Czechoslovak Mathematical Journal

Jan Pochciał

Sequential characterizations of metrizability

Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 2, 203–215

Persistent URL: `http://dml.cz/dmlcz/102453`

Czechoslovak Mathematical Journal, 41 (116) 1991, Praha

SEQUENTIAL CHARACTERIZATIONS OF METRIZABILITY

JAN POCHCIAŁ, Katowice

(Received June 15, 1988)

0. Many classical theorems concerning metrizability of topological spaces are well known (see e.g. [7], [12]). Different generalizations of metric spaces have been considered (see e.g. [8]). Presently, some general approaches to the problem of metrizability have been studied (cf. [9], [5]).

In this paper, a new approach to this subject is presented (cf. [18], [17]). This approach is based on the theory of sequential convergence understood as a subset of $X^N \times X$, where X is an arbitrary set. Apart from natural conditions F, U, S, H that are usually assumed, and an operation $G \to G^*$ which assigns to any convergence G the smallest convergence containing G and satisfying the Urysohn condition, a few conditions of diagonal type are introduced. Using these simple notions and conditions it is possible to give characterizations of convergences generated by real functions (Proposition 1) and functions having some additional properties like triangle condition (Proposition 2) and symmetry (Proposition 4). This leads to a characterization of metrizable convergences (Theorem 1) and topologies (Theorem 2) as well as to a characterization of metrizable topologies for paracompact spaces (Theorem 3). In this way characterizations of some generalized metric spaces like quasi-metrizable (Proposition 3), symmetrizable and semimetrizable (Proposition 5), and γ-spaces (Proposition 6) are also obtained.

Proofs of sufficiency of a few metrization theorems like those of Alexandroff-Urysohn, Nagata-Smirnov and Moore are shown as examples of applications.

The paper develops some ideas of [18]; Propositions 1,2 and Theorem 1 are generalizations of Theorems 1 – 3 presented there. The proofs presented in the paper are elementary; they are based only on sequential methods.

1. Let X be an arbitrary set and G a convergence on X, i.e. $G \subset X^N \times X$, where N is the set of all positive integers. If $\langle (x_n), x \rangle \in G$, then we say that the sequence (x_n) is convergent to x in G and write $x_n \to x(G)$ or simply $x_n \to x$. In the case of two or more indices, we write e.g. $x_{mn} \to^n x$ to emphasize which of them tends to infinity.

The following conditions are considered in literature (see e.g. [10] and [16]; in [16] these conditions appear as (L_i), where $i = 0, 1, 2, 3$).

F. If $x_n \to x$, then $x_{p_n} \to x$ for each subsequence (x_{p_n}) of (x_n).

U. If each subsequence of (x_n) contains a subsequence (x_{q_n}) such that $x_{q_n} \to x$, then $x_n \to x$.

S. If $x_n = x$ for $n = 1, 2, \dots$, then $x_n \to x$.

H. If $x_n \to x$ and $x_n \to y$, then $x = y$.

Given a convergence G, we define a convergence G^* in the following way:

$x_n \to x(G^*)$ if each subsequence of (x_n) contains a subsequence (x_{q_n}) such that $x_{q_n} \to x(G)$.

This operation is close to the notion of a base of convergence defined by M. Dolcher in [6] (see also [14]). Namely, if $G_0^* = G$ and B_x is the family of all sequences convergent to x in G_0, then B_x is a base of G at x. Conversely, if B_x is a base of G at x and G_0 is defined as follows: $x_n \to x(G_0)$ if (x_n) is a subsequence of some sequence belonging to B_x, then $G_0^* = G$.

Remark 1. If a convergence G satisfies condition F, then G^* is the smallest convergence containing G and satisfying the Urysohn condition U. Moreover, G satisfies conditions S and H iff G^* does.

We shall say that a convergence G is generated by a real function $f: X \times X \to R$ if

$$x_n \to x(G) \quad \text{iff} \quad f(x, x_n) \to 0\,,$$

where the convergence on the right is the usual convergence of a sequence of real numbers.

Remark 2. For any function f the convergence G_f generated by f satisfies conditions F and U. Moreover, $f(x, x) = 0$ iff G_f satisfies condition S. If G_f satisfies conditions S and H, then $f(x, y) = 0$ iff $x = y$.

In the sequel, we shall assume that all convergences satisfy condition S and all functions generating convergences are non-negative.

We shall follow [7] and [8] in using topological notions and notation.

2. The following diagonal condition has been introduced in [18]:

D_1. If $x_{mn} \to^n x$ for $m = 1, 2, \dots$, then $x_{nn} \to x$.

Additionally, consider the following weaker condition

D_1'. If $x_{mn} \to^n x$ for $m = 1, 2, \dots$, then there is an increasing sequence (p_n) of positive integers such that $x_{p_n, p_n} \to x$.

Suppose that X is a first-countable topological space and $\{U_n(x): U_{n+1}(x) \subset U_n(x)$ for $n = 1, 2, \dots\}$ is a base of neighborhoods at $x \in X$. Define the convergence G_0:

$$x_n \to x(G_0) \quad \text{if} \quad x_n \in U_n(x) \quad \text{for} \quad n = 1, 2, \dots .$$

Note that G_0 fulfils conditions F, D_1 (and so D_1'), and G_0^* is the convergence generated by the topology of X.

