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STRUCTURE OF FREE ALGEBRAS

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Free algebras in some varieties have the property that from each of their elements we can get an arbitrary generator of the free algebra by a suitable performing of operations. More precisely, let $F(X)$ be a free algebra on the set X, let $x_1, \ldots, x_n \in X$ be arbitrary elements and $(x_1, \ldots, x_n)\,\omega$ a term. Then for any element $x \in X$ there exists a term $(x, y)\,\pi$ such that $(x, (x_1, \ldots, x_n)\,\omega)\,\pi = x$. For example, free lattices and free groups have this property. Let $(x_1, \ldots, x_n)\,\omega$ be an arbitrary element of a free lattice. Then it is sufficient to put $(x, y)\,\pi = (y \vee x) \wedge x$. Let $(x_1, \ldots, x_n)\,\omega$ be an arbitrary element of a free group. Then it is sufficient to put $(x, y)\,\pi = x \,.\, y^{-1}$.

However, not all free algebras have this property. For example, free semigroups have not this property. Since in a free semigroup the length of the terms always increases by further multiplication, we cannot get any generator from any term.

We have shown just the extreme examples of free algebras where either all elements are returnable to any generator, or all elements are not returnable to any generator. Not each free algebra is so extreme and it is convenient to consider two kinds of returnability.

Definition. *Let $F(X)$ be a free algebra on the set X. Let $x, y, x_1, \ldots, x_n \in X$ and $(x_1, \ldots, x_n)\,\omega$ be a term. This term is returnable if there exists $i \in \{1, \ldots, n\}$ and a term $(x, y)\,\lambda_i$ such that*

$$(x_i(x_1, \ldots, x_n)\,\omega)\,\lambda_i = x_i \,.$$

The term $(x_1, \ldots, x_n)\,\omega$ is strongly returnable if for arbitrary $x \in X$ there exists a term $(x, y)\,\lambda_x$ such that

$$(x, (x_1, \ldots, x_n)\,\omega)\,\lambda_x = x \,.$$

In a free algebra on a one-element set the notion of a returnable term and the notion of a strongly returnable term are the same. We shall call these terms returnable, not strongly returnable.

All terms in free lattices and free groups are strongly returnable and all terms in free semigroups are unreturnable.

The aim of this paper is to find the possible structures of free algebras from the view-point of returnability.

Theorem 1. *Let $F(X)$ be a free algebra on the set X. Let $x_1, \ldots, x_n \in X$ and let*

$(x_1, \ldots, x_n)\,\omega$ be a term. If $(x_1, \ldots, x_n)\,\omega$ is returnable, then for each $j = 1, \ldots, n$ there exists a term $(x, y)\,\lambda_j$ such that

$$(x_j, (x_1, \ldots, x_n)\,\omega)\,\lambda_j = x_j\,.$$

Proof. Since $(x_1, \ldots, x_n)\,\omega$ is returnable, there exists $i \in \{1, \ldots, n\}$ and a term $(x, y)\,\lambda_i$ such that

$$(x_i(x_1, \ldots, x_n)\,\omega)\,\lambda_i = x_i\,.$$

Without any loss of generality we can assume that $i = 1$. For $j \neq 1$ the substitution x_j for x_1 and $(x_1, \ldots, x_n)\,\omega$ for x_j yields

$$((x_j, (x_j, x_2, \ldots, x_{j-1}, (x_1, \ldots, x_n)\,\omega, x_{j+1}, \ldots, x_n)\,\omega)\,\lambda_1 = x_j\,,$$

which proves the theorem.

Theorem 2. *Each at least binary returnable term is strongly returnable.*

Proof. Let $F(X)$ be a free algebra on the set X. Let $x, y, z \in X$ and let $(x, y)\,\omega$ be a returnable term. We can assume that there exists a term λ such that

$$(x, (x, y)\,\omega)\,\lambda = x\,.$$

By the substitution z for x and $(x, y)\,\omega$ for y we get

$$(z, (z, (x, y)\,\omega)\,\omega)\,\lambda = z\,,$$

which means that $(x, y)\,\omega$ is a strongly returnable term. If the term λ is of a greater arity, then the proof is quite similar.

