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# A FAMILY OF NONREGULAR DISTANCE MONOTONE GRAPHS

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## 0. INTRODUCTION AND BASIC NOTIONS

The (O, 2)-graphs and the distance monotone graphs (DM-graphs) have been introduced in [1] and [2], [4] and [5], respectively, in two characterizations of hypercubes. H. M. Mulder [2] and independently J. M. Laborde-Rao Hebbare [3] proved that (O, 2)-graphs are regular. In [5] and [6] the authors introduced the following conjecture.

**Conjecture.** *Every DM-graph $G$ with minimal degree $d(G) \geqq 3$ is regular.*

We exhibit a family of counterexamples to this conjecture.

We first recall some definitions introduced in [2] and [5].

For any two vertices $u$, $v$ in a simple graph $G$ the interval $I(u, v)$ is the set of vertices lying on a shortest $u$, $v$-path. Let $d(u, v)$ be the distance between $u$ and $v$.

A graph $G$ *is distance-monotone* (DM-graph for short) if each interval $I(u, v)$ verifies $w \in V(G) - I(u, v) \Rightarrow \exists w' \in I(u, v)$ such that $d(w, w') > d(u, v)$.

DM-graphs of diameter 3 except $P_4$ can be obtained from $K_{n,n}$ for $n \geqq 3$ by deletion of a perfect matching [5]. In [6] a matrix representation of DM-graphs of diameter 4 with $d(G) \geqq 3$ is introduced.

Consider a (0, 1)-matrix $M = (m_{i,j})$ fulfilling the following conditions

(1) $M$ has at least 4 rows and 4 columns.

(2) For any 3 different row indices $i, j, k$ there are 4 column indices $a, b, c, d$ such that

$$m_{ia} = m_{ja} \neq m_{ka} ,$$
$$m_{ib} \neq m_{jb} = m_{kb} ,$$
$$m_{ic} = m_{kc} \neq m_{jc} ,$$
$$m_{id} = m_{jd} = m_{kd} .$$

(2*) For any 3 different column indices $a, b, c$ there are 4 row indices $i, j, k, l$ such that

$$m_{ia} = m_{ib} \neq m_{ic} ,$$
$$m_{ja} \neq m_{jb} = m_{jc} ,$$
$$m_{ka} = m_{kc} \neq m_{kb} ,$$
$$m_{la} = m_{lb} = m_{lc} .$$

To every such a $m$ by $n$ matrix $M$ we associate a graph $G$ of order $2(m + n)$ in the following way:

$$V(G) = \{u_1, u_2, \ldots, u_m\} \cup \{u'_1, u'_2, \ldots, u'_m\} \cup \{v_1, v_2, \ldots, v_n\} \cup \cup \{v'_1, v'_2, \ldots, v'_n\}$$

and all the edges of $G$ are obtained by:

$$m_{ij} = 1 \Rightarrow \{u_i, v'_j\} \in E(G) \quad \text{and} \quad \{u'_i, v_j\} \in E(G)\,,$$
$$m_{ij} = 0 \Rightarrow \{u_i, v_j\} \in E(G) \quad \text{and} \quad \{u'_i, v'_j\} \in E(G)\,.$$

It is easy to verify that $G$ is a DM-graph of diameter 4, and furthermore all DM-graphs of diameter 4 with $d(G) \geqq 3$ can be obtained in this way [6].

We are going to construct an $m \times n$ matrix ($m \neq n$) fulfilling the conditions (1), (2) and (2*). In the associated DM-graph vertices $u_i$, $u'_i$ corresponding to row indices are of degree $n$ and vertices $v_i$, $v'_i$ corresponding to column indices are of degree $m$, therefore, the associated DM-graph is not regular.

Let $e_1, e_2, \ldots, e_p$ be the canonic basis of $V(p, 2)$ the vector space of dimension $p$ over $GF(2)$.

Let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_n$ be some mutually different vectors of $V(p, 2)$. We can associate to these vectors a $n \times m$ matrix $M = (m_{ij})$ in the following way: $m_{ij} = v_i u_j = v_{i1}u_{j1} + v_{i2}u_{j2} + \ldots + v_{ip}u_{jp}$ where $v_i = (v_{i1}, v_{i2}, \ldots, v_{ip})$ and $u_j = (u_{j1}, u_{j2}, \ldots, u_{jp})$ in the basis $e_1, e_2, \ldots, e_p$.

