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ON INTEGRATION IN BANACH SPACES, XIII (INTEGRATION WITH RESPECT TO POLYMEASURES)

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INTRODUCTION

In Definition 1 we introduce the product S^*-integral, a multilinear generalization of the Kolmogoroff integral for countable partitions, see Section 1 in Part VII and [27]. According to Theorem 1, roughly speaking, product S^*-integrability implies integrability, and the integrals coincide. (For $d = 1$ we stated this fact in Theorem VII.1 only for $\tilde{\mathscr{P}}$-measurable functions, nonetheless the proof given there works for $\mathscr{P}$-measurable functions.). By Theorem 2 the converse holds at least for $(f_i) \in \mathscr{L}_1(\Gamma)$.

In Theorem 12 we prove the finiteness of the multiple L_1-gauge $\hat{\Gamma}[(\cdot), (T_i)]$ on $\mathscr{L}_1(\Gamma)$, promised in Part XI, and already used many times since then. Our rather long proof is based on a Fubini Theorem 4 in the case of assumption a) in its formulation. From Theorem 4 we easily obtain not only Theorems V.9 – V.12 on integration by substitution, see Corollary 2 of Theorem 4, but also their multilinear generalizations, see Theorems 5 – 8. Theorem 9 is a generalization of Theorem V.8, while Theorem 11 gives characterization of $\mathscr{L}_1(\Gamma)$.

We note that interesting and deep results have been obtained recently in harmonic analysis of bimeasures, see [20] – [24] and [31]. Highly interesting is the approach and the results of R. C. Blei in [1] and [2].

1. INTEGRABILITY AND PRODUCT S^*-INTEGRABILITY

Definition 1. Let $f_i: T_i \to X_i$, $i = 1, \ldots, d$, and let $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. We say that a d-tuple (f_i) is *product S^*-integrable, shortly $\mathsf{X}S^*$-integrable on* (A_i), if there is a $y \in Y$ and for each $\varepsilon > 0$ a product of countable $\mathscr{P}_i$-partitions $\mathsf{X}\pi^*_{i,\varepsilon}(A_i) \in \mathsf{X}\Pi^*_i(A_i)$ such that $\sum_{j_{p(1)}=1}^{\omega_{p(1)}} \ldots \sum_{j_{p(d)}=1}^{\omega_{p(d)}} \Gamma(A_{i,j_i})(f_i(t_{i,j_i}))$ converges iteratedly unconditionally in Y for each permutation p of $\{1, \ldots, d\}$, for each product of countable $\mathscr{P}_i$-partitions

$$\mathsf{X}\pi^*_i(A_i) \geqq \mathsf{X}\pi^*_{i,\varepsilon}(A_i)\,, \quad (A_i) = \{A_{i,j_i}\}_{j_i=1}^{\omega_i}\,, \quad i = 1, \ldots, d\,,$$

and each $t_{i,j_i} \in A_{i,j_i}$, $i = 1, \ldots, d$, $j_i = 1, \ldots, \omega_i \leqq \infty$, and

$$\left| \sum_{j_{p(1)}=1}^{\omega_{p(1)}} \ldots \sum_{j_{p(d)}=1}^{\omega_{p(d)}} \Gamma(A_{i,j_i}) \left(f_i(t_{i,j_i})\right) - y \right| < \varepsilon .$$

In this case we write

$$\mathsf{X}S^*_{(A_i)}(f_i) \, \mathrm{d}\Gamma = y .$$

The following simple facts are immediate:

Let f_i: $T_i \to X_i$, $i = 1, \ldots, d$, let $(N_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and let $\hat{\Gamma}(N_i) = 0$. Then (f_i) is $\mathsf{X}S^*$-integrable on (N_i) and $\mathsf{X}S^*_{(N_i)}(f_i) \, \mathrm{d}\Gamma = 0$.

Let f_i: $T_i \to X_i$, $i = 1, \ldots, d$, let g_1: $T_1 \to X_1$, let (f_i) and $(g_1, f_2, \ldots, f_d)$ be $\mathsf{X}S^*$-integrable on $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, and let a, b be scalars. Then $(af_1 + bg_1, f_2, \ldots, f_d)$ is $\mathsf{X}S^*$-integrable on (A_i) and

$$\mathsf{X}S^*_{(A_i)}(af_1 + bg_1, f_2, \ldots, f_d) \, \mathrm{d}\Gamma = a \, . \, \mathsf{X}S^*_{(A_i)}(f_i) \, \mathrm{d}\Gamma + b \, . \, \mathsf{X}S^*_{(A_i)}(g_1, f_2, \ldots, f_d) \, \mathrm{d}\Gamma .$$

If (f_i) is $\mathsf{X}S^*$-integrable on $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, then

$$\left| \mathsf{X}S^*_{(A_i)}(f_i) \, \mathrm{d}\Gamma \right| \leqq \prod_{i=1}^{d} \|f_i\|_{A_i} \hat{\Gamma}(A_i) ,$$

and (f_i) is $\mathsf{X}S^*$-integrable on each $(B_i) \in \mathsf{X}(A_i \cap \sigma(\mathscr{P}_i))$. Hence if f_i: $T_i \to X_i$ is $\mathscr{P}_i$-measurable and $F_i = \{t_i \in T_i, \ f_i(t_i) \neq 0\} \in \sigma(\mathscr{P}_i)$, $i = 1, \ldots, d$, then (f_i) is $\mathsf{X}S^*$-integrable on (F_i) if and only if it is $\mathsf{X}S^*$-integrable on each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

The assertions of the next lemma are also evident.

Lemma 1. 1) *Let f_i: $T_i \to X_i$, $i = 1, \ldots, d$, let $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$ and let (f_i) be $\mathsf{X}S^*$-integrable on (A_i). Then the indefinite $\mathsf{X}S^*$ integral $\mathsf{X}S^*_{(\cdot)}(f_i) \, \mathrm{d}\Gamma$: $\mathsf{X}(A_i \cap \sigma(\mathscr{P}_i)) \to Y$ is a vector d-polymeasure, i.e., it is separately countably additive.*

2) *Let $(f_i) \in \mathsf{X}\bar{S}(\mathscr{P}_i, X_i) \cap \mathscr{L}_1(\Gamma)$, let $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, and let $\hat{\Gamma}(A_i) < +\infty$. Then (f_i) is $\mathsf{X}S^*$-integrable on (A_i), $(f_i \chi_{A_i}) \in \mathscr{I}_1(\Gamma)$ and*

$$\mathsf{X}S^*_{(B_i)}(f_i) \, \mathrm{d}\Gamma = \int_{(B_i)} (f_i) \, \mathrm{d}\Gamma$$

for each $(B_i) \in \mathsf{X}(A_i \cap \sigma(\mathscr{P}_i))$.

3) *Let $(f_i) \in \mathsf{X}E(\mathscr{P}_i, X_i)$. Then (f_i) is integrable if and only if it is $\mathsf{X}S^*$-integrable on (F_i), where $F_i = \{t_i \in T_i, f_i(t_i) \neq 0\} \in \sigma(\mathscr{P}_i)$, $i = 1, \ldots, d$. In this case $(f_i) \in \mathscr{I}_d(\Gamma)$ and*

$$\mathsf{X}^*_{(A_i)}S(f_i) \, \mathrm{d}\Gamma = \int_{(A_i)} (f_i) \, \mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Using these facts we now prove

Theorem 1. *Let f_i: $T_i \to X_i$ be $\mathscr{P}_i$-measurable, let $F_i = \{t_i \in T_i, f_i(t_i) \neq 0\} \in \sigma(\mathscr{P}_i)$, and let $X_{f_i} = \overline{\mathrm{sp}} \{f_i(T_i)\}$, $i = 1, \ldots, d$. Suppose either $\Gamma(\ldots)(x_i)$: $\mathsf{X}(F_i \cap \mathscr{P}_i) \to Y$ has locally a control d-polymeasure for each $(x_i) \in \mathsf{X}X_{f_i}$, or $f_i(T_i) \subset X_i$ is relatively*

σ-compact for each $i = 1, \ldots, d$. Let finally (f_i) be $\mathsf{X}S^*$-integrable on (F_i). Then $(f_i) \in \mathcal{I}(\Gamma) = \mathcal{I}_1(\Gamma)$ and

$$\mathsf{X}S^*_{(A_i)}(f_i)\, \mathrm{d}\Gamma = \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathcal{P}_i)$.

Proof. We prove the theorem under the assumption that $\Gamma(\cdot)(x_i)$: $\mathsf{X}(F_i \cap \mathcal{P}_i\} \to Y$ has locally a control d-polymeasure for each $(x_i) \in \mathsf{X}X_{f_i}$. The case when $f_i(T_i) \subset X_i$ is relatively σ-compact for each $i = 1, \ldots, d$ can be treated similarly using Theorems X.1 and X.5.

Clearly we may replace X_i by X_{f_i} for each $i = 1, \ldots, d$, see Assertion 2) of Theorem XII.3, and Γ by $\Gamma' = \Gamma$: $\mathsf{X}(F_i \cap \mathcal{P}_i) \to L^{(d)}(X_{f_i}; Y)$. By Theorems VIII.17 and VIII.19 there is a control d-polymeasure, say $\lambda_1 \times \ldots \times \lambda_d$: $\mathsf{X}(F_i \cap \sigma(\mathcal{P}_i)) \to [0, 1]$ for the semivariation $\hat{\Gamma}'$: $\mathsf{X}(F_i \cap \sigma(\mathcal{P}_i)) \to [0, +\infty]$. Since $\hat{\Gamma}' \leqq \hat{\Gamma}$ on $\mathsf{X}(F_i \cap \sigma(\mathcal{P}_i))$, and since by assumption $\hat{\Gamma}$ is σ-finite on $\mathsf{X}(F_i \cap \sigma(\mathcal{P}_i))$, see the beginning of Part IX, there are $(F_{i,k}) \in \mathsf{X}(F_i \cap \sigma(\mathcal{P}_i))$, $k = 1, 2, \ldots$ such that $F_{i,k} \nearrow F_i$ for each $i = 1, \ldots, d$, and $\hat{\Gamma}'(F_{i,k}) \leqq \hat{\Gamma}(F_{i,k}) < +\infty$ for each $k = 1, 2, \ldots$. Hence $\hat{\Gamma}'$ is σ-finite on $\mathsf{X}(F_i \cap \sigma(\mathcal{P}_i))$.

According to Assertion 2) of Theorem XII.3, for each $i = 1, \ldots, d$ there are $f_{i,n} \in S(F_i \cap \mathcal{P}_i, X_{f_i})$, $n = 1, 2, \ldots$ such that $f_{i,n}(t_i) \to f_i(t_i)$ for each $t_i \in T_i$. By the Egoroff-Lusin theorem, see Section 1.4 in Part I, there are $N_i \in F_i \cap \sigma(\mathcal{P}_i)$ and $F'_{i,k} \in F_i \cap \mathcal{P}_i$, $i = 1, \ldots, d$, $k = 1, 2, \ldots$ such that $\lambda_i(N_i) = 0$ and $F'_{i,k} \nearrow F_i - N_i$ for each $i = 1, \ldots, d$, and on each $F'_{i,k}$, $k = 1, 2, \ldots$ the sequence $f_{i,n}$, $n = 1, 2, \ldots$ converges uniformly to f_i, $i = 1, \ldots, d$. Put $F^*_{i,k} = F_{i,k} \cap F'_{i,k}$, $i = 1, \ldots, d$, $k = 1, 2, \ldots$. Then $\hat{\Gamma}'(N_1, F_2, \ldots, F_d) = \ldots = \hat{\Gamma}'(F_1, \ldots, F_{d-1}, N_d) = 0$ and $F^*_{i,k} \nearrow F_i - N_i$ for each $i = 1, \ldots, d$, and $(f_i\chi_{F^*_{i,k}}) \in \mathsf{X}\bar{S}(F_i \cap \mathcal{P}_i, X_i)$ and $\hat{\Gamma}'(F_{i,k}) < +\infty$ for each $k = 1, 2, \ldots$. Hence $(f_i\chi_{F^*_{i,k}}) \in \mathcal{I}_1(\Gamma)$ for each $k = 1, 2, \ldots$ by Corollary 3 of Theorem IX.4.