On the other hand, one can easily construct an example of a convergence G satisfying conditions F, U, S, H, and thus generated by a topology (cf. [13], see also [10]), such that an arbitrary topology that generates G is not first-countable, although $G = G_0^*$ for some convergence G_0 satisfying conditions F and D_1.

The following simple example explains the difference between conditions D_1' and D_1.

Let $X = N \cup \{x_0\}$ and let $\mathscr{B}$ be a family of (increasing) subsequences of natural numbers with the following properties: two elements of $\mathscr{B}$ have no common subsequence, each sequence of natural numbers has a common subsequence with an element of $\mathscr{B}$. (In the most essential case, when $\mathscr{B}$ is infinite, the existence of such a family follows from Kuratowski-Zorn Lemma.)

Define a convergence G_0 on X:

$$x_n \to x(G_0) \quad \text{if} \quad x_n = x \quad \text{for each} \quad n = 1, 2, \ldots \quad \text{and} \quad x \neq x_0 ,$$

$$x_n \to x_0(G_0) \quad \text{if} \quad x_n = x_0 \quad \text{for each} \quad n = 1, 2, \ldots \quad \text{or} \quad (x_n)$$

is a subsequence of an element of $\mathscr{B}$.

One can easily check that the convergence G_0 satisfies D_1' but does not satisfy D_1.

Condition D_1 for a convergence G_0 is equivalent to the following one: there is a family of subsets $\{V_n(x);\ n = 1, 2, \ldots\ x \in X\}$ such that (for each $n \in N$ and $x \in X$) $x \in V_n(x)$, $V_{n+1}(x) \subset V_n(x)$ and

$$x_n \to x(G_0) \quad \text{iff} \quad x_n \in V_n(x) \quad \text{for each} \quad n \in N .$$

This implies some relations to the ideas presented in [9] and [5], where certain families of sets have been considered. In particular, it is easy to show that in the case of a topological space X a family $\{V_n(x);\ n = 1, 2, \ldots\ x \in X\}$ inducing the convergence in X is a network but, in general, it need not be a neighbornet (cf. [8], [9]).

Proposition 1. *A convergence G is generated by a function iff there is a convergence G_0 satisfying conditions* F *and* D_1' *such that* $G_0^* = G$.

Proof. Let (c_n) be a sequence of real numbers such that

$$(1) \qquad c_{n+1} \leqq c_n \leqq 1 \quad \text{for} \quad n = 1, 2, \ldots \quad \text{and} \quad c_n \to 0 .$$

If G is generated by a function f, then the convergence G_0 defined by

$$(2) \qquad x_n \to x(G_0) \quad \text{if} \quad f(x, x_n) \leqq c_n \quad \text{for} \quad n = 1, 2, \ldots$$

satisfies conditions F and D_1, and $G_0^* = G$ for any sequence (c_n) satisfying condition (1).

Now, assume that G_0 satisfies conditions F and D_1', and $G_0^* = G$. Let $\{(x_n^\lambda): \lambda \in \Lambda\}$ be the family of all sequences convergent to x in G_0. Let (c_n) satisfy condition (1).

Define a function $f: X \times X \to R$ in the following way:

$$(3) \qquad f(x, y) = \begin{cases} \inf\{c_n: n = 1, 2, \ldots\} & \text{if } y = x_n^\lambda \text{ for some } \lambda \in \Lambda \text{ and } n \in N , \\ 1 & \text{otherwise} . \end{cases}$$

Let $x_n \to x(G_f)$ iff $f(x, x_n) \to 0$.

It is easy to check that $G_f = G_0^*$ (cf. [18], Theorem 1).

A function f is said to satisfy the *triangle inequality* if

$$(\Delta) \qquad f(x, y) \leqq f(x, z) + f(z, y).$$

If, moreover, $f(x, y) = 0$ if $x = y$, then the function f is called *quasi-metric* (see e.g. [8] p. 488).

We introduce the following diagonal condition:

D_2. There is an increasing sequence (p_n) of positive integers such that if $x_{mn} \to^n x_m$ for $m = 1, 2, \ldots$ and $x_m \to x$, then $x_{p_n, p_n} \to x$.

Proposition 2. *A convergence G is generated by a function satisfying* (Δ) *iff there is a convergence G_0 satisfying conditions* F *and* D_2 *such that $G_0^* = G$. The convergence G is quasi-metrizable* (generated by a quasi-metric) *iff G_0 additionally satisfies condition* H.

Proof. If a convergence G is generated by a function f satisfying condition (Δ), then putting $c_n = 1/2^n$ for $n = 1, 2, \ldots$ in formula (2) we get a convergence G_0 satisfying condition D_2 with $p_n = n + 1$ $(n = 1, 2, \ldots)$. In fact $f(x_{n+1}, x_{n+1,n+1}) \leqq \leqq 1/2^{n+1}$ and $f(x, x_{n+1}) \leqq 1/2^{n+1}$ imply, by (Δ), $f(x, x_{n+1,n+1}) \leqq 1/2^n$, which, by formula (2), is equivalent to $x_{n+1,n+1} \to x(G_0)$.

Now, suppose that a convergence G_0 satisfies conditions F and D_2, and let (p_n) be a sequence of positive integers that appears in D_2. Then G_0 satisfies conditions F and D_1', whence the convergence $G = G_0^*$ is generated by a function, in view of Proposition 1.

