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FACTORS OF CLAW-FREE GRAPHS

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1. INTRODUCTION

Recently, many papers have been concerned with the properties of claw-free graphs. In the present paper we follow up with the study of factors of the claw-free graphs which originated from the following well-known result by Sumner [5].

Theorem. *Every connected claw-free graph having an even number of vertices has a perfect matching.*

If the number of vertices of G is odd then obviously a perfect matching does not exist. One can easily see that such a graph always has a factor, referted to as an almost perfect matching, whose all components are single edges except for one which is a (not necessarily induced) path of length 2. Nevertheless, such a graph can fail to have some other types of factors which can be of interest, e.g. a factor which, besides single edges, contains a triangle, a bull or an induced path of length 2 as its components. In this paper we given full characterizations of such classes of graphs.

We consider simple graphs, i.e. graphs without loops and multiple edges. In general, we follow the terminology of Harary [2]. In particular, $V(G)$ and $E(G)$ stand for the set of vertices and the set of edges of G, respectively, $\langle M \rangle$ is the induced subgraph on $M \subseteq V(G)$, $G \setminus G_1 = \langle V(G) \setminus V(G_1) \rangle$ for $G_1 \subseteq G$ and $G - x = \langle V(G) \setminus \{x\} \rangle$ for $x \in V(G)$.

Throughout the paper we denote by E the complete graph on 2 vertices, by T the *triangle*, i.e. the complete graph on 3 vertices, by P the three-vertex path, by S the *claw*, i.e. the three-edge star and by B the *bull*, i.e. the graph depicted in Figure 1. By a *factor* in a graph we mean a spanning subgraph of G. If all components of a factor are isomorphic to E then the factor is called a *perfect matching*.

Let $H_1, \ldots, H_k$ be graphs, By an $\{H_1, \ldots, H_k\}$-subgraph of G we mean a subgraph each component of which is isomorphic to one of the graphs $H_1, \ldots, H_k$. An $\{H_1, \ldots, H_k\}$-factor of G is an $\{H_1, \ldots, H_k\}$-subgraph which is a factor of G. By an $H_1, \{H_2\}$*-factor* of G we mean a factor with exactly one component isomorphic to H_1 and all the others isomorphic to H_2. A factor whose every component is an induced subgraph of G is called *strong*. Clearly, a $T, \{E\}$-factor is a strong T, $\{E\}$-factor. A graph is said to be *claw-free* if it contains no copy of the claw as an induced subgraph. Clearly, every induced subgraph of a claw-free graph is claw-free, too. For short, we call a graph *odd* if it has an odd number of vertices. Otherwise we call it *even*. If G is an odd connected claw-free graph and x is a cutvertex of G then

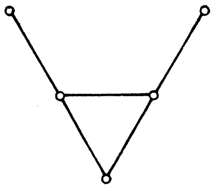

Figure 1. The bull

$G - x$ consists of exactly two components which are either both odd or both even. The cutvertex x is called *odd* or *even* according to the parity of the components of $G - x$. The *trivial* graph is the graph with one vertex only.

In our terminology, an almost perfect matching is a $P, \{E\}$-factor and, as mentioned above, it exists in every odd connected claw-free graph. Its component isomorphic to P is either an induced subgraph or a subgraph of a triangle. Consequently, every odd connected claw-free graph has a strong $P, \{E\}$-factor or a (strong) $T, \{E\}$-factor.

In Section 2 we prove that every odd connected claw-free graph has either a $T, \{E\}$-factor or a strong $B, \{E\}$-factor. In Section 3 we give full characterizations of odd connected claw-free graphs with a $T, \{E\}$-factor, and odd connected claw-free graphs with a strong $P, \{E\}$-factor.

As a byproduct of the former result we get a characterization of all odd connected claw-free graphs with a perfect 2-matching. A *perfect 2-matching* is a factor whose components are isomorphic to either an odd cycle or the graph E. This concept is an analogue of the perfect matching and plays an important role in the matching theory (cf Lovász, Plummer [3]). Our result strengthens one of the results of [4] on the existence of a perfect 2-matching in claw-free graphs.