Since (f_i) is $\mathsf{X}S^*$-integrable on (F_i) by assumption, it is also $\mathsf{X}S^*$-integrable on each $(F_{i,k})$, $k = 1, 2, \ldots$. Let $k \in \{1, 2, \ldots\}$ be fixed. Assertion 1) of Theorem XII.3 and Definition 1 easily yield that

$$\int_{(A_i)} (f_i\chi_{F^*_{i,k}})\, \mathrm{d}\Gamma = \mathsf{X}S^*_{(A_i)}(f_i\chi_{F^*_{i,k}})\, \mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(F_i \cap \mathcal{P}_i)$. But then assertion 1) of Lemma 1 and Corollary 2 of Theorem IX.4 immediately imply that $(f_i\chi_{F_i - N_i}) \in \mathcal{I}(\Gamma')$, hence also $(f_i) \in \mathcal{I}(\Gamma')$, and

$$\mathsf{X}S^*_{(A_i)}(f_i)\, \mathrm{d}\Gamma = \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma' = \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}(F_i \cap \sigma(\mathcal{P}_i))$, hence also for each $(A_i) \in \mathsf{X}\sigma(\mathcal{P}_i)$. The equality $\mathcal{I}(\Gamma) = \mathcal{I}_1(\Gamma)$ is a consequence of Theorem X.3.

A partial conversion to Theorem 1 is the following

Theorem 2. *Let $(f_i) \in \mathcal{L}_1(\Gamma)$. Then (f_i) is $\mathsf{X}S^*$-integrable on each $(A_i) \in \mathsf{X}\sigma(\mathcal{P}_i)$*

and

(1) $$\mathsf{X}S^*_{(A_i)}(f_i)\,\mathrm{d}\Gamma = \int_{(A_i)}(f_i)\,\mathrm{d}\Gamma$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Proof. Let $(f_i) \in \mathscr{L}_1(\Gamma)$ and let $F_i = \{t_i \in T_i,\ f_i(t_i) \neq 0\}$, $i = 1, \ldots, d$. It is sufficient to show that (f_i) is $\mathsf{X}S^*$-integrable on (F_i) and that (1) holds for (F_i). According to Theorem 12 below $\hat{\Gamma}[(f_i), (T_i)] < +\infty$. Let $\varepsilon_1 > 0$. Take $\varepsilon > 0$ such that $\varepsilon(1 + \varepsilon)^{d-1}\, d\hat{\Gamma}[(f_i), (T_i)] < \varepsilon_1$. By Assetion 1) of Theorem XII.3, for each $i = 1, \ldots, d$ there is a countable $\mathscr{P}_i$-partition $\pi^*_{i,\varepsilon}(F_i) = \{F_{i,j}\}_{j \in J_i}$ such that for any points $t_{i,j} \in F_{i,j}$, $i = 1, \ldots, d$, $j \in J_i$, the inequality

$$\left|f_i(t_i) - \sum_{j \in J_i} f_i(t_{i,j})\,\chi_{F_{i,j}}(t_i)\right| \leqq (\varepsilon/2)\,|f_i(t_i)|$$

holds for each $t_i \in F_i$.

Let $\{F'_{i,j}\}_{j \in J_i'} = \pi^*_i(F_i) \geqq \pi^*_{i,\varepsilon}(F_i)$, take $t'_{i,j} \in F'_{i,j}$, $i = 1, \ldots, d$, $j \in J'_i$, and put $h_{i,\varepsilon} = \sum_{j \in J_i'} f_i(t'_{i,j})\,\chi_{F'_{i,j}}$ for $i = 1, \ldots, d$. Then $h_{i,\varepsilon} \in E(\mathscr{P}_i, X_i)$ and $|h_{i,\varepsilon}| \leqq (1 + \varepsilon)(f_i)$ for each $i = 1, \ldots, d$. Hence $(h_{i,\varepsilon}) \in \mathscr{I}(\Gamma)$ by the definition of $\mathscr{L}_1(\Gamma)$, see Definition XI.3, and the assumption $(f_i) \in \mathscr{L}_1(\Gamma)$. Clearly

$$\begin{aligned}
&\left|\int_{(F_i)}(f_i)\,\mathrm{d}\Gamma - \int_{(F_i)}(h_{i,\varepsilon})\,\mathrm{d}\Gamma\right| \leqq \left|\int_{(F_i)}(f_1 - h_{1,\varepsilon}, \ldots, f_d)\,\mathrm{d}\Gamma\right| + \ldots \\
&\ldots + \left|\int_{(F_i)}(h_{1,\varepsilon}, \ldots, h_{d-1,\varepsilon}, f_d - h_{d,\varepsilon})\,\mathrm{d}\Gamma\right| \leqq \\
&\leqq \varepsilon\hat{\Gamma}[(f_i), (T_i)] + \varepsilon(1 + \varepsilon)\,\hat{\Gamma}[(f_i), (T_i)] + \ldots \\
&\ldots + \varepsilon(1 + \varepsilon)^{d-1}\,\hat{\Gamma}[(f_i), (T_i)] \leqq \varepsilon(1 + \varepsilon)^{d-1}\, d\hat{\Gamma}[(f_i), (T_i)] < \varepsilon_1 .
\end{aligned}$$

The separate countable additivity of the indefinite integral

$$\int_{(\cdot)}(h_{i,\varepsilon})\,\mathrm{d}\Gamma\colon \mathsf{X}(F_i \cap \sigma(\mathscr{P}_i)) \to Y$$

implies that

$$\int_{(F_i)}(h_{i,\varepsilon})\,\mathrm{d}\Gamma = \sum_{j_{p(1)} \in J_{p(1)}'} \cdots \sum_{j_{p(d)} \in J'_{p(d)}} \Gamma(F'_{i,j_i})\,(f_i(t'_{i,j}))\,,$$

where we have iterated unconditional convergence for any permutation p of $\{1, \ldots, d\}$. Since $\varepsilon_1 > 0$ was arbitrary, the theorem is proved.

It remains an open problem whether Theorem 2 holds for arbitrary $(f_i) \in \mathscr{I}(\Gamma)$.

2. SOME FUBINI THEOREMS AND INTEGRATION BY SUBSTITUTION

In Theorem XI.6 we showed that $\mathscr{L}_1(\Gamma)$ is one of the natural classes, in author's opinion the most important one, for the validity of the Fubini theorem for integration with respect to polymeasures, i.e., of multilinear integration. Trivially the Fubini theorem holds for elements of $\mathsf{X}S(\mathscr{P}_i, X_i)$. The next theorem is also evident.

Theorem 3. *Let* $(f_i) \in \mathsf{X}\bar{S}(\mathscr{P}_i, X_i)$ *let* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, *and let* $\hat{\Gamma}(A_i) < +\infty$. *Then all* *assertions of Theorem* XI.6 *and its Corollary* 1, i.e., *of the Fubini theorem, hold for* $(f_i\chi_{A_i})$.

Since for finite dimensional X_i a bounded $\mathscr{P}_i$-measurable $f_i: T_i \to X_i$ belongs to $\bar{S}(\mathscr{P}_i, X_i)$, and since the scalar semivariation of a separately countably additive vector d-polymeasure on a Cartesian product of σ-rings is bounded, see Assertion 4) of Theorem VIII.3, we immediately obtain the next corollary. Let us note that this corollary also follows from Theorem XI.6.

Corollary. *For each* $i = 1, \ldots, d$ *let* X_i *be finite dimensional, let* $\mathscr{P}_i$ *be a* σ*-ring, and let* $f_i: T_i \to X_i$ *be a bounded* $\mathscr{P}_i$*-measurable function. Then all assertions of Theorem* XI.6 *and its Corollary* 1 *hold for* (f_i).

Let us recall that Theorem XI.6 was proved under the additional assumption of finiteness of the multiple L_1-gauge $\hat{\Gamma}[(g_i), (T_i)]$ for $(g_i) \in \mathscr{L}_1(\Gamma)$. In Section 3 we will prove this finiteness. Our proof is based on the Fubini type result of Theorem 4 below. First we need some notions.

For $\mathscr{P}_i$-measurable $f_i: T_i \to X_i$, $i = 1, \ldots, d$, put $X_{f_i} = \overline{\mathrm{sp}}\,\{f_i(T_i)\}$, $F_i = \{t_i \in T_i, f_i(t_i) \neq 0\}$, and $\mathscr{P}_{f_i} = \bigcup_{k=1}^{\infty} \{t_i \in T_i, |f_i(t_i)| \leqq k^{-1}\} \cap \mathscr{P}_i$.

Definition 2. Let $\Gamma_\tau: \mathsf{X}\mathscr{P}_i \to L^{(d)}(X_i; Y)$, $\tau \in \mathscr{T}$, be separately additive. We say that the semivariations $\hat{\Gamma}_\tau$, $\tau \in \mathscr{T}$, are locally uniformly σ-finite on $\mathsf{X}\sigma(\mathscr{P}_i)$ if the set function $(A_i) \to \sup_{\tau\in\mathscr{T}} \hat{\Gamma}_\tau(A_i)$, $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, is σ-finite on $\mathsf{X}(F_i \cap \sigma(\mathscr{P}_i))$ for each $(F_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$.

Theorem 4. *Let* $d > 1$, *let* $f_i: T_i \to X_i$ *be* $\mathscr{P}_i$*-measurable for each* $i = 1, \ldots, d$, *and suppose either* a) *or* b) *or* c) *below*:

a) $\Gamma(\cdot)(x_i): \mathsf{X}\mathscr{P}_{f_i} \to Y$ *has locally a control d-polymeasure for each* $(x_i) \in \mathsf{X}X'_i$, *where* $X'_i = X_{f_i}$;

b) $f_i(T_i) \subset X_i$ *is relatively* σ*-compact for each* $i = 1, \ldots, d$;

c) $c_0 \not\subset Y$.