Example. Let $p = 3$, $m = 8$ (then the $u_i$ are all the vectors of $V(3, 2)$) and $n = 7$ with $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$, $v_4 = e_1 + e_2$, $v_5 = e_1 + e_3$, $v_6 = e_2 + e_3$, $v_7 = e_1 + e_2 + e_3$. We obtain the following $7 \times 8$ matrix:

(I)
$$\begin{bmatrix} 0&0&0&0&1&1&1&1 \\ 0&0&1&1&0&0&1&1 \\ 0&1&0&1&0&1&0&1 \\ 0&0&1&1&1&1&0&0 \\ 0&1&0&1&1&0&1&0 \\ 0&1&1&0&0&1&1&0 \\ 0&1&1&0&1&0&0&1 \end{bmatrix}$$

**Proposition 1.** *The matrix* (I) *verifies the properties* (2) *and* (2*). *The associated DM-graph is therefore of order* 30 *with* 14 *vertices of degree* 8 *and* 16 *of degree* 7.

**Proposition 2.** *For* $p \geqq 4$ *let* $u_1, u_2, \ldots, u_m$ *and* $v_1, v_2, \ldots, v_n$ *be some mutually different vectors of* $V(p, 2)$ *such that the* $p$ *basis vectors* $e_1, e_2, \ldots, e_p$ *and the* $p(p - 1)/2$ *sums of two basis vectors belong to the sets* $\{u_1, u_2, \ldots, u_m\}$ *and* $\{v_1, v_2, \ldots, v_n\}$.

*Then the matrix associated to these vectors verifies the properties* (2) *and* (2*).

The proof is common to the propositions:

Without loss of generality by permutations of rows and by permutation of columns we can assume that the first $p$ rows and the first $p$ columns are associoted to the basis vectors.

We first prove the property (2*). Choose 3 columns $i, j, k$ and consider the values of the first $p$ rows corresponding to the basis vectors. These are nothing else than that the vectors $u_i, u_j, u_k$ expressed in the basis $e_1. e_2, \ldots, e_p$.

*First case*: we have a row index $\alpha$, $\alpha \leqq p$, with $m_{\alpha i} = m_{\alpha j} = m_{\alpha k}$. Then there is an other row $\beta$ (still in the first $p$ rows) such that for example $m_{\beta i} = m_{\beta j} \neq m_{\beta k}$ (because $u_i, u_j, u_k$ are distinct).

Therefore there is a third one $\gamma$ with $m_{\gamma i} \neq m_{\gamma j} = m_{\gamma k}$ or $m_{\gamma i} \neq m_{\gamma i} \neq m_{\gamma k}$; in both cases the row associated to $e_\beta + e_\gamma$ is the fourth required.

*Second case*: there is no row index $\alpha$, $\alpha \leqq p$ with $m_{\alpha i} = m_{\alpha j} = m_{\alpha k}$.

If $p = 3$ (therefore we study the matrix (I)) we may have:

$$m_{1i} = m_{1j} \neq m_{1k},$$
$$m_{2i} \neq m_{2j} = m_{2k},$$
$$m_{3i} = m_{3k} \neq m_{3j}.$$

In this case the 7th row is associated to $e_1 + e_2 + e_3$ and we have $m_{7i} = m_{7j} = m_{7k}$.

If we are not in this case, or if $p > 3$, we have two indices (in the first $p$ ones), say $\alpha$ and $\beta$, with

$$m_{\alpha i} = m_{\alpha j} \neq m_{\alpha k},$$
$$m_{\beta i} = m_{\beta j} \neq m_{\beta k}$$
(possibly permuting $i, j, k$).

The vectors $u_i$ and $u_j$ are different, therefore there exists an index $\gamma \leqq p$ with

$$m_{\gamma i} \neq m_{\gamma j} \neq m_{\gamma k}$$
or
$$m_{\gamma i} \neq m_{\gamma j} = m_{\gamma k}$$

and we have the property (2*) with the row indices $\alpha, \gamma, \delta$ and $\varepsilon$, with $e_\delta = e_\alpha + e_\beta$ and $e_\varepsilon = e_\beta + e_\gamma$.

Thus the matrix verifies (2*), and by transposition verifies (2).

Therefore we have the following property:

**Proposition 3.** *For $p \geqq 4$ and every pair of integers $n, m$ such that*

$$p(p+1)/2 \leqq n \leqq 2^p,$$
$$p(p+1)/2 \leqq m \leqq 2^p$$

there exists a DM-graph with some vertices of degree $n$ and others of degree $m$.

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