Define the following sequences:

$$(4) \qquad q_{n,1} = p_n \qquad (n = 1, 2, \ldots),$$
$$q_{n,i} = p_{q_{n,i-1}} \quad (i = 2, 3, \ldots, \; n = 1, 2, \ldots)$$

and

$$(5) \qquad q_n = q_{2,n} \quad (n = 1, 2, \ldots).$$

Obviously, (q_n) is an increasing sequence of positive integers.

Moreover, put

$$(6) \qquad c_n = \begin{cases} 1 & \text{if} \quad n < q_1 \\ 1/2^k & \text{if} \quad q_k \leqq n < q_{k+1} \end{cases} \quad (n = 1, 2, \ldots).$$

Assume that the function f that generates G is given by formula (3) with the sequence (c_n) given by formula (6).

We shall show that

$$(7) \qquad f(x, y) \leqq \sqrt{2}) \max [f(x, z), f(z, y)] \quad \text{for every} \quad x, y, z \in X.$$

Following [18] (Theorem 2), we shall consider three cases.

First case: $\max [f(x, z), f(z, y)] = 1$.
Inequality (7) is evident.

Second case: $\max[f(x,z), f(z,y)] = 0$.
By (3), $z_n \to x(G_0)$ and $y_n \to z(G_0)$ for $z_n = z$ and $y_n = y$ $(n = 1, 2, \ldots)$. By condition D_2, $y_n \to x(G_0)$, so $f(x, y) = 0$.

Third case: $\max[f(x,z), f(z,y)] = 1/\sqrt{2^k}$ for some $k \in N$.
By conditions F, S and formula (3) it follows that there are a matrix (x_{nm}) and a sequence (x_m) such that

$$x_m \to x(G_0), \quad x_{q_k} = z$$

and

$$x_{mn} \to^n x_m(G_0) \ (m = 1, 2, \ldots), \quad x_{q_k, q_k} = y .$$

Let $y_n = x_{p_n, p_n}$ for $n = 1, 2, \ldots$. By condition D_2, $y_n \to x(G_0)$. Moreover, by (4) and (5),

$$y = x_{q_k, q_k} = x_{q_{2,k}, q_{2,k}} = x_{pq_{2,k-1}, pq_{2,k-1}} = x_{pq_{k-1}, pq_{k-1}} = y_{q_{k-1}} .$$

Therefore, by (3) and (6), we have $f(x, y) \leqq 1/2^{k-1}$ which completes the proof of (7).

From (7) it follows that

$$(8) \qquad f(x, y) \leqq 2 \max[f(x,t), f(t,z), f(z,y)] \quad \text{for every} \quad x, t, z, y \in X .$$

Inequality (8) implies by induction (see e.g. similar proofs in [4], [2] p. 300, [7] p. 527) that for each positive integer k and $t_i \in X$ $(i = 0, 1, \ldots, k)$ such that $t_0 = x$ and $t_k = y$ we have

$$(9) \qquad f(x, y) \leqq 2(f(t_0, t_1) + f(t_1, t_2) + \ldots + f(t_{k-1}, t_k)) .$$

Define

$$g(x, y) = \inf\{f(t_0, t_1) + f(t_1, t_2) + \ldots + f(t_{k-1}, t_k) : t_0, t_1, \ldots, t_k \in X ,\ t_0 = x,\ t_k = y,\ k \in N\} .$$

Evidently, the function g fulfils condition (Δ). Moreover, by inequality (9), we have

$$\tfrac{1}{2} f(x, y) \leqq g(x, y) \leqq f(x, y) .$$

This proves that the functions f and g generate the same convergence. This, by Remark 2, completes the proof.

Since a quasi-metric generating a convergence G induces the finest of all topologies that generate G, hence in the case of topological spaces we obtain:

Proposition 3. *A topological space X is quasi-metrizable iff it is sequential and there is a convergence G_0 satisfying conditions* F, H *and* D_2 *such that G_0^* is the convergence generated by the topology of X.*

Remark 3. In [8] (p. 489) the following condition equivalent to quasi-metrizability of a topological space (X, τ) is considered:

There is a function $g: N \times X \to \tau$ such that

(i) $\{g(n, x);\ n \in N\}$ is a base at x,

(ii) $y \in g(n + 1, x) \Rightarrow g(n + 1, y) \subset g(n, x)$.

Putting $x_n \to x(G_0)$ iff $x_n \in g(n, x)$ for all $n \in N$, we get conditions D_1 and D_2 (with $p_n = n + 1$ for $n = 1, 2, \ldots$).

A symmetric function g, i.e. such that $g(x, y) = g(y, x)$ is called *a symmetric* if $g(x, y) = 0$ implies $x = y$ (see e.g. [8] p. 480).

There are many ways of describing convergences that are generated by symmetric functions (cf. [18]).

In this paper, we shall use the following condition:

(∗) If $x_n \to x$, then there is an increasing sequence (p_n) of positive integers and a matrix (x_{mn}) $(m, n = 1, 2, \ldots)$ such that $x_{mn} \to^n x_{p_n}$ and $x_{nn} = x$ for each positive integer n.

Proposition 4. *A convergence G is generated by a symmetric function iff there is a convergence G_0 satisfying conditions* F, D_1' *and* (∗) *such that $G_0^* = G$. Moreover, if G satisfies condition* H, *then it is generated by a symmetric.*

Proof. If a convergence G is generated by a symmetric function f, then evidently the convergence G_0 given by formula (2) with (c_n) given by (1) satisfies conditions F, D_1, (∗) and $G_0^* = G$.