Let further d_1, $1 \leqq d_1 < d$, *be a positive integer, and let* $(f_1, \ldots, f_{d_1}, x_{d_1+1}\chi_{A_{d_1+1}}, \ldots, x_d\chi_{A_d}) \in \mathscr{I}(\Gamma)$ *for each* $(x_{d_1+1}, \ldots, x_d) \in X'_{d_1+1} \times \ldots \times X'_d$, *where* $X'_i = X_{f_i}$, *and for each* $(A_{d_1+1}, \ldots, A_d) \in \mathscr{P}_{f_{d_1+1}} \times \ldots \times \mathscr{P}_{f_d}$. *For these arguments put* $\Gamma_{(A_1,\ldots,A_{d_1})}(A_{d_1+1}, \ldots, A_d)\,(x_{d_1+1}, \ldots, x_d) = \int_{(A_i)}(f_1, \ldots, f_{d_1}, x_{d_1+1}\chi_{A_{d_1+1}}, \ldots, x_d\chi_{A_d})\,\mathrm{d}\Gamma$ *for* $(A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \ldots \times \sigma(\mathscr{P}_{f_{d_1}})$. *Then for any given* $(A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \ldots \times \sigma(\mathscr{P}_{f_{d_1}})$ *we have* $\Gamma_{(A_1,\ldots,A_{d_1})}: \mathscr{P}_{f_{d_1+1}} \times \ldots \times \mathscr{P}_{f_d} \to L^{(d-d_1)}(X'_{d_1+1}, \ldots, X'_d; Y)$, *where* $X'_i = X_{f_i}$, *it is separately countably additive in the strong operator topology, and has a control* $(d - d_1)$*-polymeasure provided* a) *holds.*

Suppose finally that the semivariations $\hat{\Gamma}_{(A_1,\ldots,A_{d_1})}$, $(A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \ldots \times \sigma(\mathscr{P}_{f_{d_1}})$, *are uniformly* σ*-finite on* $\sigma(\mathscr{P}_{f_{d_1+1}}) \times \ldots \times \sigma(\mathscr{P}_{f_d})$. *In Particular, this happens if* $\hat{\Gamma}[(f_i), (T_i)] < +\infty$. *Then* $(f_i) \in \mathscr{I}(\Gamma)$ *if and only if* $(f_{d_1+1}, \ldots, f_d) \in$

$\in \mathscr{I}(\Gamma_{(A_1,\dots,A_{d_1})})$ *for each* $(A_1,\dots,A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \dots \times \sigma(\mathscr{P}_{f_{d_1}})$. *In this case*

$$\int_{(A_i)} (f_i)\, \mathrm{d}\Gamma = \int_{(A_{d_1+1},\dots,A_d)} (f_{d_1+1},\dots,f_d)\, \mathrm{d}\Gamma_{(A_1,\dots,A_{d_1})}$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_{f_i})$.

Proof. According to Assertion 2) of Theorem XII.3, $f_i\colon T_i \to X_{f_i}$ is $\mathscr{P}_{f_i}$-measurable for each $i = 1,\dots,d$. Put $\Gamma' = \Gamma\colon \mathsf{X}\mathscr{P}_{f_i} \to L^{(d)}(X_{f_i}; Y)$. Since $F_i \cap \sigma(\mathscr{P}_i) = F_i \cap \sigma(\mathscr{P}_{f_i})$ for each $i = 1,\dots,d$, the semivariation $\hat{\Gamma}'$ is σ-finite on $\mathsf{X}(F_i \cap \sigma(\mathscr{P}_{f_i}))$ by the assumed local σ-finiteness of the semivariation $\hat{\Gamma}$ on $\mathsf{X}\sigma(\mathscr{P}_i)$, see the beginning of Part IX. Hence we may replace Γ by Γ'. Take $(F'_{i,k}) \in \mathsf{X}\mathscr{P}_{f_i}$, $k = 1,2,\dots$ such that $F'_{i,k} \nearrow F_i$ for each $i = 1,\dots,d$ and $\hat{\Gamma}'(F'_{i,k}) < +\infty$ for each $k = 1,2,\dots$.

Consider now the first assertion of the theorem.

Assume a). Since X_{f_i} is separable for each $i = 1,\dots,d$, according to Theorems VIII.17 and VIII.19 there is a control d-polymeasure, say $\lambda_1 \times \dots \times \lambda_d\colon \mathsf{X}\sigma(\mathscr{P}_{f_i}) \to [0,1]$, for the semivariation $\hat{\Gamma}'\colon \mathsf{X}\sigma(\mathscr{P}_{f_i}) \to [0,+\infty]$. For each $i = 1,\dots,d$ take a sequence $f_{i,n} \in S(\mathscr{P}_{f_i}, X_{f_i})$, $n = 1,2,\dots$ such that $f_{i,n} \to f_i$. Applying the Egoroff-Lusin theorem coordinatewise, see Section 1.4 in Part I, there are $N_i \in \sigma(\mathscr{P}_{f_i})$ and $F^*_{i,k} \in \mathscr{P}_{f_i}$, $i = 1,\dots,d$, $k = 1,2,\dots$ such that $\hat{\Gamma}'(N_1, F_2,\dots,F_d) = \dots = \hat{\Gamma}'(F_1,\dots,F_{d-1},N_d) = 0$, $F^*_{i,k} \nearrow F_i - N_i$ for each $i = 1,\dots,d$, and on each $F^*_{i,k}$, $k = 1,2,\dots$ the sequence $f_{i,n}$, $n = 1,2,\dots$ converges uniformly to the function f_i, $i = 1,\dots,d$. For $i = 1,\dots,d$ and $k = 1,2,\dots$ put $F_{i,k} = F'_{i,k} \cap F^*_{i,k}$, and for $k, n = 1,2,\dots$ and $(A_1,\dots,A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \dots \times \sigma(\mathscr{P}_{f_{d_1}})$ define $\Gamma_{k,n,(A_1,\dots,A_{d_1})}$ similarly as $\Gamma_{(A_1,\dots,A_{d_1})}$ replacing $f_1,\dots,f_{d_1}$ by $f_{1,n}\chi_{F_{1,k}},\dots,f_{d_1,n}\chi_{F_{d_1,k}}$, respectively. Then by Theorem VIII.1 and Corollary 4 of Theorem IX.4 we obtain

$$\begin{aligned}
&\Gamma_{(A_1,\dots,A_{d_1})}(A_{d_1+1},\dots,A_d)\,(x_{d_1+1},\dots,x_d) = \\
&= \int_{(A_i)}(f_1,\dots,f_{d_1}, x_{d_1+1}\chi_{A_{d_1+1}},\dots,x_d\chi_{A_d})\,\mathrm{d}\Gamma' = \\
&= \int_{(A_i - N_i)}(f_1,\dots,f_{d_1}, x_{d_1+1}\chi_{A_{d_1+1}},\dots,x_d\chi_{A_d})\,\mathrm{d}\Gamma' = \\
&= \lim_{k\to\infty} \int_{(A_i \cap F_{i,k})}(f_1,\dots,f_{d_1}, x_{d_1+1}\chi_{A_{d_1+1}},\dots,x_d\chi_{A_d})\,\mathrm{d}\Gamma' = \\
&= \lim_{k\to\infty} \int_{(A_i)}(f_1\chi_{F_{1,k}},\dots,f_{d_1}\chi_{F_{d_1,k}}, x_{d_1+1}\chi_{F_{d_1+1}},\dots,x_d\chi_{F_{d,k}})\,\mathrm{d}\Gamma' = \\
&= \lim_{k\to\infty}\lim_{n\to\infty} \int_{(A_i)} (f_{i,n}\chi_{F_{1,k}},\dots,f_{d_1,n}\chi_{F_{d_1,k}}, x_{d_1+1}\chi_{F_{d+1,k}},\dots,x_d\chi_{F_{d,k}})\,\mathrm{d}\Gamma = \\
&= \lim_{k\to\infty}\lim_{n\to\infty} \Gamma_{k,n,(A_1,\dots,A_{d_1})}(A_{d_1+1},\dots,A_d)\,(x_{d_1+1},\dots,x_d)
\end{aligned}$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_{f_i})$ and each $(x_{d_1+1},\dots,x_d) \in X_{f_{d_1+1}} \times \dots \times X_{f_d}$. Since evidently $\Gamma_{k,n,(A_1,\dots,A_{d_1})}\colon \mathscr{P}_{f_{d_1+1}} \times \dots \times \mathscr{P}_{f_d} \to L^{(d-d_1)}(X_{f_{d_1+1}},\dots,X_{f_d}; Y)$ for each $k, n = 1,2,\dots$ and each $(A_1,\dots,A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \dots \times \sigma(\mathscr{P}_{f_{d_1}})$, we conclude that $\Gamma_{(A_1,\dots,A_{d_1})}\colon \mathscr{P}_{f_{d_1+1}} \times \dots \times \mathscr{P}_{f_d} \to L^{(d-d_1)}(X_{f_{d_1+1}},\dots,X_{f_d}; Y)$ by the Banach-Steinhaus theorem for each $(A_1,\dots,A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \dots \times \sigma(\mathscr{P}_{f_{d_1}})$. Since $\Gamma_{k,n,(A_1,\dots,A_{d_1})}\colon \mathscr{P}_{f_{d_1+1}} \times \dots \times \mathscr{P}_{f_d} \to L^{(d-d_1)}(X_{f_{d_1+1}},\dots,X_{f_d}; Y)$ is separately countably additive in the strong operator topology for each $k, n = 1,2,\dots$ and each

$(A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \ldots \times \sigma(\mathscr{P}_{f_{d_1}})$, $\Gamma_{(A_1,\ldots,A_{d_1})}$ has the same property for each $(A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \ldots \times \sigma(\mathscr{P}_{f_{d_1}})$ by the (VHSN) theorem for polymeasures, see the beginning of Part VIII. Obviously $\lambda_{d_1+1} \times \ldots \times \lambda_d$: $\sigma(\mathscr{P}_{f_{d_1+1}}) \times \ldots$ $\ldots \times \sigma(\mathscr{P}_{f_d}) \to [0, +\infty)$ is a control $(d - d_1)$-polymeasure for each $\Gamma_{(A_1,\ldots,A_{d_1})}$, $(A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \ldots \times \sigma(\mathscr{P}_{f_{d_1}})$.

Assume b). We take $f_{i,n} \in S(\mathscr{P}_{f_i}, X_{f_i})$ and $F^*_{i,k}$ in accordance with Theorem X.1 and proceed as in a) above.

Assume c). For each $i = 1, \ldots, d$ take $f_{i,n} \in S(\mathscr{P}_{f_i}, X_{f_i})$, $n = 1, 2, \ldots$ such that $f_{i,n} \to f_i$ and $|f_{i,n}| \nearrow |f_i|$, and put $F_{i,k} = F'_{i,k} \cap \{t_i \in T_i, |f_i(t_i)| \leq k\}$ for $k = 1, 2, \ldots$. Since $\hat{\Gamma}[(f_i), (F'_{i,k})] < +\infty$ for each $k = 1, 2, \ldots$, we obtain that $(f_i \chi_{F_{i,k}}) \in \mathscr{L}_1(\Gamma) \subset \mathscr{I}(\Gamma)$ for each $k = 1, 2, \ldots$ by Theorem XI.5. Now the first assertion of the theorem follows similarly as in a) above.