In Section 4 we examine connections between our results and some edge-partition problems studied by Favaron *et al.* [1]. Our results are, in a sense, generalizations of some theorems in [1].

2. EXISTENCE

Theorem 1. *Every odd connected claw-free graph with at least one triangle has an* $\{E, T, B\}$*-factor.*

Proof. Let C be the largest $\{T\}$-subgraph of G and $A = G \setminus C$. If $V(A) = \emptyset$ then we are done, so suppose that $V(A)$ is nonempty. No vertex of A has degree greater than 2 in A since otherwise, as this vertex cannot centre a claw, it would belong together with two of its neighbours to a triangle in A which contradicts the maximality of C. Hence A consists of paths, cycles and isolated vertices only.

Let the largest $\{T\}$-subgraph C in G be chosen such that the number of odd components of A is minimal.

Since in each component of A at least one vertex is adjacent to some vertex in C, we can easily see that there is a largest $\{E\}$-subgraph H in A such that

(i) $V(H)$ contains all vertices belonging to even components of A, and

(ii) in each of the odd components of A the only vertex which does not belong to $V(H)$ has a neighbour in $V(C)$.

This is seen from the following simple observation. If $z \in V(A)$ is of degree 2 in A, z_1 and z_2 are the neighbours of z in A and $u \in V(C)$ is a neighbour of z in G then since z cannot centre a claw, u is adjacent to z_1 or z_2.

Denote $X = V(A) \setminus V(H)$. We prove that every vertex of C has at most one neighbour in X. Let, on the contrary, $y \in V(C)$ be adjacent to $x_1, x_2 \in X$. Denote by y_1, y_2 the two neighbours of y in C. Clearly, x_1, x_2 are non-adjacent but then, considering $\langle\{x_1, x_2, y, y_1\}\rangle$, we see that y_1 is adjacent to x_1 or x_2. By symmetry we can assume, without loss of generality, that y_1 is adjacent to x_1. Consider $\langle\{x_1, x_2, y, y_2\}\rangle$. We see that y_2 is adjacent to x_1 or x_2. If we replace in C the triangle $\langle\{y, y_1, y_2\}\rangle$ by $\langle\{x_1, y_1, y_2\}\rangle$ in the first case and by $\langle\{x_1, y, y_1\}\rangle$ in the latter case, then in both cases the number of odd components of A is decreased which contradicts the choice of A.

Hence every vertex of C has at most one neighbour in X. Joining every vertex of X to one of its neighbours in $V(C)$ with an edge and considering the resulting subgraphs together with the components of H and the remaining components of C, we get a factor of G each component of which is isomorphic to either E, T or one of the graphs depicted in Figure 2. As the second graph in Figure 2 is the bull and the

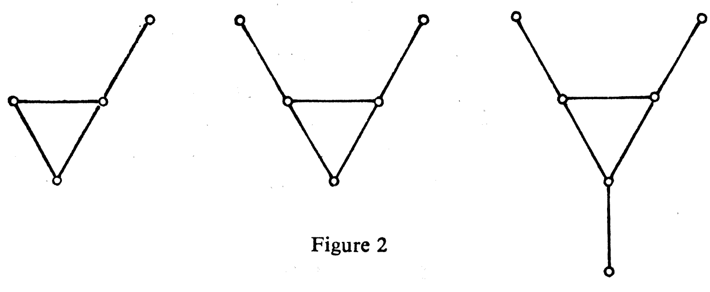

Figure 2

first and the third have a perfect matching, we can easily construct an $\{E, T, B\}$-factor in G. ■

We turn our attention to the existence of factors with components of two types only.

Proposition 2. *Let G be an odd connected claw-free graph. Then*

a) *G has an $\{E, T\}$-factor if and only if G has a T, $\{E\}$-factor and*

b) *G has an $\{E, B\}$-factor if and only if G has a B, $\{E\}$-factor.*

Proof. The assertion follows immediately from the following lemma, proved in [4].