Corollary. *Each returnable term which is not strongly returnable, is unary.*

Theorem 3. *Let there exist a strongly returnable term ω in a free algebra $F(X)$ where $\operatorname{card} X > 1$ and ω is not nullary. Then all terms in $F(X)$ are strongly returnable.*

Proof. Let ω be a strongly returnable term. First we suppose that ω is at least binary. Without any loss of generality we can suppose that ω is precisely binary and denote it by $(x, y)\,\omega$ where $x, y \in X$, $x \neq y$. It follows that for each $z \in X$ there exists a term λ such that

$$(z, (x, y)\,\omega)\,\lambda = z\,.$$

Let π be an arbitrary term, we show that it is also strongly returnable. Let $t \in X$. If $z \neq x$, then the substitution π for x and t for z yields

$$(t, (\pi, y)\,\omega)\,\lambda = t\,,$$

and if $z = x$ then $z \neq y$ and the substitution π for y and t for z yields

$$(t, (t, \pi)\,\omega)\,\lambda = t\,.$$

According to Theorem 2, π is strongly returnable.

Now suppose that ω is a unary strongly returnable term, which is denoted by $x\omega$. We show that there exists a binary strongly returnable term. Choose $t \in X$, $t \neq x$.

By the definition there exists a term λ such that

$$(t, x\omega)\,\lambda = t\,.$$

λ is a binary returnable term and according to Theorem 2 it is strongly returnable. According to the first part of this proof each term in $F(X)$ is strongly returnable.

Remark. A similar theorem for nullary strongly returnable terms does not hold. For example in a free monoid the unit is a nullary strongly returnable term and all other terms are unreturnable.

Theorem 4. *Let $F(X)$ be a free algebra on the set X and $x, y \in X$, $x \neq y$. Each term in $F(X)$ is strongly returnable if and only if there exists a binary term $(x, y)\,\omega$ with the property $(x, y)\,\omega = x$.*

Proof. If there exists a binary term $(x, y)\,\omega$ with the property $(x, y)\,\omega = x$, then it is returnable and according to Theorem 2 it is strongly returnable. According to Theorem 3 all terms in $F(X)$ are strongly returnable.

Conversely, let each term in $F(X)$ be strongly returnable. It is sufficient to show the existence of a binary term. Let us suppose that all terms are at most unary and let $x\omega$ be such a term. Since it is strongly returnable, for $y \in X$ there exists a term λ with the property $x\omega\lambda = y$. If λ is also unary, then by the substitution x for y we get $x\omega\lambda = x$, which contradicts the assumption $x \neq y$. Thus the term λ must be at least binary, it means $(y, x\omega)\,\lambda = y$, which proves the theorem.

Corollary. *A unary term $x\omega$ is returnable and not strongly returnable if and only if there exists a unary term λ with the property $x\omega\lambda = x$.*

Proof. If there exists a binary term $(x, y)\,\lambda$ such that $(x\omega, y)\,\lambda = x$, then according to Theorem 4 all terms are strongly returnable.

Now we describe the possible structures of free algebras from the view-point of returnability of terms. It is convenient to distinguish nullary, unary and at least binary terms.

First we shall consider only free algebras without nullary operations. Theorem 3 implies that each free algebra has one of the following structures.

(i) All terms are strongly returnable.

According to Theorem 4 this is the case if and only if there exists a binary term $(x, y)\,\omega$ with the property $(x, y)\,\omega = x$. Examples for this case are free lattices and free groups.

(ii) All terms are returnable but not strongly returnable.

According to Corollary to Theorem 2 in this case all the terms are unary and according to Corollary to Theorem 4, for each term $x\omega$ there exists a term λ with the property $x\omega\lambda = x$. An example for this case is a free algebra with one unary operation ω where the identity $x\omega\omega = x$ holds.

(iii) All terms are unreturnable.

In this case there can be terms of any arity and the necessary and sufficient condition for this case is that there exists no term ω (apart from a variable) with the property $\omega = x$. An example is any absolutely free algebra.

(iv) There are returnable and also unreturnable terms.

According to Corollary to Theorem 4, in this case returnable terms are precisely those unary terms $x\omega$ for which there exists a unary term λ with the property $x\omega\lambda = x$. An example for this case is a free algebra with two unary operations ω and λ with the property $x\omega\omega = x$. The returnable terms are precisely those which do not contain any λ.

Now we shall consider also nullary terms. It is clear that each nullary term can be only strongly returnable or unreturnable. Theorem 3 implies that if there exist returnable or unreturnable terms in a free algebra, then strongly returnable terms can be only nullary.

To each free algebra in the case (i) a nullary strongly returnable term can be added, for which no non-trivial equation holds.

To each free algebra in the cases (ii)–(iv) a nullary unreturnable term can be added, for which no non-trivial equation holds, and a binary unreturnable term $(x, y)\,\omega$ with a nullary strongly returnable term α can be added, for which the equation $(x, \alpha)\,\omega = x$ holds.

It follows that free algebras in all cases can be with nullary strongly returnable terms or without them and free algebras in the cases (ii)–(iv) can be with nullary unreturnable terms or without them.

References

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