Let the semivariations $\Gamma_{(A_1,\ldots,A_{d_1})}$: $\sigma(\mathscr{P}_{f_{d_1+1}}) \times \ldots \times \sigma(\mathscr{P}_{f_d}) \to [0, +\infty]$, $(A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \ldots + \sigma(\mathscr{P}_{f_{d_1}})$, be uniformly σ-finite. Then there are $(G_{d_1+1,k}, \ldots, G_{d,k}) \in \mathscr{P}_{f_{d_1+1}} \times \ldots \times \mathscr{P}_{f_d}$, $k = 1, 2, \ldots$ such that $G_{i,k} \nearrow F_i$ for each $i = d_1 + 1, \ldots, d$ and $\sup\{\hat{\Gamma}_{(A_1,\ldots,A_{d_1})}(G_{d_1+1,k}, \ldots, G_{d,k}), (A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \ldots$ $\ldots \times \sigma(\mathscr{P}_{f_{d_1}})\} < +\infty$ for each $k = 1, 2, \ldots$. For $i = d_1 + 1, \ldots, d$ and $k = 1, 2, \ldots$ put $H_{i,k} = F_{i,k} \cap G_{i,k}$. Then $(f_{d_1+1}\chi_{H_{d_1+1,k}}, \ldots, f_d \chi_{H_{d,k}}) \in \mathscr{I}(\Gamma_{(A_1,\ldots,A_{d_1})})$ for each $k = 1, 2, \ldots$ and each $(A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \ldots \times \sigma(\mathscr{P}_{f_{d_1}})$, and

$$\int_{(A_{d_1+1},\ldots,A_d)} (f_{d_1+1}\chi_{H_{d_1+1,k}}, \ldots, f_d \chi_{H_{d,k}}) \, d\Gamma_{(A_1,\ldots,A_d)} =$$
$$= \lim_{n\to\infty} \int_{(A_{d_1+1},\ldots,A_d)} (f_{d_1+1,n}\chi_{H_{d_1+1,k}}, \ldots, f_{d,k}\chi_{H_{d,k}}) \, d\Gamma_{(A_1,\ldots,A_{d_1})} =$$
$$= \lim_{n\to\infty} \int_{(A_i)} (f_1, \ldots, f_{d_1}, f_{d_1+1,n}\chi_{H_{d_1+1,k}}, \ldots, f_{d,n}\chi_{H_{d,k}}) \, d\Gamma$$

for each $(A_i) \in X\sigma(\mathscr{P}_{f_i})$ and each $k = 1, 2, \ldots$ by Corollary 4 of Theorem IX.4 in the cases of assumptions a) and b), and by Theorem XI.10, i.e. by LDCT, in the case of assumption c). Hence $(f_1, \ldots, f_{d_1}, f_{d_1+1}\chi_{H_{d_1+1,k}}, \ldots, f_d\chi_{H_{d,k}}) \in \mathscr{I}(\Gamma)$ and

$$\int_{(A_i)} (f_1, \ldots, f_{d_1}, f_{d_1+1}\chi_{H_{d_1+1,k}}, \ldots, f_d\chi_{H_{d,k}}) \, d\Gamma =$$
$$= \int_{(A_{d_1+1},\ldots,A_d)} (f_{d_1+1}\chi_{H_{d_1+1,k}}, \ldots, f_d\chi_{H_{d,k}}) \, d\Gamma_{(A_1,\ldots,A_{d_1})}$$

for each $(A_i) \in X\sigma(\mathscr{P}_{f_i})$ and each $k = 1, 2, \ldots$ by Theorem IX.4. If now $(f_i) \in \mathscr{I}(\Gamma)$, then

$$\int_{(A_i)} (f_i) \, d\Gamma' = \lim_{k\to\infty} \int_{(A_i)} (f_1, \ldots, f_{d_1}, f_{d_1+1}\chi_{H_{d_1+1,k}}, \ldots, f_d\chi_{H_{d,k}}) \, d\Gamma' =$$
$$= \lim_{k\to\infty} \int_{(A_{d_1+1},\ldots,A_d)} (f_{d_1+1}\chi_{H_{d_1+1,k}}, \ldots, d_d\chi_{H_{d,k}}) \, d\Gamma_{(A_1,\ldots,A_{d_1})}$$

for each $(A_i) \in X\sigma(\mathscr{P}_{f_i})$ by Theorem VIII.1. Hence $(f_{d_1+1}, \ldots, f_d) \in \mathscr{I}(\Gamma_{(A_1,\ldots,A_{d_1})})$ and

$$\int_{(A_{d_1+1},\ldots,A_d)} (f_{d_1+1}, \ldots, f_d) \, d\Gamma_{(A_1,\ldots,A_{d_1})} = \int_{(A_i)} (f_i) \, d\Gamma'$$

for each $(A_i) \in X\sigma(\mathscr{P}_{f_i})$ by Theorem IX.4.

The converse assertion follows analogously by using Theorems VIII.1 and IX.4 and Corollary 4 of Theorem IX.4.

Corollary 1. *Let the assumptions of the theorem be fulfilled and suppose moreover that* $(x_1\chi_{A_1}, \ldots, x_{d_1}\chi_{A_{d1}}, f_{d_1+1}, \ldots, f_d) \in \mathscr{I}(\Gamma)$ *for each* $(x_1, \ldots, x_{d_1}) \in X_1 \times \ldots \times X_{d_1}$ *and each* $(A_1, \ldots, A_{d_1}) \in \mathscr{P}_{f_1} \times \ldots \times \mathscr{P}_{f_{d_1}}$. *For these arguments put*

$$\Gamma_{(A_{d1+1},\ldots,A_d)}(A_1, \ldots, A_{d_1})(x_1, \ldots, x_{d_1}) =$$
$$= \int_{(A_i)}(x_1\chi_{A_1}, \ldots, x_{d_1}\chi_{A_{d1}}, f_{d_1+1}, \ldots, f_d)\,\mathrm{d}\Gamma$$

for $(A_{d_1+1}, \ldots, A_d) \in \sigma(\mathscr{P}_{f_{d_1+1}}) \times \ldots \times \sigma(\mathscr{P}_{f_d})$, and suppose the semivariations $\hat{\Gamma}_{(A_{d_1+1},\ldots,A_d)}$, $(A_{d_1+1}, \ldots, A_d) \in \sigma(\mathscr{P}_{f_{d_1+1}}) \times \ldots \times \sigma(\mathscr{P}_{f_d})$, are uniformly σ-finite on $\sigma(\mathscr{P}_{f_1}) \times \ldots \times \sigma(\mathscr{P}_{f_{d1}})$; in particular, let $\hat{\Gamma}[(f_i), (T_i)] < +\infty$. Then the following conditions are equivalent:

a) $(f_{d_1+1}, \ldots, f_d) \in \mathscr{I}(\Gamma_{(A_1,\ldots,A_{d1})})$ for each $(A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \ldots \times \sigma(\mathscr{P}_{f_{d1}})$,
b) $(f_i) \in \mathscr{I}(\Gamma)$, and
c) $(f_1, \ldots, f_{d_1}) \in \mathscr{I}(\Gamma_{(A_{d1+1},\ldots,A_d)})$ for each $(A_{d_1+1}, \ldots, A_d) \in \sigma(\mathscr{P}_{f_{d_1+1}}) \times \ldots$ $\ldots \times \sigma(\mathscr{P}_{f_d})$,

and if they hold, then

$$\int_{(A_{d1+1},\ldots,A_d)}(f_{d_1+1}, \ldots, f_d)\,\mathrm{d}\Gamma_{(A_1,\ldots,A_{d1})} = \int_{(A_i)}(f_i)\,\mathrm{d}\Gamma =$$
$$= \int_{(A_1,\ldots,A_{d_1})}(f_1, \ldots, f_{d_1})\,\mathrm{d}\Gamma_{(A_{d_1+1},\ldots,A_d)}$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_{f_i})$.

The following result is somewhat unexpected.

Corollary 2. *Theorems* V.11, V.12 *and* VII.3, *and Corollary of Theorem* V.9 *easily follow from the theorem. The same is true for Theorems* V.9 *and* V.10 *provided the functions g in their formulations are* $\mathscr{P}$*-measurable.*

Proof. It suffices to put $T_1 = T_2 = T$, $\mathscr{P}_1 = \mathscr{P}_2 = \mathscr{P}_f$, and $\Gamma(A_1, A_2)(x_1, x_2) = n(A_1 \cap A_2)\, x_2x_1$ in Theorems V.9 and V.12, where $X_1 = X$ and $X_2 = L(X, Z)$, $\Gamma(A_1, A_2)(x_1, x_2) = n(A_1 \cap A_2)\, x_1x_2$ in Theorems V.10 and V.11, where $X_1 = L(Z_1, Z)$ and $X_2 = Z_1$ in Theorem V.10, and $X_1 = X$ and $X_2 = Z$ in Theorem V.11, and $\Gamma(A_1, A_2)(x_1, x_2) = x_2n(A_1 \cap A_2)\, x_1 = x_1n(A_1 \cap A_2)\, x_2$ in Theorem VII.3, where $X_1 = X$ and $X_2 = L(Y, Z)$.

In all these cases $\Gamma(\cdot)(x_1, x_2)\colon \mathscr{P}_1 \times \mathscr{P}_2 \to Y$ has a control bimeasure because countably additive vector measures have locally control measures. Thus a) of the theorem is fulfilled.

Let us note that in Theorem V.9 we may suppose that the function g is $\mathscr{P}_f$-measurable. Namely, it is easy to see that in the proof of Theorem V.9 we may replace X by any separable subspace X' which contains the ranges of f_n, $n = 1, 2, \ldots$. However, by the well known theorem of Pettis, see Theorem 3.5.5 in [25], the assumed n-essential $\mathscr{P}_f$-measurability of $g(\cdot)\, x'$ for each $x' \in X'$ implies the n-essential $\mathscr{P}_f$-measurability of $g\colon T \to L(X', Z)$.

Theorem 4 is essential for our proof of the forthcoming generalizations of Theorems V.9.–V.12.

Theorem 5. *Let Z_i, $i = 1, \ldots, d$, be Banach spaces, and let $\Phi: X\mathscr{P}_{f_i} \to L^{(d)}(Z_i; Y)$ be separately countably additive in the strong operator topology with a locally σ-finite semivariation $\hat{\Phi}$ on $X\sigma(\mathscr{P}_{f_i})$. For each $i = 1, \ldots, d$ let $g_i: T_i \to L(X_i, Z_i)$ be $\mathscr{P}_i$-measurable, and let $(g_i(\cdot)\, x_i \chi_{A_i}) \in \mathscr{I}(\Phi)$ and*

$$\Gamma(A_i)(x_i) = \int_{(A_i)} (g_i(\cdot)\, x_i)\, \mathrm{d}\Phi$$

for each $(A_i) \in X\mathscr{P}_{f_i}$ and each $(x_i) \in XX_{f_i}$. Then $\hat{\Gamma}[(f_i), (A_i)] \leqq \hat{\Phi}[(g_i f_i), (A_i)]$ for each $(A_i) \in X\sigma(\mathscr{P}_{f_i})$, and if $(f_i) \in \mathscr{I}(\Gamma)$, then $(g_i f_i) \in \mathscr{I}(\Phi)$ and

$$(1) \qquad \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma = \int_{(A_i)} (g_i f_i)\, \mathrm{d}\Phi$$

for each $(A_i) \in X\sigma(\mathscr{P}_{f_i})$. The converse holds at least in the following cases:

a) *$\Phi(\cdot)(z_i): X\mathscr{P}_{f_i} \to Y$ has locally a control d-polymeasure for each $(z_i) \in XZ_i$;*

b) *$c_0 \not\subset Y$, and*

c) *$f_i(T_i) \subset X_i$ is relatively σ-compact for each $i = 1, \ldots, d$, in particular if each X_i, $i = 1, \ldots, d$ is finite dimensional.*

Proof. The inequality $\hat{\Gamma}[(f_i), (A_i)] \leqq \hat{\Phi}[(g_i f_i), (A_i)]$ is an immediate consequence of the definition of $\hat{\Gamma}[(f_i), (A_i)]$, see Definition VIII.3. If $(f_i) \in \mathscr{I}_0 = XS(\mathscr{P}_i, X_i)$, then obviously $(g_i f_i) \in \mathscr{I}(\Phi)$ and (1) holds. Using transfinite induction we obtain these assertion for $(f_i) \in \mathscr{I}(\Gamma) = \bigcup_{\alpha < \Omega} \mathscr{I}_\alpha(\Gamma)$, see Definition IX.2.