Conversely, suppose that a convergence G_0 satisfies conditions F, D_1', (∗)and $G = G_0^*$. Let f be given by formula (3) and

$$g(x, y) = \max\,[f(x, y), f(y, x)]\,. \tag{10}$$

Denote by G_g the convergence generated by the function g. We shall show that $G_g = G$. Evidently, $G_g \subset G$. On the other hand, if $x_n \to x(G)$, then $x_{q_n} \to x(G_0)$ for an increasing sequence (q_n) of positive integers. By (∗) and (3) we have $f(x_{qp_n}, x) \to 0$, so, by (10), $g(x, x_{qp_n}) \to 0$ or, equivalently, $x_{qp_n} \to x(G_g)$. Since G_g satisfies condition U we get $G_g = G$ as desired. It is easy to see that under condition H the convergence G is generated by a symmetric.

Recall (see e.g. [8] p. 480) that a topological space X is symmetrizable if there is a symmetric g on X satisfying the following condition: $U \subset X$ is open iff for each $x \in U$ there exists $\varepsilon > 0$ which $B(x, \varepsilon) \subset U$, where $B(x, \varepsilon) = \{y \in X;\ g(x, y) < \varepsilon\}$. If $\{B(x, \varepsilon);\ \varepsilon > 0\}$ forms a neighborhood base at x, then X is called semi-metrizable.

Proposition 5. *A topological space X is symmetrizable (semi-metrizable) iff it is sequential (Fréchet) and the convergence in X is generated by a symmetric.*

Proof. Since semi-metrizable spaces are just symmetrizable and Fréchet (cf. [8] Theorem 9.6), it remains to prove only the first part.

Sufficiency. It follows directly from Lemma 9.3 in [8].

Necessity. Let g be a symmetric for G, i.e. $x_n \to x(G)$ iff $g(x, x_n) \to 0$. Let U be open in a sequential topology generating G and let $x \in U$. Then $x_n \to x(G)$ implies $x_n \in U$ for almost all n. We shall show that there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. Indeed, in the opposite case for each $\varepsilon_n > 0$ there is $x_n \notin U$ such that $g(x, x_n) < \varepsilon_n$, i.e. $x_n \to x(G)$ (if $\varepsilon_n \to 0$), a contradiction.

If U is not open then there is $x \in U$ and a sequence (x_n) such that $x_n \to x$ $(g(x, x_n) \to 0)$ and $x_n \notin U$. It follows that $B(x, \varepsilon) \not\subset U$ for each $\varepsilon > 0$. This completes the proof.

If we combine Propositions 2 and 4, we easily obtain

Theorem 1. *A convergence G is metrizable (i.e. generated by a metric) iff there is a convergence G_0 satisfying conditions* F, H, D_2 *and* $(*)$ *such that* $G_0^* = G$.

Using the same arguments as in the case of Proposition 3, we have

Theorem 2. *A topological space X is metrizable iff it is sequential and there is a convergence G_0 satisfying conditions* F, H, D_2 *and* $(*)$ *such that G_0^* is the convergence generated by the topology of X.*

Corollary 1. (The Alexandroff-Urysohn metrization theorem, cf. [1], see also [7] p. 413.) *A topological space X is metrizable iff it is a T_0-space and has a development $\mathscr{W}_1, \mathscr{W}_2, \ldots$ such that*

(11) *for every positive integer n and any two sets $W_1, W_2 \in \mathscr{W}_{n+1}$ with non-empty intersection there exists a set $W \in \mathscr{W}_n$ such that $W_1 \cup W_2 \subset W$.*

Proof of sufficiency. Notice that X is first-countable, so it is sequential. Define the convergence G_0:

$$x_n \to x(G_0) \quad \text{if} \quad x_n \in \mathrm{St}(x, \mathscr{W}_n) \quad \text{for each positive integer } n\,.$$

Evidently, G_0^* is the convergence generated by the topology of X. By (11), $\mathrm{St}(x, \mathscr{W}_{i+1}) \subset \mathrm{St}(x, \mathscr{W}_i)$ for each $i \in N$, hence G_0 satisfies condition F. Condition $(*)$ follows from the fact that $x \in \mathrm{St}(y, \mathscr{W}_n)$ iff $y \in \mathrm{St}(x, \mathscr{W}_n)$.

Assume that $x_n \to x(G_0)$ and $x_n \to y(G_0)$, i.e. $x_n \in \mathrm{St}(x, \mathscr{W}_n)$ and $x_n \in \mathrm{St}(y, \mathscr{W}_n)$ for $n = 1, 2, \ldots$ or, equivalently, $x \in \mathrm{St}(x_n, \mathscr{W}_n)$ and $y \in \mathrm{St}(x_n, \mathscr{W}_n)$ $(n = 1, 2, \ldots)$. Therefore, by (11), $x \in \mathrm{St}(y, \mathscr{W}_{n-1})$ and $y \in \mathrm{St}(x, \mathscr{W}_{n-1})$ for $n = 2, 3, \ldots$ which, by T_0, implies $x = y$.

To prove condition D_2 assume that $x_{mn} \to^n x(G_0)$ $(m = 1, 2, \ldots)$ and $x_m \to x(G_0)$. In particular, $x_{kk} \in \mathrm{St}(x_k, \mathscr{W}_k)$ and $x_k \in \mathrm{St}(x, \mathscr{W}_k)$ for each $k \in N$. By (11) we have $x_{kk} \in \mathrm{St}(x, \mathscr{W}_{k-1})$ for $k = 2, 3, \ldots$. Therefore, putting $p_n = n + 1$ for $n = 1, 2, \ldots$, we have $x_{p_n, p_n} \to x$, which proves D_2. It remains to apply Theorem 2.