Lemma. *Let G be an odd connected claw-free graph and suppose that F is a factor of G each component of which is either a single edge or odd. Then there exists a factor F′ in G such that the only odd component of F′ coincides with some of the odd components of F, and all the other components of F′ are single edges.* ■

Corollary 3. *Every odd connected claw-free graph with at least one triangle has a $T, \{E\}$-factor or a strong $B, \{E\}$-factor.*

Proof. Suppose that G does not have a $T, \{E\}$-factor. By Theorem 1 and Lemma in [4] (see the proof of Proposition 2), G has a $B, \{E\}$-factor. Denote by H the component of G isomorphic to B. Moreover, denote by α_1 and α_2 the vertices of degree 1, by b_1 and b_2 the neighbours of α_1 and α_2, respectively, and by c the only vertex of degree 2 in H.

The vertices α_1 and α_2 are non-adjacent since otherwise, replacing H by $\langle\{\alpha_1, \alpha_2\}\rangle$ and $\langle\{b_1, b_2, c\}\rangle$ we get a $T, \{E\}$-factor in G. For the same reason the vertex c is adjacent to neither α_1 nor α_2. The vertex α_1 is not adjacent to b_2 since otherwise b_2 would centre a claw. Similarly, $a_2b_1 \notin E(G)$. Hence, H is an induced subgraph of G and we have a strong $B, \{E\}$-factor in G. ■

Example. The graph in Figure 3a has a $T, \{E\}$-factor but has no $B, \{E\}$-factor. The graph in Figure 3b has a strong $B, \{E\}$-factor but has no $T, \{E\}$-factor. The graph in Figure 3c has both a $T, \{E\}$-factor and a strong $B, \{E\}$-factor.

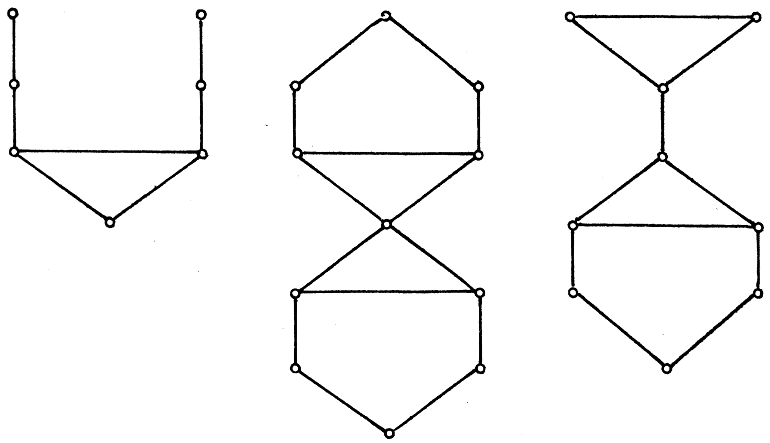

Figure 3

3. CHARACTERIZATIONS

In this section we give full characterizations of classes of odd connected claw-free graphs which fail to have a $T, \{E\}$-factor, a perfect 2-matching, and a strong $P, \{E\}$-factor.

We first prove two auxiliary assertions.

Lemma 4. *An odd connected claw-free graph G has a $T, \{E\}$-factor if and only if there is a triangle T in G such that all components of $G \setminus T$ are even.*

Proof. If G has a $T, \{E\}$-factor then $G \setminus T$ has a perfect matching. Consequently, all components of $G \setminus T$ are even. Conversely, if all components of $G \setminus T$ are even then taking a perfect matching in each of them (its existence follows by the Sumner Theorem) and adding T, we get a $T, \{E\}$-factor in G. ■

Lemma 5. *Let G be an odd connected claw-free graph having no $T, \{E\}$-factor. Then for every triangle $C \subseteq G$ at least one vertex of C is a curvertex of G.*

Proof. For triangle-free graphs the theorem is trivial so we can suppose that G contains at least one triangle. Let, on the contrary, $C \subseteq G$ be a triangle no vertex of which is a cutvertex of G, and denote by $G_1, \ldots, G_k$ the components of $G \setminus C$. Then, by Lemma 4, $k \geqq 2$ and for every j, $1 \leqq j \leqq k$, at least two distinct vertices of C have a neighbour in G_j.