For the converse assertion define $\Psi: X\mathscr{P}_{f_i} \times X\mathscr{P}_{f_i} \to L^{(2d)}(X_1, \ldots, X_d, L(X_1, Z_1), \ldots, L(X_d, Z_d); Y)$ by the equality $\Psi(A_1, \ldots, A_d, B_1, \ldots, B_d)(x_1, \ldots, x_d, u_1, \ldots, u_d) = \Phi(A_i \cap B_i)(u_i x_i)$, where $(A_i), (B_i) \in X\mathscr{P}_{f_i}$, $(x_i) \in XX_{f_i}$ and $(u_i) \in XL(X_i, Z_i)$. Applying Theorem 4 to Ψ we obtain the converse assertion in the cases a) and b). Using the functionals $y^* \in Y^*$, by b) we obtain the converse assertion for the weak integral, see Section 2 in Part XII. Namely, if $(g_i f_i) \in \mathscr{I}(\Phi)$, then $(f_i) \in w\,\mathscr{I}(\Gamma)$ and

$$w \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma = \int_{(A_i)} (g_i f_i)\, \mathrm{d}\Phi \in \dot{Y} \subset Y^{**}$$

for each $(A_i) \in X\sigma(\mathscr{P}_{f_i})$, where $\dot{Y}$ is the image of Y in Y^{**} by the natural embedding. Now by Assertion 3) of Theorem XII.9 we have $(f_i) \in \mathscr{I}(\Gamma)$ if c) holds.

The following corollary is a generalization of Theorem IX.5.

Corollary. *Let $(f_i) \in \mathscr{I}(\Gamma)$, let $\gamma(A_i) = \int_{(A_i)} (f_i)\, \mathrm{d}\Gamma$ for $(A_i) \in X(\mathscr{P}_i)$, and let $\varphi_i: T_i \to K$ be $\mathscr{P}_i$-measurable for each $i = 1, \ldots, d$. If $(\varphi_i) \in \mathscr{I}(\gamma)$, then $(\varphi_i f_i) \in \mathscr{I}(\Gamma)$ and*

$$\int_{(A_i)} \varphi_i\, \mathrm{d}\gamma = \int_{(A_i)} (\varphi_i f_i)\, \mathrm{d}\Gamma$$

for each $(A_i) \in X\sigma(\mathscr{P}_i)$. The converse assertion holds at least in the cases a), b) *and* c) *stated in the theorem.*

The following generalization of Theorem V.10 can be proved analogously as Theorem 5.

Theorem 6. *For each $i = 1, \ldots, d$ let W_i, Z_i be Banach spaces and let $X_i \subset L(W_i, Z_i)$. Let $\Phi: X\mathscr{P}_{f_i} \to L^{(d)}(Z_i; Y)$ be as in Theorem 5, and let $g_i: T_i \to W_i$*

be $\mathscr{P}_i$-measurable for $i = 1, \ldots, d$. *Let finally* $(x_i g_i(\cdot)\, \chi_{A_i}) \in \mathscr{I}(\Phi)$ *and*

$$\Gamma(A_i)(x_i) = \int_{(A_i)} (x_i\, g_i(\cdot))\, \mathrm{d}\Phi$$

for each $(x_i) \in \mathsf{X} X_{f_i}$ *and each* $(A_i) \in \mathsf{X}\mathscr{P}_{f_i}$. *Then* $\hat{\Gamma}[(f_i), (A_i)] \leqq \hat{\Phi}[(f_i g_i), (A_i)]$ *for each* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$, *and if* $(f_i) \in \mathscr{I}(\Gamma)$, *then* $(f_i g_i) \in \mathscr{I}(\Phi)$ *and*

$$\int_{(A_i)} (f_i)\, \mathrm{d}\Gamma = \int_{(A_i)} (f_i g_i)\, \mathrm{d}\Phi$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_i)$. *The converse assertion holds at least in the cases* a), b) *and* c) *stated in Theorem* 5.

Directly from Theorem 4 we obtain the next generalization of Theorem V.11.

Theorem 7. *For each* $i = 1, \ldots, d$ *let* Z_i *be a Banach space and* $g_i \colon T_i \to Z_i$ *a $\mathscr{P}_i$-measurable function. Let* $\Phi \colon \mathsf{X}\mathscr{P}_{f_i} \to L^{(d)}(X_i; L^{(d)}(Z_i; Y))$ *be separately countably additive in the strong operator topology with* $\hat{\hat{\Phi}} \colon \mathsf{X}\sigma(\mathscr{P}_{f_i}) \to [0, +\infty]$ *σ-finite, where*

$$\hat{\hat{\Phi}}(A_i) = \sup \{ \left| \int_{(A_i)} (h_i)\, \mathrm{d}(\int_{(\cdot)} (e_i)\, \mathrm{d}\Phi) \right|,\ (h_i) \in \mathsf{X} S(\mathscr{P}_i, Z_i),$$
$$(e_i) \in \mathsf{X} S(\mathscr{P}_i, X_i) \quad and \quad \|h_i\|_{T_i}, \|e_i\|_{T_i} \leqq 1 \}$$

for $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_{f_i})$. *Let further* $(f_i \chi_{A_i}) \in \mathscr{I}(\Phi)$, $(g_i \chi_{A_i}) \in \mathscr{I}(\Phi(\cdot)(x_i))$ *and*

$$\Gamma(A_i)(x_i) = \int_{(A_i)} (g_i)\, \mathrm{d}(\Phi(\cdot)(x_i))$$

for each $(A_i) \in \mathsf{X}\mathscr{P}_{f_i}$ *and each* $(x_i) \in \mathsf{X} X_{f_i}$. *Let finally one of the following conditions be fulfilled*:

a′) $\Phi(\cdot)(x_i)(z_i) \colon \mathsf{X}\mathscr{P}_{f_i} \to Y$ *has locally a control d-polymeasure for each* $(x_i) \in \mathsf{X} X_{f_i}$ *and each* $(z_i) \in \mathsf{X} Z_{g_i}$;

b) $c_0 \not\subset Y$, *and*

c′) $f_i(T_i) \subset X_i$ *and* $g_i(T_i) \subset Z_i$, $i = 1, \ldots, d$, *are relatively σ-compact subsets.*

Then $(f_i) \in \mathscr{I}(\Gamma)$ *if and only if* $(g_i) \in \mathscr{I}(\int_{(\cdot)} (f_i)\, \mathrm{d}\Phi)$. *In this case*

$$\int_{(A_i)} (f_i)\, \mathrm{d}\Gamma = \int_{(A_i)} (g_i)\, \mathrm{d}(\int_{(\cdot)} (f_i)\, \mathrm{d}\Phi)$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_{f_i})$. *Moreover,* $\hat{\Gamma}[(f_i), (A_i)] \leqq \hat{\hat{\Phi}}[(g_i), (f_i), (A_i)]$ *for each* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_{f_i})$, *where* $\hat{\hat{\Phi}}[(g_i), (f_i), (A_i)]$ *is defined analogously to* $\hat{\hat{\Phi}}(A_i)$ *(now* $|h_i| \leqq |g_i|$ *and* $|e_i| \leqq |f_i|$ *for each* $i = 1, \ldots, d$).

Proof. It suffices to define $\Psi \colon \mathsf{X}\mathscr{P}_{f_i} \times \mathsf{X}\mathscr{P}_{f_i} \to L^{(2d)}(X_1, \ldots, X_d, Z_1, \ldots, Z_d; Y)$ by the equality

$$\Psi(A_1, \ldots, A_d, B_1, \ldots, B_d)(x_i)(z_i) = \Phi(A_i \cap B_i)(x_i)(z_i)$$

and to apply Theorem 4. The last inequality immediately follows from definitions.

The following generalization of Theorem V.12 can be proved analogously as Theorem 5.

Theorem 8. *For each* $i = 1, \ldots, d$ *let* Z_i *be a Banach space such that* $Z_i \subset L(X_i, Z_i)$, *and let* $g_i \colon T_i \to L(X_i, Z_i)$ *be $\mathscr{P}_i$-measurable. Let* $\Phi \colon \mathsf{X}\mathscr{P}_{f_i} \to L^{(d)}(L(X_i, Z_i);$ $L^{(d)}(X_i; Y))$ *be separately countably additive in the strong operator topology with* *a locally σ-finite semivariation* $\hat{\Phi}$ *on* $\mathsf{X}\sigma(\mathscr{P}_{f_i})$. *Let* $(g_i\chi_{A_i}) \in \mathscr{I}(\Phi)$ *and*

$$\Gamma(A_i) = \int_{(A_i)} (g_i)\, \mathrm{d}\Phi$$

for each $(A_i) \in \mathsf{X}\mathscr{P}_{f_i}$. *Then* $\hat{\Gamma}[(f_i), (A_i)] \leqq \hat{\Phi}'[(g_i f_i), (A_i)]$ *for each* $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_{f_i})$, *where* $\Phi' = \Phi\colon \mathsf{X}\mathscr{P}_{f_i} \to L^{(d)}(Z_i;\ L^{(d)}(X_i;\ Y))$, *and if* $(f_i) \in \mathscr{I}(\Gamma)$, *then* $(g_i f_i) \in \mathscr{I}(\Phi)$ *and*

$$\int_{(A_i)} (f_i)\, \mathrm{d}\Gamma = \int_{(A_i)} (g_i f_i)\, \mathrm{d}\Phi$$

for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}_{f_i})$. *The converse assertion holds at least in the cases* a), b) *and* c) *stated in Theorem* 5.

Let us give also a generalization of the important Theorem V.8.

Theorem 9. *For each* $i = 1, \ldots, d$ *let* S_i *be a non empty set,* $\mathscr{D}_i \subset 2^{S_i}$ *a δ-ring and* $\varphi_i\colon T_i \to S_i$ *a* $(\mathscr{P}_i, \mathscr{D}_i)$*-measurable transformation, i.e.,* $\varphi_i^{-1}(\mathscr{D}_i) \subset \mathscr{P}_i$. *For* $(B_i) \in \mathsf{X}\mathscr{D}_i$ *put* $\Gamma_{(\varphi_i^{-1})}(B_i) = \Gamma(\varphi_i^{-1}(B_i))$, *let* $\Gamma' = \Gamma\colon \mathsf{X}\varphi_i^{-1}(\mathscr{D}_i) \to L^{(d)}(X_i;\ Y)$, *and suppose the semivariation* $\hat{\Gamma}'$ *is finite on* $\mathsf{X}\varphi_i^{-1}(\mathscr{D}_i)$. *Let finally* $f_i\colon S_i \to X_i$ *be* $\mathscr{D}_i$*-measurable (then* $f_i(\varphi_i(\cdot))$ *is* $\varphi_i^{-1}(\mathscr{D}_i)$*-measurable) for each* $i = 1, \ldots, d$. *Then, if* $(f_i) \in \mathscr{I}(\Gamma_{(\varphi_i^{-1})})$, *then* $(f_i(\varphi_i(\cdot))) \in \mathscr{I}(\Gamma')$ *and*

$$\int_{(B_i)} (f_i)\, \mathrm{d}(\Gamma_{(\varphi_i^{-1})}) = \int_{(\varphi_i^{-1}(B_i))} (f_i(\varphi_i(\cdot)))\, \mathrm{d}\Gamma'$$

for each $(B_i) \in \mathsf{X}\sigma(\mathscr{D}_i)$. *The converse holds at least in the following cases:*

a) $\Gamma'(\cdot)(x_i)\colon \mathsf{X}\varphi_i^{-1}(\mathscr{D}_{f_i}) \to Y$ *has locally a control d-polymeasure for each* $(x_i) \in \mathsf{X}X_{f_i}$;

b) $f_i(S_i) \subset X_i$ *is relatively σ-compact for each* $i = 1, \ldots, d$;

c) $c_0 \not\subset Y$.