3. Let us introduce the following two conditions:

D_2'. If $x_{mn} \to^n x_m$ for $m = 1, 2, \ldots$ and $x_m \to x$, then $x_{p_n, p_n} \to x$ for an increasing sequence (p_n) of positive integers.

$(**)$ If $x_n \to x$ and $y_n \to x$, then there is an increasing sequence (p_n) of positive integers and a matrix (x_{mn}) $(m, n = 1, 2, \ldots)$ such that $x_{mn} \to^m x_{p_n}$ and $x_{nn} = y_{p_n}$ for each positive integer n.

Remark 4. Evidently, D_2 implies D_2' and $(**)$ implies $(*)$. Moreover, if f is a metric, then the convergence G_0 defined by formula (2) (with (c_n) given by (1)) fulfils condition $(**)$.

Lemma 1. *If a convergence G_0 satisfies conditions* F *and* D'_1, *then there is a convergence G_1 satisfying conditions* F *and* D_1 *such that $G_0^* = G_1^*$.*

Moreover, G_1 can be chosen in such a way that

(i) *if G_0 satisfies condition D'_2, then G_1 does;*

(ii) *if G_0 satisfies condition $(**)$, then G_1 does.*

Proof. Let a convergence G_0 satisfy conditions F and D'_1 and let a function f be given by formula (3) with a sequence (c_n) satisfying (1).

Define a convergence G_1 as follows:

$$x_n \to x(G_1) \quad \text{if} \quad f(x, x_n) \leqq c_n \quad \text{for} \quad n = 1, 2, \ldots .$$

Then $G_0 \subset G_1$ and G_1 satisfies condition D_1.

Moreover, we have

(12) if $x_n \to x(G_1)$, then there is a matrix (x_{mn}) $(m, n = 1, 2, \ldots)$ such that $x_{mn} \to^m x(G_0)$ and $x_{nn} = x_n$ for each positive integer n.

By (12) and due to condition D'_1 for G_0, it follows that if $x_n \to x(G_1)$, then there is an increasing sequence (p_n) of positive integers such that $x_{p_n} \to x(G_0)$. Therefore $G_0^* = G_1^*$.

Now, suppose that $x_{mn} \to^n x_m(G_1)$ for $m = 1, 2, \ldots$ and $x_m \to x(G_1)$. Since $x_{p_m} \to x(G_0)$ for an increasing sequence (p_m) of positive integers, there exists a matrix (y_{mn}) such that $y_{mn} \to^n x_{p_m}(G_0)$ and $y_{nn} = x_{p_n,p_n}$ $(n = 1, 2, \ldots)$, in view of (12) and condition F for G_0. Therefore, if G_0 satisfies condition D'_2 then there is an increasing sequence (q_n) such that $x_{qp_n,qp_n} \to x(G_0)$. Consequently, G_1 satisfies condition D'_2 because $G_0 \subset G_1$.

Similarly we see that if G_0 satisfies $(**)$, then G_1 does.

Lemma 2. *If a convergence G_0 satisfies conditions* F, D_1 *and* D'_2, *then*

(13) *for every x there is an increasing sequence (p_n) of positive integers such that if $x_{mn} \to^n x_m(G_0)$ $(m = 1, 2, \ldots)$ and $x_m \to x(G_0)$, then $x_{p_n,p_n} \to x(G_0)$.*

Proof. Let (x_{mn}) be a matrix such that $x_{mn} \to^n x_m(G_0)$ for $m = 1, 2, \ldots$ and $x_m \to x(G_0)$. Notice that $x_{nn} \to x(G_0)$ by F and D'_2. By F, D_1 and D'_2 it follows that for each $k \in N$ there is an index i_k such that for each $i \geqq i_k$ there is a sequence (y_{ni}) such that $y_{ni} \to^n x(G_0)$ and $y_{ki} = x_{ii}$.

Indeed, in the opposite case there would be an increasing sequence (r_n) of positive integers such that no subsequence of (x_{r_n,r_n}) would be convergent to x in G_0.

Therefore there is an increasing sequence (q_n) of positive integers (chosen for the matrix (x_{mn})) such that

(14) $$x_{q_n',q_n'} \to x(G_0) \quad \text{iff} \quad q_n' \geqq q_n \quad \text{for each} \quad n = 1, 2, \ldots ,$$

where (q_n') is an arbitrary increasing sequence of positive integers.

Now, suppose that (13) does not hold, i.e. there is a sequence of matrices (x_{mn}^k) $(k = 1, 2, \dots)$ such that $x_{mn}^k \to^n x_m^k(G_0)$ $(m, k = 1,2, \dots)$, $x_m^k \to^m x(G_0)$ $(k = 1, 2, \dots)$ and $q_{n_0}^k \to^k \infty$ for some index n_0, where (q_n^k) are the sequences having property (14) and chosen for the matrices (x_{mn}^k), respectively $(k = 1, 2, \dots)$.