If $k \geqq 4$ then some vertex of C has two neighbours in three distinct components of $G \setminus C$ and hence centres a claw; thus $k \leqq 3$.

At least one of the G_j's is odd since otherwise, by Lemma 4, G would have a $T, \{E\}$-factor. From the fact that G is odd we then see that exactly two of the components of $G \setminus C$ (say G_1 and G_2) are odd and, if $k = 3$, the component G_3 is even. In the latter case, G_3 is an even connected claw-free graph, so by the Summer Theorem it has a perfect matching. Clearly, the graph $G \setminus G_3$ cannot have a $T, \{E\}$-factor since otherwise it could be extended to a $T, \{E\}$-factor of G. Therefore it is sufficient to consider the case $k = 2$.

Choose vertices x and x_1 of C such that x has its neighbours in both G_1 and G_2 and x_1 has a neigbour in G_1 (such vertices exist since none of the vertices of C is a cutvertex of G). Denote by x_2 the third vertex of C and by z_i the neighbours of x in G_i, $i = 1, 2$. If x_2 has no neighbour in G_2 then x_1 is adjacent to some vertex in G_2 and, since x cannot centre a claw, x_2 has a neighbour in G_1; in this case we interchange the notation of x_1 and x_2. Hence, in each case we have vertices d_1 and d_2 such that $d_i \in G_i$ and $d_i x_i \in E(G)$, $i = 1, 2$ (not excluding the possible cases $d_1 = z_1$ and $d_2 = z_2$).

We prove that, for $i = 1, 2$, x has a neighbour in G_i which is not an odd cutvertex of G_i. Suppose, on the contrary, that in some of G_i's (say in G_1) each of the neighbours of x in G_1 is an odd cutvertex of G_1. Let $z_1^1, \ldots, z_1^k$ be all neighbours of x in G_1. The subgraph $K_1 = \langle\{z_1^1, \ldots, z_1^k\}\rangle$ is a clique for, if $z_1^i z_1^j \notin E(G)$ for some i, j, $i \neq j$, then $\{z_1^i, z_1^j, z_2, x\}$ induces a claw. For $i = 1, \ldots, k$ denote by B^i the only component of $G_1 \setminus z_1^i$ which contains no vertex of K_1 (if there were two such components then z_1^i would centre a claw). In every B^i choose a vertex b^i adjacent to z_1^i.

If there is a vertex in G_1 which is neither in K_1 nor in any of $B^1, \ldots, B^k$ then it cannot be adjacent to any vertex of $B^1, \ldots, B^k$. Thus, we can find j and a vertex

$a \notin B^j \cup K_1$ such that $az_1^j \in E(G)$. Since $b^j x \notin E(G)$ (otherwise b^j would be some of z_1^j's and, consequently, $b^j \in K_1$) and similarly, $ax \notin E(G)$, we see, observing $\langle\{a, b^j, x, z_1^j\}\rangle$, that $ab^j \in E(G)$ which implies $a \in B^j$, a contradiction. Thus $G_1 = \langle V(K_1) \cup \bigcup_{i=1}^{k} V(B^i)\rangle$. This implies that $V(G_1)$ is the union of the disjoint even sets $V(B^j) \cup \{z_1^j\}$ $(j = 1, \dots, k)$ and hence it is even. This contradiction proves that at least one of the vertices $z_1^1, \dots, z_1^k$ (say z_1^1) is not an odd cutvertex of G_1.

Similarly we can show that some of the neighbours of x in G_2 (say z_2^1) is not an odd cutvertex of G_2.