Proof. The first assertion follows by transfinite induction from its validity for $(f_i) \in \mathsf{X}S(\mathscr{D}_i, X_i)$ and the definition of $\mathscr{I}(\Gamma) = \bigcup_{\alpha<\Omega} \mathscr{I}_\alpha(\Gamma)$, see Definition IX.2. Let us prove the converse assertion in the individual cases:

a) According to Theorem XII.3 there are $(f_{i,n}) \in \mathsf{X}S(\mathscr{D}_{f_i}, X_{f_i})$, $n = 1, 2, \ldots$ such that $f_{i,n} \to f_i$ and $|f_{i,n}| \nearrow |f_i|$ for each $i = 1, \ldots, d$. Owing to Theorems VIII.17 and VIII.19 there is a control d-polymeasure, say $\lambda_1 \times \ldots \times \lambda_d$ for $\hat{\Gamma}'\colon \mathsf{X}\varphi_i^{-1}(\sigma(\mathscr{D}_i)) \to$ $\to [0, +\infty]$, see Definition VIII.2 for $\hat{\Gamma}'$. Using now the Egoroff-Lusin theorem coordinatewise we obtain the desired assertion analogously as in the proof of Theorem V.8 in [6].

b) In this case we use Theorem X.1 instead of the Egoroff-Lusin theorem and proceed as in the proof of Theorem V.8 in [6].

c) Take $(f_{i,n}) \in \mathsf{X}S(\mathscr{D}_i, X_i)$, $n = 1, 2, \ldots$ such that $f_{i,n} \to f_i$ and $|f_{i,n}| \nearrow |f_i|$ for each $i = 1, \ldots, d$. For $F_i = \{s_i \in S_i,\ f_i(s_i) \neq 0\}$, $i = 1, \ldots, d$, take $F^*_{i,k} \in \mathscr{D}_i$, $k = 1, 2, \ldots$ such that $F^*_{i,k} \nearrow F_i$, and put $F_{i,k} = F^*_{i,k} \cap \{s_i \in S_i,\ |f_i(s_i)| \leqq k\}$. Then

$$(*) \qquad \int_{(\varphi_i^{-1}(B_i))} (f_i(\varphi_i(\cdot)))\, \mathrm{d}\Gamma' = \lim_{k\to\infty} \int_{(\varphi_i^{-1}(B_i \cap F_{i,k}))} (f_i(\varphi_i(\cdot)))\, \mathrm{d}\Gamma'$$

for each $(B_i) \in \mathsf{X}\sigma(\mathscr{D}_i)$ by Theorem VIII.1. By assumption $\hat{\Gamma}'(\varphi_i^{-1}(F^*_{i,k})) < +\infty$, hence $\hat{\Gamma}'[(f_i(\varphi_i(\cdot))), (\varphi_i^{-1}(F_{i,k}))] < +\infty$.) However, since $c_0 \not\subset Y$, we have

$(f_i(\varphi_i(\cdot))\, \chi_{\varphi_i^{-1}(F_{i,k})}) \in \mathscr{L}_1(\Gamma')$ by Theorem XI.5 for each $k = 1, 2, \ldots$. Let k be fixed. Then

$$\int_{(\varphi_i^{-1}(B_i \cap F_{i,k}))} (f_i(\varphi_i(\cdot)))\, \mathrm{d}\Gamma' = \lim_{n\to\infty} \int_{(\varphi_i^{-1}(B_i \cap F_{i,k}))} (f_{i,n}(\varphi_i(\cdot)))\, \mathrm{d}\Gamma'$$

for each $(B_i) \in \mathsf{X}\sigma(\mathscr{D}_i)$ by the Lebesgue dominated convergence theorem in $\mathscr{L}_1(\Gamma)$, see Theorem XI.10. Since obviously

$$\int_{(\varphi_i^{-1}(B_i \cap F_{i,k}))} (f_{i,n}(\varphi_i(\cdot)))\, \mathrm{d}\Gamma' = \int_{(B_i \cap F_{i,k})} (f_{i,n})\, \mathrm{d}\Gamma_{(\varphi_i^{-1})}\,,$$

we have

$$(f_i\chi_{F_{i,k}}) \in \mathscr{I}(\Gamma_{(\varphi_i^{-1})}) \quad \text{and} \quad \int_{(B_i \cap F_{i,k})} (f_i)\, \mathrm{d}\Gamma_{(\varphi_i^{-1})} = $$
$$= \int_{(\varphi_i^{-1}(B_i \cap F_{i,k}))} (f_i(\varphi_i(\cdot)))\, \mathrm{d}\Gamma'$$

for each $(B_i) \in \mathsf{X}\sigma(\mathscr{D}_i)$. But then $(*)$ implies that $(f_i) \in \mathscr{I}(\Gamma_{(\varphi_i^{-1})})$ and the equality (1) holds for each $(B_i) \in \mathsf{X}\sigma(\mathscr{D}_i)$.

Returning to the Fubini theorems, analogously as Theorem 4 we obtain

Theorem 10. *Let $d > 1$ and let d_1, $1 \leqq d_1 < d$ be a positive integer. Let $f_i\colon T_i \to X_i$ be $\mathscr{P}_i$-measurable for each $i = 1, \ldots, d$ and let $(f_1, \ldots, f_{d_1}, x_{d_1+1}\chi_{A_{d_1+1}}, \ldots, x_d\chi_{A_d}) \in \mathscr{L}_1(\Gamma)$ for each $(x_{d_1+1}, \ldots, x_d) \in X_{f_{d_1+1}} \times \ldots \times X_{f_d}$ and each $(A_{d_1+1}, \ldots, A_d) \in \mathscr{P}_{f_{d_1+1}} \times \ldots \times \mathscr{P}_{f_d}$. Then*

1) *$\Gamma_{(A_1, \ldots, A_{d_1})}\colon \mathscr{P}_{f_{d_1+1}} \times \ldots \times \mathscr{P}_{f_d} \to L^{(d-d_1)}(X_{f_{d_1+1}}, \ldots, H_{f_d}; Y)$, and it is separately countably additive in the strong operator topology for each $(A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \ldots \times \sigma(\mathscr{P}_{f_{d_1}})$.*
2) *The semivariations $\hat{\Gamma}_{(A_1, \ldots, A_{d_1})}$, $(A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \ldots \times \sigma(\mathscr{P}_{f_{d_1}})$, are uniformly σ-finite on $\sigma(\mathscr{P}_{f_{d_1+1}}) \times \ldots \times \sigma(\mathscr{P}_{f_d})$.*
3) *$(f_i) \in \mathscr{I}(\Gamma)$ if and only if $(f_{d_1+1}, \ldots, f_d) \in \mathscr{I}(\Gamma_{(A_1, \ldots, A_{d_1})})$ for each $(A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \ldots \times \sigma(\mathscr{P}_{f_{d_1}})$. In this case*

$$\int_{(A_i)} (f_i)\, \mathrm{d}\Gamma = \int_{(A_{d_1+1}, \ldots, A_d)} (f_{d_1+1}, \ldots, f_d)\, \mathrm{d}\Gamma_{(A_1, \ldots, A_{d_1})} \quad \textit{for each}$$
$$(A_i) \in \mathsf{X}\sigma(\mathscr{P}_{f_i})\,.$$

Proof. 1) follows analogously as the first assertion of Theorem 4 in the case of assumption c).

2) For $i = d_1 + 1, \ldots, d$ take $x_i \in X_{f_i}$ with $|x_i| = 1$, $H_{i,k} \in \mathscr{P}_{f_i}$, $k = 1, 2, \ldots$ such that $H_{i,k} \nearrow F_i$, and put $G_{i,k} = H_{i,k} \cap \{t_i \in T_i,\ |f_i(t_i)| \leqq k\}$ for $k = 1, 2, \ldots$. Then $G_{i,k} \nearrow F_i$ for each $i = d_1 + 1, \ldots, d$ and $(f_1, \ldots, f_{d_1}, x_{d_1+1}\chi_{G_{d_1+1,k}}, \ldots, x_d\chi_{G_{d,k}}) \in \mathscr{L}_1(\Gamma)$ for each $k = 1, 2, \ldots$ by assumption. Hence
$\hat{\Gamma}_{(A_1, \ldots, A_{d_1})}(G_{d_1+1}, \ldots, G_{d,k}) \leqq \hat{\Gamma}[(f_1, \ldots, f_{d_1}, x_{d_1+1}\chi_{G_{d_1+1,k}}, \ldots, x_d\chi_{G_{d,k}}), (T_i)] < +\infty$ for each $k = 1, 2, \ldots$ and each $(A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \ldots \times \sigma(\mathscr{P}_{f_{d_1}})$ by Theorem 12 below.

3) follows analogously as in the proof of Theorem 4.

From this theorem and from the Fubini theorem in $\mathscr{L}_1(\Gamma)$, i.e., from Theorem XI.6, we immediately obtain the following characterization of $\mathscr{L}_1(\Gamma)$ for $d > 1$.

Theorem 11. *Let $d > 1$ and let $f_i: T_i \to X_i$ be $\mathscr{P}_i$-measurable for each $i = 1, \ldots, d$. Then $(f_i) \in \mathscr{L}_1(\Gamma)$ if and only if the following two conditions hold for any positive integer d_1, $1 \leqq d_1 < d$:*

(i) $(f_1, \ldots, f_{d_1}, x_{d_1+1}\chi_{A_{d_1+1}}, \ldots, x_d\chi_{A_d}) \in \mathscr{L}_1(\Gamma)$ *for each* $(x_{d_1+1}, \ldots, x_d) \in \in X_{f_{d_1+1}} \times \ldots \times X_{f_d}$ *and each* $(A_{d_1+1}, \ldots, A_d) \in \mathscr{P}_{f_{d_1+1}} \times \ldots \times \mathscr{P}_{f_d}$, *and*

(ii) $(f_{d_1+1}, \ldots, f_d) \in \mathscr{L}_1(\Gamma_{(A_1, \ldots, A_{d_1})})$ *for each* $(A_1, \ldots, A_{d_1}) \in \sigma(\mathscr{P}_{f_1}) \times \ldots \times \sigma(\mathscr{P}_{f_{d_1}})$.

3. FINITENESS OF $\hat{\Gamma}[(\cdot), (T_i)]$ ON $\mathscr{L}_1(\Gamma)$

It is important to note that our proof of the following theorem exploits Theorem 4 only in its formulation with assumption a).

Theorem 12. *Let $(f_i) \in \mathscr{L}_1(\Gamma)$. Then $\hat{\Gamma}[(f_i), (T_i)] < +\infty$.*

Proof. We prove the theorem by induction with respect to the dimension d. For $d = 1, 2$ the theorem was already proved, see Corollary of Theorem II.5 and the beginning of Part XI for $d = 1$, and Theorem XI.8 for $d = 2$. Hence suppose the theorem holds for the dimensions $1, \ldots, d - 1$, $d > 1$.