We may assume that

$$(15) \qquad q_{n_0}^k = k + 1 \quad \text{for} \quad k > n_0 \quad \text{and} \quad q_{n_0-1}^k < M < \infty \quad \text{if} \quad n_0 \neq 1 .$$

Consider the matrix (x_{kn}^k). Since $x_{kn}^k \to^n x_k^k(G_0)$ $(k = 1, 2, \dots)$ and, by D_1, $x_k^k \to x(G_0)$, conditions D_2' and F imply the existence of an increasing sequence (p_n) of positive integers such that

$$x_{p_n,p_n}^{p_n} \to x \quad \text{and} \quad p_{n_0} = l > M .$$

Put

$$z_n = \begin{cases} x_{q_n^l, q_n^l}^l & \text{for} \quad n \neq n_0 , \\ x_{ll}^l & \text{for} \quad n = n_0 . \end{cases}$$

By D_1 we have $z_n \to x(G_0)$. By (15), $q_{n_0}^l - 1 = l > q_{n_0-1}^l$. We have got a contradiction to the definition of the sequence (q_n^l).

Lemmas 1 and 2 make a characterization of γ-spaces in terms of condition D_2' possible. Recall (see e.g. [8] p. 491) that a topological space (X, τ) is said to be a γ-space if there exists a function $g: N \times X \to \tau$ such that

(i) $\{g(n, x); n \in N\}$ is a base at x,

(ii) for each $n \in N$ and $x \in X$ there exists $m \in N$ such that $y \in g(m, x)$ implies $g(m, y) \subset g(n, x)$.

Proposition 6. *A topological space (X, τ) is a γ-space iff it is sequential and there is a convergence G_0 satisfying conditions* F *and* D_2' *such that G_0^* is the convergence generated by the topology τ.*

Proof. Assuming, in a γ-space, $x_n \to x(G_0)$ iff $x_n \in g(n, x)$ for each $n \in N$ we get a convergence with the required properties.

Conversely, suppose that a convergence G_0 satisfies conditions F and D_2'. By Lemma 1 we may assume that G_0 satisfies also condition D_1. For each $x \in X$ and $k = 1, 2, \dots$ define $V_k(x) = \{y \in X;$ there is (x_n), $x_n \to x(G_0)$, $y = x_k\}$. Then $x_n \to x(G_0)$ iff $x_n \in V_n(x)$ for all $n \in N$. By Lemma 2 it follows that for each $n \in N$ there is $m \in N$ such that $y \in V_m(x)$ implies $V_m(y) \subset V_n(x)$. Then in the topology introduced by the family of sets $\{V_n(x)\}$ (see e.g. [11] p. 19) the sets $V_n(x)$ form a base of neighborhoods at x and for each open set U and $x \in U$ there is $k \in N$ such that $y \in V_k(x)$ implies $V_k(y) \subset U$. By the result 4.3 in [9] this completes the proof.

Lemma 3. *Let a convergence G_0 satisfy conditions* F, D_1, (13) *and* (**). *If $x_k \to x_0(G_0)$, then there is an increasing sequence (p_n) of positive integers such that $x_{p_n,p_n} \to x_k(G_0)$ whenever $x_{mn} \to^n x_m(G_0)$ $(m = 1, 2, \dots)$ and $x_m \to x_k(G_0)$ for some $k = 0, 1, \dots$.*

Proof. Suppose that a convergence G_0 satisfies conditions F, D_1 and (13) with a sequence (p_n) at a point x.

Given a matrix (x_{mn}) and a sequence (x_m) such that $x_{mn} \to^n x_m(G_0)$ for $m = 1, 2, \ldots$ and $x_m \to x(G_0)$, we have $x_{q_n, q_n} \to x(G_0)$ for an arbitrary increasing sequence (q_n) such that $q_n \geqq p_n$ for all $n = 1, 2, \ldots$. Moreover, given a point $x \in X$, there are an increasing sequence (p_n) of positive integers, a sequence (y_m) and a matrix (y_{mn}) with the following properties:

1° $y_{mn} \to^n y_m(G_0)$ for $m = 1, 2, \ldots$;

2° $y_m \to x(G_0)$;

3° $y_{q_n, q_n} \to x(G_0)$ iff $q_n \geqq p_n$ for each $n = 1, 2, \ldots$.

For arbitrary $k = 0, 1, \ldots$ denote by (p_n^k), (y_m^k) and (y_{mn}^k) sequences and a matrix, respectively, satisfying 1°–3° for the point x_k.

Now, suppose that the assertion of the lemma does not hold. Then there is an index n_0 such that $\sup \{p_{n_0}^k : k \in N\} = +\infty$.

We may assume that

$$(16) \qquad p_{n_0}^k > k \quad \text{and} \quad \sup \{p_{n_0 - 1}^k : k \in N\} < M < \infty .$$

Then, putting

$$z_n^k = \begin{cases} x_k & \text{if} \quad n \neq n_0 \\ y_{kk}^k & \text{if} \quad n = n_0 \end{cases}$$

we get

$$(17) \qquad z_n^k \not\to^n x_k(G_0) \quad \text{for} \quad k > M ,$$

in view of (16) and D_1.

Since $y_{kn}^k \to^n y_k^k(G_0)$ for $k = 1, 2, \ldots$, $y_n^k \to^n x_k(G_0)$ for $k = 1, 2, \ldots$ and $x_k \to x_0(G_0)$, we have $y_{q_n, q_n}^{q_n} \to x_0(G_0)$, where $q_n = p_{p_n^0}^0$ $(n = 1, 2, \ldots)$, by applying condition (13) twice. Since $x_{q_n} \to x_0(G_0)$, condition (17) yields a contradiction to (**). This completes the proof.