Now, if $z_1^1 x_1 \in E(G)$, then all components of $G \setminus \langle\{z_1^1, x_1, x\}\rangle$ are even. Thus $z_1^1 x_1 \notin E(G)$ and, consequently, $z_2^1 x_1 \in E(G)$ (otherwise $\{z_1^1, x_1, z_2^1, x\}$ induces a claw). Similarly we get $z_2^1 x_2 \notin E(G)$ and, conseuently, $z_1^1 x_2 \in E(G)$. Finally, it is easily seen that each component of $G \setminus \langle\{z_1^1, x_2, x\}\rangle$ is even and hence G has a $T, \{E\}$-factor. This contradiction completes the proof of the lemma. ■

Now we can proceed to the main goal of this section, i.e. the characterizations of classes of graphs which fail to have some types of factors.

We call a vertex of a graph *simplicial* if the neighbours of the vertex induce a clique in a graph.

Let G and H be graphs and let C_G and C_H be either the set of vertices of a maximal clique or a one-element set the only element of which is a simplicial vertex in G and H, respectively. By a *gluing* of the graphs G and H we mean a graph obtained from the disjoint union of the graphs G and H by adding a new vertex x and the edges xy for every $y \in C_G \cup C_H$.

Let $\mathscr{A}$ be the minimal class of graphs closed under gluing and containing the trivial graph and every odd cycle different from T.

Clearly, all graphs in $\mathscr{A}$ are odd and connected. Although, in general, gluing of two claw-free graphs need not be claw-free, one can easily see that there is a system of cliques in each $G \in \mathscr{A}$ such that every edge of G lies in exactly one and every vertex of G lies in exactly two of the cliques. Hence every $G \in \mathscr{A}$ is a line graph and, consequently, is claw-free.

Theorem 6. *Let G be an odd connected claw-free graph. The graph G has a $T, \{E\}$-factor if and only if $G \notin \mathscr{A}$.*

Proof. Necessity follows immediately from the observation that neither an odd cycle different from T nor the trivial graph has a $T, \{E\}$-factor. Moreover, every gluing of two odd graphs without a $T, \{E\}$-factor yields an odd graph without a $T, \{E\}$-factor. Thus, there is no graph with a $T, \{E\}$-factor in $\mathscr{A}$.

To prove sufficiency suppose, on the contrary, that there is an odd connected claw-free graph $G \notin \mathscr{A}$ without a $T, \{E\}$-factor. Assume that G has the smallest possible number of vertices.

Notice that odd cycles and odd paths are the only odd connected claw-free graphs without a triangle. All these graphs belong to $\mathscr{A}$, so by Lemma 5 there is a cutvertex in G. Consider two cases.

Case 1. At least one of the cutvertices of G, say x, is odd.

Let G_1 and G_2 be the components of $G - x$. Clearly, both G_1 and G_2 are odd, connected and claw-free. Moreover, they have no $T, \{E\}$-factor since otherwise it could be easily extended to a $T, \{E\}$-factor of G. By the minimality of G, $G_1, G_2 \in \mathscr{A}$.

Denote by N the set of neighbours of x in G_1. Clearly, $\langle N \rangle$ is a clique for otherwise there would be a claw in G.

We claim that N is either the set of vertices of a maximal clique in G_1 or a one-element set the only element of which is a simplicial vertex in G_1.

If N consists of at least 3 vertices then according to Lemma 5 at least one of its elements, say y, is a cutvertex of G. If $\langle N \rangle$ were a subclique of some larger clique in G_1 then y would centre a claw. Thus, $\langle N \rangle$ is a maximal clique.

If N is a one-element set, say $N = \{z\}$, then z must be simplicial in G_1 for otherwise z would centre a claw.