Let $(f_i) \in \mathscr{L}_1(\Gamma)$ and suppose $\hat{\Gamma}[(f_i), (T_i)] = \hat{\Gamma}[(f_i), (F_i)] = +\infty$, where $F_i = \{t_i \in T_i, f_i(t_i) \neq 0\} \in \sigma(\mathscr{P}_i)$, $i = 1, \ldots, d$. By the definition of the multiple L_1-gauge, see Definition VIII.3, for each $n = 1, 2, \ldots$ there is a d-tuple $(h_{i,n}) \in \mathsf{X}S(\mathscr{P}_i, X_i)$ such that $|h_{i,n}| \leqq |f_i|$ for each $i = 1, \ldots, d$ and each $n = 1, 2, \ldots$, and $|\int_{(F_i)} (h_{i,n})\, d\Gamma| > n$ for each $n = 1, 2, \ldots$. Since the semivariation $\hat{\Gamma}: \mathsf{X}(F_i \cap \sigma(\mathscr{P}_i)) \to \to [0, +\infty]$ is σ-finite by assumption, see the beginning of Part IX, there are $(F^*_{i,k}) \in \in \mathsf{X}\mathscr{P}_i$, $k = 1, 2, \ldots$ such that $F^*_{i,k} \nearrow F_i$ for each $i = 1, \ldots, d$, and $\hat{\Gamma}(F^*_{i,k}) < +\infty$ for each $k = 1, 2, \ldots$. According to Lemma III.3 for each $i = 1, \ldots, d$ there is a countable family $\{A_{i,n}\} \subset \mathscr{P}_i \cap F_i$ such that f_i is $\delta(\{A_{i,n}\})$-measurable, where $\delta(\mathscr{A})$ denotes the δ-ring generated by the family $\mathscr{A}$. Let $h_{i,n}$ be of the form $h_{i,n} = = \sum_{j=1}^{r_{i,n}} x_{i,n,j}\chi_{H_{i,n,j}}$ where $x_{i,n,j} \in X_i$, $H_{i,n,j} \in F_i \cap \mathscr{P}_i$ and $H_{i,n,j} \cap H_{i,n,k} = \emptyset$ for $j \neq k$, $j, k = 1, \ldots, r_{i,n}$. For $i = 1, \ldots, d$ put $P'_i = \delta(\{A_{i,n,n}, \{H_{i,n,j}\}_{j,n}, \{F_{i,k}\}_k)$ and let $\Gamma' = \Gamma: \chi(\mathscr{P}'_i \cap F_i) \to L^{(d)}(X_i; Y)$. Then f_i is $\mathscr{P}'_i$-measurable and $h_{i,n} \in S(\mathscr{P}'_i, X_i)$ for each $i = 1, \ldots, d$ and each $n = 1, 2, \ldots$. Hence $\hat{\Gamma}'[(f_i), (F_i)] = +\infty$. Since $(F^*_{i,k}) \in \mathsf{X}\mathscr{P}'_i$ for each $i = 1, \ldots, d$ and each $k = 1, 2, \ldots$, the semivariation $\hat{\Gamma}': \mathsf{X}(F_i \cap \sigma(\mathscr{P}'_i)) \to [0, +\infty]$ is σ-finite. Hence $(f_i) \in \mathscr{L}_1(\Gamma')$ by Theorem XI.9. Since each P'_i, $i = 1, \ldots, d$, is generated by a countable family of sets, $\Gamma'(\cdot)(x_i)$: $\mathsf{X}(F_i \cap \mathscr{P}'_i) \to Y$ has a control d-polymeasure for each $(x_i) \in \mathsf{X}X_i$ by Corollary of Theorem VIII.11.

For $i = 1, \ldots, d$ and $k = 1, 2, \ldots$ put $F_{i,k} = \{t_i \in T_i,\ k^{-1} \leqq |f_i(t_i)| \leqq k\} \cap F^*_{i,k}$, $\mathscr{P}^*_{f_i} = \bigcup_{k=1}^{\infty} F_{i,k} \cap \mathscr{P}'_i$, and let $\Gamma^* = \Gamma': \mathsf{X}\mathscr{P}^*_{f_i} \to L^{(d)}(X_i; Y)$. Obviously $\mathscr{P}^*_{f_i} \subset \mathscr{P}'_i \subset \subset \sigma(\mathscr{P}^*_{f_i}) = \sigma(\mathscr{P}'_i)$ for each $i = 1, \ldots, d$, hence $(f_i) \in \mathscr{L}_1(\Gamma^*)$ and $\Gamma^*[(f_i), (F_i)] =$

$= +\infty$. Since

$$\hat{\Gamma}^*[(f_i),(F_i)] = \hat{\Gamma}^*[(f_i),(F_{i,k} \cup (F_i - F_{i,k}))] \leqq$$
$$\leqq \hat{\Gamma}^*[(f_i),(F_i - F_{i,k})] + \hat{\Gamma}^*[(f_i),(F_{1,k}, F_2 - F_{2,k}, \dots, F_d - F_{d,k})] +$$
$$+ \dots + \hat{\Gamma}^*[(f_i),(F_1 - F_{1,k}, \dots, F_{d-1} - F_{d-1,k}, F_{d,k})] +$$
$$+ \dots + \hat{\Gamma}^*[(f_i),(F_{1,k}, F_{2,k}, F_3 - F_{3,k}, \dots, F_d - F_{d,k})] +$$
$$+ \hat{\Gamma}^*[(f_i),(F_{i,k})]$$

for each $k = 1, 2, \dots$, since $\hat{\Gamma}^*[(f_i),(F_{i,k})] < +\infty$ for each $k = 1, 2, \dots$, and since $\hat{\Gamma}^*[(f_i),(F_i - F_{i,k})] \to 0$ as $k \to \infty$ by Theorem XI.7, owing to the symmetry in coordinates we may suppose that there is a positive integer d_1, $1 \leqq d_1 < d$ and a subsequence $\{k_j\} \subset \{k\}$ such that

$$(*) \qquad \hat{\Gamma}^*[(f_i),(F_{1,k_j}, \dots, F_{d_1,k_j}, F_{d_j+1} - F_{d_1+1,k_j}, \dots, F_d - F_{d,k_j})] = +\infty$$

for each $j = 1, 2, \dots$.

Put $j_0 = 1$. By $(*)$ and the definition of the multiple L_1-gauge there are $u'_{i,1} \in \in S(\mathscr{P}'_i, X_i)$, $i = 1, \dots, d$ such that

$$|u'_{i,1}| \leqq |f_i| \chi_{F_{i,j_0}} \quad \text{for} \quad 1 \leqq i \leqq d_1,$$
$$|u'_{i,1}| \leqq |f_i| \chi_{F_i - F_{i,j_0}} \quad \text{for} \quad d_1 < i \leqq d,$$

and

$$\left|\int_{(F_i)} (u'_{i,1})\, \mathrm{d}\Gamma^*\right| > 2^{d_1} . d_1 . 2 .$$

Since $F_{i,k} \nearrow F_i$ for each $i = 1, \dots, d$ and since the indefinite integral $\int_{(\cdot)} (u'_{i,1})\, \mathrm{d}\Gamma^*$: $\mathsf{X}\sigma(\mathscr{P}^*_{fi}) \to Y$ is separately countably additive, according to Theorem VIII.1 there is a $j_1 > j_0$ such that

$$\left|\int_{(F_{1,k_{j_0}}, \dots, F_{d_1,k_{j_0}}, F_{d_1+1,k_{j_1}} - F_{d_1+1,k_{j_0}}, \dots, F_{d,k_{j_1}} - F_{d,k_{j_0}})} (u'_{i,1})\, \mathrm{d}\Gamma^*\right| >$$
$$> 2^{d_1} . d_1 . 2 .$$

Put

$$u_{i,1} = \begin{cases} 2^{-1} . u'_{i,1} & \text{for} \quad 1 \leqq i \leqq d_1, \\ u'_{i,1}\chi_{F_{i,k_{j_1}} - F_{i,k_{j_0}}} & \text{for} \quad d_1 < i \leqq d. \end{cases}$$

Obviously $(x_1\chi_{A_1}, \dots, x_{d_1}\chi_{A_{d_1}}, u_{d_1+1,1}, \dots, u_{d,1}) \in \mathscr{L}_1(\Gamma^*)$ for each $x_i \in X_i$, $A_i \in \mathscr{P}^*_{fi}$, $i = 1, \dots, d_1$. For these arguments and $(A_{d_1+1}, \dots, A_d) \in \sigma(\mathscr{P}^*_{f\,d_1+1}) \times \dots \times \sigma(\mathscr{P}^*_{fd})$ put $\Gamma^*_{(u_{d_1+1,1}, \dots, u_{d,1})(A_{d_1+1}, \dots, A_d)}(A_1, \dots, A_{d_1})(x_1, \dots, x_{d_1}) =$ $= \int_{(A_i)} (x_1\chi_{A_i}, \dots, x_{d_1}\chi_{A_{d_1}}, u_{d_1+1,1}, \dots, u_{d,1})\, \mathrm{d}\Gamma^*$. Then $\Gamma^*_{(u_{d_1+1}, \dots, u_d)(A_{d_1+1}, \dots, A_d)}$: $: \mathsf{X}_{i=1}^{d_1} \mathscr{P}^*_{fi} \to L^{(d_1)}(X_1, \dots, X_{d_1}; Y)$, and it is separately countably additive in the strong operator topology by the first assertion of Theorem 4 with assumption a) in its formulation. Let us note that we cannot use Theorem 10 since its proof used Theorem XI.10 (the proof of which through Theorem XI.6 exploits the finiteness of the multiple L_1-gauge on $\mathscr{L}_1(\Gamma)$). From the definitions of $F_{i,k}$ and $\mathscr{P}^*_{fi}$ it is easy to see that the semivariations $\Gamma^*_{(u_{d_1+1,1},\ldots,u_{d,1})(A_{d_1+1},\ldots,A_d)}$, $(A_{d_1+1}, \ldots, A_d) \in \sigma(\mathscr{P}^*_{f_{d_1+1}}) \times \ldots \times \sigma(\mathscr{P}^*_{f_d})$ are bounded on each $(A_1, \ldots, A_{d_1}) \in \mathscr{P}^*_{f_1} \times \ldots \times \mathscr{P}^*_{f_{d_1}}$. Hence they are uniformly σ-finite on $\mathscr{P}^*_{f_1} \times \ldots \times \mathscr{P}^*_{f_{d_1}}$. Thus the final assumption of Theorem 4 is satisfied. Since evidently $(f_1, \ldots, f_{d_1}, u_{d_1+1,1}, \ldots, u_{d,1}) \in \mathscr{L}_1(\Gamma^*)$, we have $(f_1, \ldots, f_{d_1}) \in \in \mathscr{L}_1(\Gamma^*_{(u_{d_1+1,1},\ldots,u_{d,1})(A_{d_1+1},\ldots,A_d)}$ for each $(A_{d_1+1}, \ldots, A_d) \in \sigma(\mathscr{P}^*_{f_{d_1+1}}) \times \ldots \times \sigma(\mathscr{P}^*_{f_d})$ and $\int_{(A_i)} (f_1, \ldots, f_{d_1}, u_{d_1+1,1}, \ldots, u_{d,1}) \, d\Gamma^* = \int_{(A_1,\ldots,A_{d_1})} (f_1, \ldots, f_{d_1}) \, .$ $. \, d\Gamma^*_{(u_{d_1+1,1},\ldots,u_{d,1})(A_{d_1+1},\ldots,A_d)}$ for each $(A_i) \in \mathsf{X}\sigma(\mathscr{P}^*_{f_i})$.

By the induction hypothesis

$$b_1 = \hat{\Gamma}^*_{(u_{d_1+1,1},\ldots,u_{d,1})(F_{d_1+1},\ldots,F_d)}[(f_1, \ldots, f_{d_1}), (F_1, \ldots, F_{d_1})] < +\infty \, .$$

Analogously

$$a_{1,1} = \hat{\Gamma}^*_{(u_{1,1})(F_1)} \, [(f_2, \ldots, f_d, (F_2, \ldots, F_d)] < +\infty$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{d_1,1} = \hat{\Gamma}^*_{(u_{d_1,1})(F_{d_1})} \, [(f_1, \ldots, f_{d_1-1}, f_{d_1+1}, \ldots, f_d),$$
$$(F_1, \ldots, F_{d_1-1}, F_{d_1+1}, \ldots, F_d)] < +\infty \, .$$

Put $a_1 = \max \{a_{i,1}, \; i \in \{1, \ldots, d_1\}\}$.

The equality $(*)$ holds for j_1. We now describe how to proceed to obtain $u_{i,n}$, b_n, and a_n when we have already found $u_{i,n-1}$, b_{n-1}, and a_{n-1}.