By Lemmas 1, 2, 3, Remarks 1, 4 and Theorem 1 we directly obtain the following propositions:

Proposition 7. *Suppose that a convergence G is compact, i.e. every sequence contains a convergent subsequence in G. Then G is metrizable iff there is a convergence G_0 satisfying conditions* F, D_2', H *and* (**) *such that $G_0^* = G$.*

Proposition 8. *If a convergence G_0 satisfies conditions* F, D_2', H *and* (**), *then the convergence G_0^* is locally metrizable, i.e. for every x there is a sequential neighborhood $U(x)$ of x and a metric that generates the convergence G_0 on $U(x)$.*

The converse is not true. It is easy to give an example of locally metrizable space with a convergence G such that there is no convergence G_0 satisfying condition D_2' for which $G_0^* = G$.

Proposition 9. *If a topological space X is sequential and there is a convergence G_0*

satisfying conditions F, D_2', H *and* (**) *such that* G_0^* *is the convergence generated by the topology of* X, *then* X *is locally metrizable.*

Since any paracompact locally metrizable space is metrizable (cf. [19]), we obtain the following characterization:

Theorem 3. *Let* X *be a paracompact space. Then* X *is metrizable iff it is sequential and there is a convergence* G_0 *satisfying conditions* F, D_2', H *and* (**) *such that* G_0^* *is the convergence generated by the topology of* X.

Corollary 2. (The Nagata-Smirnov metrization theorem, cf. [15], [19], see also [7] p. 351.) *A topological space is metrizable iff it is regular and has a* σ*-locally finite base.*

Proof of sufficiency. Let $\mathscr{U}_1, \mathscr{U}_2, \ldots$ be a sequence of locally finite, open covers forming a base and such that $\mathscr{U}_i \subset \mathscr{U}_{i+1}$ for $i = 1, 2, \ldots$.

It is easy to see that the space X is paracompact (see e.g. [7] p. 376, Th. 5.1.11). Moreover, X is first-countable; the sequence $U_n = \{\bigcap U : x \in U, U \in \mathscr{U}_n\}$ $(n = 1, 2, \ldots)$ is a base of open neighborhoods at x.

Let $\mathscr{V}_i = \{V_{i\lambda} : \lambda \in \Lambda_i\}$ for $i = 1, 2, \ldots$ be the sequence of closed covers formed by the closures of sets from $\mathscr{U}_i$, respectively. Evidently, $\mathscr{V}_i$ $(i = 1, 2, \ldots)$ are locally finite.

Define

$$\mathscr{Z}_i = \{\bigcap_{\lambda \in \Lambda_i} Z_{i\lambda} : Z_{i\lambda} = V_{i\lambda} \text{ or } Z_{i\lambda} = \mathrm{cl}(X - V_{i\lambda}), \text{ where } V_{i\lambda} \in \mathscr{V}_i\} \quad \text{for} \quad i = 1, 2, \ldots.$$

It is easy to check that $\mathscr{Z}_i$ is a locally finite, closed refinement of $\mathscr{V}_i$ for each $i = 1, 2, \ldots$.

Now, we introduce a family of convergences G_k $(k = 1, 2, \ldots)$:

$x_n \to x(G_k)$ if $x_n \in \mathrm{St}_k(x, \mathscr{Z}_n)$ for each $n = 1, 2, \ldots$, where $\mathrm{St}_1(x, \mathscr{Z}_n) = \mathrm{St}(x, \mathscr{Z}_n)$ and

$$\mathrm{St}_k(x, \mathscr{Z}_n) = \mathrm{St}(\mathrm{St}_{k-1}(x, \mathscr{Z}_n), \mathscr{Z}_n) \quad \text{for} \quad k = 2, 3, \ldots.$$

Evidently, the convergences G_k satisfy condition F. We shall show that G_k^* is the convergence generated by the topology of X for each $k = 1, 2, \ldots$. This and the regularity of X imply, in particular, that G_k satisfy condition H.

Notice that if $U \in \mathscr{U}_n$ and $\overline{V} \subset U$, then $\mathrm{St}(\overline{V}, \mathscr{Z}_n) \subset \overline{U}$. Moreover, since $\mathscr{Z}_n$ is locally finite, we have $x \in \mathrm{int}\, \mathrm{St}(x, \mathscr{Z}_n)$. Hence, by the regularity of X, G_k^* is the convergence generated by the topology of X.

To show that G_k fulfils condition D_2' $(k = 1, 2, \ldots)$ notice that if $y \in \mathrm{St}_k(x, \mathscr{Z}_n)$ and $z \in \mathrm{St}_k(y, \mathscr{Z}n)$, then $z \in \mathrm{St}_{2k}(x, \mathscr{Z}_n)$. In other words, if $x_{mn} \to^n x_m(G_k)$ and $x_m \to x(G_k)$, then $x_{nn} \to x(G_{2k})$. This, by $G_k^* = G_{2k}^*$, implies the existence of an increasing sequence (p_n) of positive integers such that $x_{p_n, p_n} \to x(G_k)$.

In a similar way we can show that G_k fulfils condition (**).

Hence each of the convergences G_k satisfies all conditions that are required in Theorem 3. This completes the proof.

In particular, we have got

Corollary 3. (The Urysohn theorem, cf. [21], [22], see also [7] p. 325 Thms. 4.2.8, 4.2.9.) *A regular (or Hausdorff and compact) second-countable space is metrizable.*

Finally, we shall show an application of Theorem 3 to a proof of the Moore theorem.