It remains to consider the case when N consists of 2 vertices, say u and w. Suppose that $\langle N \rangle$ is a proper subgraph of some maximal clique K in G. Clearly u and w are not cutvertices since otherwise we should get a claw with the centre at u or w, respectively. Consequently, by Lemma 5, each vertex in $K \setminus \langle N \rangle$ is a cutvertex. If some vertex $v \notin V(K) \cup \{x\}$ is a neighbour of u then v must be adjacent to every vertex of $K - w$ to avoid claws. Finally, v and w must be adjacent because every vertex in $K \setminus \langle N \rangle$ is a cutvertex and G_1 is claw-free .Hence $\langle V(K) \cup \{v\} \rangle$ is a clique, which contradicts the maximality of K. Thus, all neighbours of u (and similarly all neighbours of w) in G_1 belong to $V(K)$. Let L be the block in G_1 containing K and suppose that $V(L) - V(K) \neq \emptyset$. Then some vertices $t \in V(L) - V(K)$ and $s \in V(K) - \{u, w\}$ are adjacent in G_1 which again yields a claw centred at s. Thus $L = K$. For every $v \in V(K) - \{u, w\}$ let C_v be the component of $G_1 - v$ disjoint from K. Since G_1 is odd and $V(G_1)$ is the disjoint union of the set $\{u, w\}$ and the sets $V(C_v) \cup \{v\}$ for all $v \in V(K) - \{u, w\}$, at least one of the graphs C_v, say C_p, must be even. The graph $\langle \{u, w, p\} \rangle$ is a triangle and all components of $G \setminus \langle \{u, w, p\} \rangle$ are even, so by Lemma 4, G has a $T, \{E\}$-factor which contradicts the definition of G. Thus, $\langle N \rangle$ is a maximal clique as claimed.

We show analogously that the set N' of neighbours of x in G_2 is either the set of vertices of a maximal clique or a one-element set the only element of which is a simplicial vertex in G_2.

By the definition of the class $\mathscr{A}$, $G \in \mathscr{A}$. We have come to a contradiction in the case 1.

Case 2. Every cutvertex in G is even.

Let x be a cutvertex in G such that one of the components of $G - x$, say G_1, is a block. The graph $H = \langle V(G_1) \cup \{x\} \rangle$ is odd, connected, claw-free and without a $T, \{E\}$-factor. By minimality of G, $H \in \mathscr{A}$. Moreover, H does not have a cutvertex because all cutvertices in G are even. By the definition of the class $\mathscr{A}$, the only graphs without cutvertices in $\mathscr{A}$ are the odd cycles different from T and the trivial graph. Thus, H is an odd cycle and $H \neq T$. Then, however, G_1 is not a block. This contradiction completes the proof. ∎

From this theorem we obtain the following characterization of odd connected claw-free graphs having a perfect 2-matching.

Let $\mathscr{B}$ be the minimal class of graphs closed under gluing and containing the trivial graph.

Obviously, $\mathscr{B} \subseteq \mathscr{A}$ and every $G \in \mathscr{B}$ is odd.

Theorem 7. *Let G be an odd connected claw-free graph. The following statements are equivalent*:

(i) *G has a perfect 2-matching*,

(ii) *G has a strong perfect 2-matching with at most one odd cycle*,

(iii) $G \notin \mathscr{B}$.

Proof. For every even graph G the theorem follows immediately from the Sumner Theorem, so suppose that G is odd.

(i) $\Rightarrow$ (ii) If G has a perfect 2-matching, then, by Lemma of [4], G has a perfect 2-matching with exactly one odd cycle. The rest of the proof follows from the observation that every odd cycle with a chord has a perfect 2-matching with exactly one smaller odd cycle.

(ii) $\Rightarrow$ (*iii*) Let, on the contrary, $G \in \mathscr{B}$. First observe that if two graphs F and H fail to have an induced cycle of length greater than 3 then so does their gluing. This proves that if $G \in \mathscr{B}$ then the only induced cycles in G are triangles. Hence, by (ii), G has a $T, \{E\}$-factor which, by Theorem 6, contradicts the fact that $\mathscr{B} \subseteq \mathscr{A}$.