The equality $(*)$ is valid for j_{n-1}. Then, the argument being the same as above, there are $u'_{i,n} \in S(\mathscr{P}^*_{f_i}, X_i)$, $i = 1, \ldots, d$, and $j_n > j_{n-1}$ such that

$$|u'_{i,n}| \leqq |f_i| \, \chi_{F_{i,k_{j_{n-1}}}} \quad \text{for} \quad 1 \leqq i \leqq d_1 \, ,$$
$$|u'_{i,n}| \leqq |f_i| \, \chi_{F_{i,k_{j_n}} - F_{i,k_{j_{n-1}}}} \quad \text{for} \quad d_1 < i \leqq d \, ,$$

and

$$\left|\int_{(F_i)} (u'_{i,n}) \, d\Gamma^*\right| > 2^{nd_1} \, . \, d_1(2 + a_{n-1}) (1 + b_1)^{d_1} \, . \ldots . (1 + b_{n-1})^{d_1} \, .$$

Put

$$u_{i,n} = \begin{cases} 2^{-n}[(1 + b_1) \, . \ldots . (1 + b_{n-1})]^{-1} \, u'_{i,n} & \text{for} \quad 1 \leqq i \leqq d_1 \, , \\ u'_{i,n}\chi_{F_{i,k_{j_n}} - F_{i,k_{j_{n-1}}}} & \text{for} \quad d_1 < i \leqq d \, , \end{cases}$$

$$b_n = \hat{\Gamma}^*_{(u_{d_1+1,n},\ldots,u_{d,n})(F_{d_1+1},\ldots,F_d)}[(f_1, \ldots, f_{d_1}), (F_1, \ldots, F_{d_1})] < +\infty \, ,$$

$$a_{1,n} = \hat{\Gamma}^*_{(v_{1,n})(F_1)} \, [(f_2, \ldots, f_d), (F_2, \ldots, F_d)] < +\infty \, ,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{d_1,n} = \hat{\Gamma}^*_{(v_{d_1,n})(F_{d_1})} \, [(f_1, \ldots, f_{d_1-1}, f_{d_1+1}, \ldots, f_d),$$
$$(F_1, \ldots, F_{d_1-1}, F_{d_1+1}, \ldots, F_d)[< +\infty \, ,$$

where $v_{i,n} = \sum_{r=1}^{n} u_{i,r}$ for $i = 1, \ldots, d_1$, and $a_n = \max \{a_{i,n}, \; i \in \{1, \ldots, d_1\}\}$.

Having $u_{i,n}$, b_n and a_n for $n = 1, 2, \ldots$, put $u_i = \sum_{n=1}^{\infty} u_{i,n}$ for $i = 1, \ldots, d$. Clearly $|u_i| \leqq |f_i|$ for each $i = 1, \ldots, d$, hence $(u_i) \in \mathscr{L}_1(\Gamma^*) \subset \mathscr{I}(\Gamma^*)$. For $(A_i) \in \mathsf{X}\sigma(\mathscr{P}^*_{f_i})$ put $\gamma(A_i) = \int_{(A_i)} (u_i) \, d\Gamma^*$. Then $\gamma \colon \mathsf{X}\sigma(\mathscr{P}^*_{f_i}) \to Y$ is a separately countably additive vector d-polymeasure. For $1 \leqq i < d_1$ put $G_{i,n} = F_i$, and let $G_{i,n} = F_{i,k_{j_n}} - F_{i,k_{j_{n-1}}}$ for $d_1 < i \leqq d$. Since $F_{i,k} \nearrow$, $G_{i,n} \to \emptyset$ for $i = d_1 + 1, \ldots, d$. Hence $\gamma(G_{i,n}) \to 0$ as $n \to \infty$ by Theorem VIII.1. Take a positive integer $n_0 > 2$ such that $(\gamma(G_{i,n_0})| < 1$. Since

$$\sum_{r=n_0+1}^{\infty} |u_{i,r}| = \sum_{r=n_0+1}^{\infty} 2^{-r}[(1 + b_1) \ldots . (1 + b_{r-1})]^{-1} |u'_{i,r}| \leqq$$

$$\leqq \sum_{r=n_0+1}^{\infty} 2^{-r}[(1 + b_1) \ldots . (1 + b_{r-1})]^{-1} |f_i| \leqq (1 + b_{n_0})^{-1} |f_i|$$

for $i = d_1 + 1, \ldots, d$, we obtain

$$d_1(2 + a_{n_0-1}) < \left|\int_{(G_{i,n_0})} (u_{i,n_0}) \, \mathrm{d}\Gamma^*\right| =$$

$$= \left|\int_{(G_{i,n_0})} (u_1 - \sum_{r=1}^{n_0-1} u_{1,r} - \sum_{r=n_0+1}^{\infty} u_{1,r}, u_{2,n_0}, \ldots, u_{d,n_0}) \, \mathrm{d}\Gamma^*\right| \leqq$$

$$\leqq \left|\int_{(G_{i,n_0})} (u_1, u_{2,n_0}, \ldots, u_{d,n_0}) \, \mathrm{d}\Gamma^*\right| +$$

$$+ \left|\int_{(G_{i,n_0})} (\sum_{r=1}^{n_0-1} u_{1,r}, u_{2,n_0}, \ldots, u_{d,n_0}) \, \mathrm{d}\Gamma^*\right| +$$

$$+ \left|\int_{(G_{i,n_0})} (\sum_{r=n_0+1}^{\infty} u_{1,r}, u_{2,n_0}, \ldots, u_{d,n_0}) \, \mathrm{d}\Gamma^*\right| \leqq$$

$$\leqq \left|\int_{(G_{i,n_0})} (u_1, u_2 - \sum_{r=1}^{n_0-1} u_{2,r} - \sum_{r=n_0+1}^{\infty} u_{2,r}, u_{3,n_0}, \ldots, u_{d,n_0}) \, \mathrm{d}\Gamma^*\right| +$$

$$+ a_{n_0-1} + b_{n_0}(1 + b_{n_0})^{-1} \leqq \ldots \leqq d_1 a_{n_0-1} + d_1 + 1 \,,$$

a contradiction.

From this theorem and from Theorem XI.5 we immediately obtain

Theorem 13. *Let* $c_0 \not\subset Y$. *Then* $(f_i) \in \mathscr{L}_1(\Gamma)$ *if and only if* $\hat{\Gamma}[(f_i), (T_i)] < +\infty$.

Let us close the paper with the following

Problem. Let $X_i = K$ for each $i = 1, \ldots, d$. Is then $\mathscr{I}(\Gamma) = \mathscr{L}_1(\Gamma)$?

References

[1] *Blei, R. C.:* Fractional dimensions and bounded fractional forms, Mem. Amer. Math. Soc. *57*, (1985), No. 331.

[2] *Blei, R. C.:* Multi-linear measure theory and the Grothendieck factorization theorem, Proc. London Math. Soc. (3) *56* (1988), 529—546.

[3] *Chang, D. K., Rao, M. M.:* Special representation of weakly harmonizable processes, Stochastic Anal. Appl. *6* (1988), 169—189.

[4] *Dobrakov, L.:* On integration in Banach spaces, I, and II, Czech. Math. J. *20 (95)* (1970), 511—536, and 680—695.

[5] *Dobrakov, I.:* On integration in Banach spaces, III, Czech. Math. J. *29 (104)* (1979), 478—499.

[6] *Dobrakov, I.:* On integration in Banach spaces, IV, and V, Czech. Math. J. *30 (105)* (1980), 259—279, and 610—628.
[7] *Dobrakov, I., Morales, P.:* On integration in Banach spaces, VI, Czech. Math. J. *35 (110)* (1985), 173—187.
[8] *Dobrakov, I.:* On integration in Banach spaces, VII, Czech. Math. J. *38 (113)* (1988), 434—449.
[9] *Dobrakov, I.:* On integration in Banach spaces, VIII (Polymeasures), Czech. Math. J. *37 (112)* (1987), 487—506.
[10] *Dobrakov, I.:* On integration in Banach spaces, IX, and X (Integration with respect to polymeasures), Czech. Math. J. *38 (113)* (1988), 589—601, and 713—725.
[11] *Dobrakov, I.:* On integration in Banach spaces, XI (Integration with respect to polymeasures), Czech. Math. J. *39 (114)* 1989,
[12] *Dobrakov, I.:* On integration in Banach spaces, XII (Integration with respect to polymeasures), Czech. Math. J.
[13] *Dobrakov, I.:* On extension of vector polymeasures, Czech. Math. J. *38 (113)*, (1988), 88—94.
[14] *Dobrakov, I.:* On representation of linear operators on $C_0(T, X)$, Czech. Math. J. *21 (96)*, (1971), 13—30.
[15] *Dobrakov, I.:* On Lebesgue pseudonorms on $C_0(T)$, Math. Slovaca *32*, (1982), 327—335.
[16] *Dobrakov, I.:* Representation of multilinear operators on $XC_0(T_i)$, Czech. Math. J. *39 (114)*, (1989),
[17] *Dobrakov, I.:* Representation of multilinear operators on $X_0(T_i, X_i)$, I, Atti del Semin. di Mat. Fis. Universita di Modena.
[18] *Dobrakov, I.:* Representation of multilinear operators on $XC_0(T_i, X_i)$, II, Czech. Math. J.
[19] *Fréchet, M.:* Sur les fonctionnelles bilineares, Trans. Amer. Math. Soc. *16* (1915), 215—224.
[20] *Gilbert, J. E., Leih, T. J.:* Factorization, tensor products, and bilinear forms in Banach space theory, Notes in Banach spaces, Univ. of Texas Press, Austin, 1980, 182—305.
[21] *Gilbert, J. E., Ito, T., Schreiber, B. M.:* Bimeasure algebras on locally compact groups, J. Functional Anal. *64* (1985), 134—162.
[22] *Graham, C. C., Schreiber, B. M.:* Bimeasure algebras on LCA groups, Pacific J. Math. *115* (1984), 91—127.
[23] *Graham, C. C., Schreiber, B. M.:* Sets of interpolation for Fourier transforms of bimeasures, Colloq. Math. *51* (1987), 149—154.
[24] *Graham, C. C., Schreiber, B. M.:* Projections in spces of bimeasures, Canad. Math. Bull. *31* (1988), 19—25.
[25] *Hille, E. Phillips, R.:* Functional Analysis and Semigroups, Amer. Math. Soc. Coll. Publ., Providence, 1957.
[26] *Johnson, B. E., Kadison, R. V., Ringrose, J. R.:* Cohomology of operator algebras, III. Reduction to normal cohomology, Bull. Soc. math. France *100* (1972), 73—96.
[27] *Kolmogoroff, A.:* Untersuchungen über den Integralbegriff, Math. Ann. *103* (1930), 654—696.
[28] *Rao, M. M.:* Bimeasures and harmonizable processes (Analysis, Classification, and Representation), UCR Tech. Report No. *14*, 1988.
[29] *Saeki, S.:* The ranges of certain isometries of tensor products of Banach spaces, J. Math. Soc. Japan *23* (1971), 27—39.
[30] *Saeki, S.:* Tensor products of $C(X)$-spaces and their conjugate spaces, J. Math. Soc. Japan *28* (1976), 33—47.
[31] *Varopoulos, N. Th.:* Tensor algebras and harmonic analysis, Acta Math. *119* (1967), 51—112.
[32] *Ylinen, K.:* Noncommutative Fourier transformations of bounded bilinear forms and completely bounded multilinear operators, (UCR Tech. Report No. 12 1987), J. Functional Anal. (to appear)

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