Corollary 4. (The Moore metrization theorem, cf. [3], [20], see also [7] p. 409.) *A topological space is metrizable iff it is a T_0-space and has a strong development.*

Proof of sufficiency. Let $\mathscr{U}_1, \mathscr{U}_2, \dots$ be a strong development such that $\mathscr{U}_{i+1}$ is a refinement of $\mathscr{U}_i$ for each $i = 1, 2, \dots$.

Define $x_n \to x(G_0)$ if $x_n \in \mathrm{St}(x, \mathscr{U}_n)$ for each $n = 1, 2, \dots$.

Evidently, G_0 satisfies condition F and G_0^* is the convergence generated by the topology of X. In a similar way as in Corollary 1 we can show that G_0 satisfies condition H. Notice that for every x and each positive integer n there is a number p_n such that $\mathrm{St}(\mathrm{St}(x, \mathscr{U}_{p_n}), \mathscr{U}_{p_n}) \subset \mathrm{St}(x. \mathscr{U}_n)$. This directly implies condition D_2'.

Now, suppose that $x_n \to x(G_0)$ and $y_n \to x(G_0)$, i.e. $x_n \in \mathrm{St}(x, \mathscr{U}_n)$ and $y_n \in \in \mathrm{St}(x, \mathscr{U}_n)$ for each $n = 1, 2, \dots$. Hence there are sets $U_{1n}, U_{2n} \in \mathscr{U}_n$ such that $x_n \in U_{1n}$, $y_n \in U_{2n}$ and $x \in U_{1n} \cap U_{2n} = U_n$. Since $x_n \to x(G_0)$ and $y_n \to x(G_0)$, there is a number p_n such that $x_{p_n} \in U_n$ and $y_{p_n} \in U_n$, so $x_{p_n} \in \mathrm{St}(y_{p_n}, \mathscr{U}_n)$.

Consequently, the convergence G_0 fulfils condition $(**)$.

To complete the proof, it remains to apply first-countability and paracompactness of X.

I wish to thank Professor A. Kamiński for his valuable remarks.

References

[1] *Alexandroff, P., Urysohn, P.,* Une condition nécessaire et suffisante pour qu' une classe (L) soit une classe (D), C. R. Acad. Paris *177* (1923), 1274—1276.

[2] *Alexiewicz, A.,* Functional Analysis (in Polish) Warszawa, 1969.

[3] *Arhangelskij, A.,* New criteria of paracompactness and metrisability of an arbitrary T_1 -space (in Russian) DAN SSSR *141* (1961), 13—15.

[4] *Burzyk, J., Mikusiński, P.,* On normability of semigroups, Bull. Ac. Pol. Math. *28* (1980), 33—35.

[5] *Collins, P. J., Reed, G. M., Roscoe, A. W., Rudin, M. E.,* A lattice of conditions on topological spaces, Proc. Amer. Math. Soc., *94* (1985), 487—496.

[6] *Dolcher, M.,* Topologie e strutture di convergenza, Ann. Scuola Norm. Sup. di Pisa, Ser III, *14* (1960), 63—92.

[7] *Engelking, R.,* General topology, Warszawa, 1977.

[8] *Gruenhage, G.,* Generalized Metric Spaces, Handbook of set theoretic topology, chapter 10, Elsevier, 1984.

[9] *Junnila, H.,* Neighbornets, Pacific J. Math., 76 (1978), 83—108.

[10] *Kamiński, A.*, On multivalued topological convergences, Bull. Ac. Pol. Math. *29* (1981), 605—608.
[11] *Kantorovič, L., Akilov, G.*, Functional Analysis (in Russian) Moskva 1977.
[12] *Kelley, J. L.*, General topology, New York, 1955.
[13] *Kisyński, J.*, Convergence du type *L*, Colloquium Math. *7* (1960), 205—211.
[14] *Mikusiński, P., Pochcial, J.*, On bases of convergence, Proc. Conf. Schwerin 1983, Abh. Akad. Wiss. DDR Abt. Math. Naturwiss. Techn. 1984, 163—166.
[15] *Nagata, J.*, On necessary and sufficient condition of metrizability, Journ. Inst. Polyt. Osaka City Univ. *1* (1950), 93—100.
[16] *Novák, J.*, On convergence spaces and their sequential envelopes, Czechoslovak Math. J. *15* (*90*), (1965), 74—100.
[17] *Pochcial, J.*, On metrizability of convergence, Proc. Conf. Schwerin 1983, Abh. Akad. Wiss. DDR Abt. Math. Naturwiss. Techn. 1984, 177—179.
[18] *Pochcial, J.*, On functional convergences, Rend. Ist. Matem. Univ. di Trieste *17* (1985), 47—54.
[19] *Smirnov, J.*, On metrization of topological spaces (in Russian) Uspehi Mat. Nauk *6* (1951), wyp. 6, 100—111.
[20] *Stone, A. H.*, Sequences of coverings, Pacific Journ. of Math. *10* (1960), 689—691.
[21] *Urysohn, P.*, Über die Metrisation der kompakten topologischen Räume, Math. Ann. *92* (1924), 275—293.
[22] *Urysohn, P.*, Zum Metrisationsproblem, Math. Ann. *94* (1925), 309—315.

Author's address: Institute of Mathematics, Polish Academy of Sciences, Wieczorka 8, 40-013 Katowice, Poland.