(iii) $\Rightarrow$ (i) Let $G \notin \mathscr{B}$. If $G \notin \mathscr{A}$ then we are done by Theorem 6. Suppose that $G \in \mathscr{A} \setminus \mathscr{B}$ is an odd connected claw-free graph without a perfect 2-matching and with the smallest possible number of vertices. By the definition of $\mathscr{A}$ and $\mathscr{B}$, G is either an odd cycle or a gluing of some odd connected claw-free graphs F and H. In the former case G itself is a perfect 2-matching while in the latter one F has a perfect 2-matching, by the minimality of G. This 2-matching can be extended to a perfect 2-matching in G because $G \setminus F$ is even, connected and claw-free. ∎

We pass on to the graphs with a strong $P, \{E\}$-factor.

Let G and H be vertex-disjoint graphs. A graph obtained by identifying a vertex x of G and a vertex y of H is a *sticking* of G and H provided that the vertices x and y are not cutvertices of G and H, respectively.

Let $\mathscr{C}$ be the minimal class of graphs closed under sticking and containing all odd complete graphs. Clearly, all graphs in $\mathscr{C}$ are odd, connected and claw-free.

Theorem 8. *Let G be an odd connected claw-free graph. The graph G has a strong $P, \{E\}$-factor if and only if $G \notin \mathscr{C}$.*

We first prove the following simple auxiliary assertion.

Claim. *Let an odd connected claw-free graph G be a sticking of G_1 and G_2. Then G has no strong $P, \{E\}$-factor if and only if both G_1 and G_2 are odd and have no strong $P, \{E\}$-factors.*

Proof of Claim. Suppose that G has no strong $P, \{E\}$-factor. If both G_1 and G_2 are even then there is a perfect matching M_1 in G_1 and M_2 in G_2. The components of M_1 and M_2 which contain the common vertex x of G_1 and G_2 in G apparently determine an induced path in G which together with the remaining components of M_1 and M_2 forms a strong $P, \{E\}$-factor in G.

Hence both G_1 and G_2 are odd. If one of them (say G_1), has a strong $P, \{E\}$-factor R, then $G_2 - x$ is an even connected claw-free graph, so it has a perfect matching which, together with R, forms a strong $P, \{E\}$-factor of G.

Conversely, suppose that both G_1 and G_2 are odd and have no strong $P, \{E\}$-factor. If G has a strong $P, \{E\}$-factor R, then the only vertex of degree 2 in R coincides with the common vertex x of G_1 and G_2. One of the neighbours of x in R, say y, belongs to G_1 and the other one, say z, belongs to G_2. This is, however, a contradiction because the graphs $G_1 \setminus \langle\{x, y\}\rangle$ and $G_2 \setminus \langle\{x, z\}\rangle$ are odd so that they do not have a perfect matching.

Proof of Theorem 8. To prove necessity, we observe that no odd complete graph has a strong $P, \{E\}$-factor and hence, by the claim, neither does any graph from $\mathscr{C}$.

We prove sufficiency by induction on the number of vertices in G. The assertion is true for the trivial graph, so assume that $G \notin \mathscr{C}$ is a nontrivial odd connected claw-free graph without a strong $P, \{E\}$-factor.

Suppose G has a cutvertex x. Denote by G_1 and G_2 the components of $G \setminus x$ and let $G_1' = G \setminus G_2$ and $G_2' = G \setminus G_1$. Then G is a sticking of G_1' and G_2' and hence, by the claim, both G_1' and G_2' are odd and have no strong $P, \{E\}$-factor. By the induction hypothesis, both G_1' and G_2' belong to $\mathscr{C}$. Thus, by the definition of $\mathscr{C}$, we have $G \in \mathscr{C}$, a contradiction. Hence G is a block.

As $G \notin \mathscr{C}$, G is not a complete graph. We prove that G contains an induced subgraph H isomorphic to P such that $G \setminus H$ is connected.

