 4.4. Let us only note that $(R^{(g)}_{\mathscr{K}})_{X,\mathscr{K}} = (R_{X,\mathscr{K}})^{(g)}_{\tilde{\mathscr{K}}} = R_{X,\mathscr{K}}$ because $R_{X,\mathscr{K}}$ is regular with regard to $\tilde{\mathscr{K}}$.

4.12. Theorem. *Let $Q, R \subseteq G^H$ be relations with* card $H \geqq 3$ *and let* $\mathscr{K} = \{K_i\}_{i=1}^4$ *be a t-decomposition of H. Let Q, R be regular with regard to $\mathscr{K}$. If $Q_{X,\mathscr{K}}$ is the reflexive (symmetric, transitive) hull of $R_{X,\mathscr{K}}$ with regard to $\tilde{\mathscr{K}}$ for each subset $X \subseteq G$ fulfilling* $0 <$ card $X \leqq$ card K_3, *then Q is the same hull of R, however, with regard to $\mathscr{K}$.*

Proof. The assertion is a consequence of 3.3, 4.4 and 4.9.

5. CONCLUSION

Clearly, for binary relations the properties defined above (with regard to any one of the two b-decompositions of the index set $H = \{1, 2\}$) coincide with the well-known ones (see e.g. [4]). Hence, the notions of tolerance, quasi-order, equivalence, order, and the well-known results concerning these notions can be extended in a natural way to relations in the general sense.

Provided $R \subseteq G^H$ is a relation with card $H \geqq 3$ and $\mathscr{K}$ is a t-decomposition of H, let R be called a *cyclic order with regard to* $\mathscr{K}$ if R is asymmetric, transitive and cyclic with regard to $\mathscr{K}$. If, moreover, card $G \geqq 3$ and R is complete with regard to $\mathscr{K}$, then let R be called a *complete cyclic order with regard to* $\mathscr{K}$. Because of 2.16, for ternary relations the notion of the (complete) cyclic order with regard to the t-decomposition $\mathscr{K} = \{\{1\}, \{2\}, \{3\}, \emptyset\}$ of the index set $H = \{1, 2, 3\}$ coincides with the notion of the (complete) cyclic order from [2] and [3]. In [1] a cyclic order means a complete cyclic order. Thus, the results presented in [1], § 5, as well as those obtained by V. Novák in the series of papers dealing with cyclic orders and including [2] and [3], can be generalized to cyclic orders in the general sense defined above.

In order that we might rightly consider the funamental investigations of relations in the general sense as complete, we have to define the relational systems and to study their morphisms. This will be done in another paper.

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