If the connectivity of G is 2 then there is a vertex $v \in V(G)$ such that the graph $G - v$ has at least two different terminal blocks, say L_1 and L_2. Moreover, for some vertices $v_1 \in V(L_1)$ and $v_2 \in V(L_2)$ that are not cutvertices in $G - v$, there are edges vv_1 and vv_2 in G. The graph $H = \langle\{v_1, v, v_2\}\rangle$ is isomorphic to P and $G \setminus H$ is connected.

If the connectivity of G is 3 then let H' be any induced path $\langle\{y, v, z\}\rangle$ in G. If $G \setminus H'$ is connected then we are done. Otherwise, denote by F_1 and F_2 any two components of $G \setminus H'$. If the middle vertex v of the path H' is not adjacent to at least one vertex of F_1 and at least one vertex of F_2 then the set $\{y, z\}$ is a cutset in G, and consequently the connectivity of G is at most 2. Hence the degree of v in G is at least 4. The connectivity of $G - v$ is 2, so by the argument given above there is a path H in $G - v$ such that $(G - v) \setminus H$ is connected. Since the degree of v is at least 4 in G, $G \setminus H$ is connected, too.

Finally, if G is 4-connected then $G \setminus H$ is connected for any subgraph H of G isomorphic to P.

Since $G \setminus H$ has a perfect matching, G has a strong $P, \{E\}$-factor. ■

4. LINE GRAPHS

Favaron et al. [1] considered some edge-partition problems for graphs. A *decomposition* of a graph G into subgraphs $G_1, \ldots, G_n$ is a partition of the edge set of G into subsets $E_1, \ldots, E_n$ such that G_i is induced by E_i for $i = 1, \ldots, n$. To make the terminology of [1] consistent with ours let us define, for a graph H, and $H, \{P\}$-*decomposition* of a graph G to be a decomposition of G into such subgraphs that exactly one of them is isomorphic to H and the others are isomorphic to P.

In [1] the authors give full characterizations of the graphs with an $S, \{P\}$-decomposition (recall that S denotes the claw) and the graphs with a $P', \{P\}$-decomposition, where P' is the four-vertex path.

Notice that every decomposition of a graph G defines a factor in the line graph $L(G)$. For example, a $P', \{P\}$-decomposition in G gives a strong $P, \{E\}$-factor in $L(G)$. Conversely, a strong $P, \{E\}$-factor in a line graph $L(G)$ gives a $P', \{P\}$-decomposition in G. Clearly, this correspondence between decompositions and factors is not always one-to-one. For example, a $T, \{E\}$-factor in a line graph $L(G)$ corresponds to either a $T, \{P\}$-decomposition or an $S, \{P\}$-decomposition of G.

The results of Favaron et al. [1] can be reformulated in terms of line graphs as follows.

(i) A connected graph G with an odd number of edges has a $P', \{P\}$-decomposition (or, equivalently, an odd connected line graph $L(G)$ has a strong $P, \{E\}$-factor) if and only if $L(G) \notin \mathscr{C}$.

(ii) A connected graph G with an odd number of edges has an $S, \{P\}$-decomposition if and only if $L(G) \notin \mathscr{A}'$, where $\mathscr{A}'$ is the minimal superclass of $\mathscr{A}$ containing the triangle T and closed under gluing.

Since the class of line graphs is a proper subclass of the class of claw-free graphs, (i) is a special case of our Theorem 8. On the other hand, our Theorem 7 and the above remark give the following proposition.

Proposition 9. *A connected graph G with an odd number of edges has a $T, \{P\}$-decomposition or an $S, \{P\}$-decomposition if and only if $L(G) \notin \mathscr{A}$.* ■

The following problem is still open.

Problem. *Give a characterization of the graphs with a $T, \{P\}$-decomposition.*

Notice that both $\mathscr{A}$ and $\mathscr{C}$ consist of line graphs only. This observation together with our Theorems 7 and 8 give the last proposition.

Proposition 10. *If an odd connected claw-free graph is not a line graph of a graph then G has both a T, $\{E\}$-factor and a strong P, $\{E\}$-factor.* ■

References

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