Czechoslovak Mathematical Journal

Lynn H. Erbe; Zhong Chao Liang

Qualitative behavior of a generalized Emden-Fowler differential system

Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 3, 454–466

Persistent URL: http://dml.cz/dmlcz/102480

Czechoslovak Mathematical Journal, 41 (116) 1991, Praha

QUALITATIVE BEHAVIOR OF A GENERALIZED EMDEN-FOWLER DIFFERENTIAL SYSTEM

LYNN H. ERBE and ZHONGCHAO LIANG*), Edmonton

(Received February 1, 1990)

0. INTRODUCTION

Consider the two-dimensional nonlinear system

$$\text{(S)} \qquad x' = a(t) f_1(x)\, g_1(y)\,, \quad y' = b(t) f_2(x)\, g_2(y)$$

where $a, b : I = [\tau, \infty) \to R^+ = (0, \infty)$, $f_1, (-g_2) : R = (-\infty, \infty) \to R^+$ and $f_2, g_1 : R \to R$ are continuous; $x f_2(x) > 0$ for $x \neq 0$, $y\, g_1(y) > 0$ for $y \neq 0$. We also assume that the solution of any Cauchy problem is unique and exists on I.

The following nonlinear systems are special forms of (S):

$$x' = a_1(t) f_1(y)\,, \quad y' = -a_2(t) f_2(x)\,, \quad \text{(cf. [4])}\,,$$

$$(r(t)\, x')' + a(t) f(x) = 0 \quad \text{(cf. [3], [5])}\,,$$

$$x'' + a(t) f(x)\, g(x') = 0 \quad \text{(cf. [9], [2], [11], [12-14], [6-8])}\,.$$

The limiting system of (S)

$$\text{(LS)} \qquad x' = \alpha f_1(x)\, g_1(y)\,, \quad y' = \beta f_2(x)\, g_2(y)$$

is also considered, where

$$H_0\colon \lim_{t\to\infty} a(t) = \alpha > 0\,, \quad \lim_{t\to\infty} b(t) = \beta > 0$$

α, β const.

The purpose of the present paper is to establish necessary and sufficient conditions for all solutions of (S) to be oscillatory, bounded, asymptotically periodic or vanishing as $t \to \infty$, and for all orbits of (LS) to be periodic. We also find conditions assuring that all solutions of (S) are almost asymptotically periodic or the trivial solution of (S) is uniformly stable. Our results improve and extend some theorems in 1, 2, 5-13].

Research supported by NSERC Canada and the University of Alberta.

*) On leave from Ocean University of Qingdao, Gingdao, Shandong, P.R.C.

1. OSCILLATION

In this section we give a necessary and sufficient condition for the oscillation of (S). This is further discussed in the following sections.

Let $M(t) = (x(t), y(t))$ be a nontrivial solution of (S). $x(t)$ (or $y(t)$) is said to be *oscillatory* if it has arbitrarily large zeros. $M(t)$ is said to be *oscillatory* if $x(t)$ and $y(t)$ are oscillatory. (S) is said to be *oscillatory* if all solutions of (S) are oscillatory.

It is easy to see that under the assumptions on (S), the oscillation of $M(t)$ is equivalent to that of $x(t)$ or $y(t)$ and that the zeros of $x(t)$ and $y(t)$ separate one another.

To see this, suppose that $y(t) \neq 0$ for $t_1 \leqq t \leqq t_2$ and $x(t_1) = x(t_2) = 0$. Then

$$0 = \frac{x(t_2)}{y(t_2)} - \frac{x(t_1)}{y(t_1)} = \int_{t_1}^{t_2} \frac{x'y - y'x}{y^2} \, dt =$$

$$= \int_{t_1}^{t_2} \frac{a(t) f_1(x) \, g_1(y) \, y - b(t) f_2(x) g_2(y) \, x}{y^2} \, dt > 0$$

and this contradiction shows that the zeros of $x(t)$ and $y(t)$ separate each other.

In this section and the later sections we will need the following hypotheses:

$$\mathrm{H}_1\colon \int_\tau^\infty a(t) \, dt = \infty ,$$

$$\mathrm{H}_2\colon \int_\tau^\infty b(t) \, dt = \infty ,$$

$$\mathrm{H}_3\colon 0 < A \leqq b(t)/a(t) \leqq B , \quad A, B \text{ const.},$$

$$\mathrm{H}_4\colon F(x) = \int_0^x \frac{f_2(u)}{f_1(u)} \, du \to \infty \quad \text{as} \quad |x| \to \infty ,$$

$$\mathrm{H}_5\colon G(y) = \int_0^y \frac{g_1(v)}{g_2(v)} \, dv \to -\infty \quad \text{as} \quad |y| \to \infty .$$

Theorem 1. *Suppose that* H_2 *and* H_3 *are satisfied. Then* (S) *is oscillatory if and only if* H_4 *and* H_5 *hold.*

Proof. Sufficiency: Suppose that $(x(t), y(t))$ is a nonoscillatory solution of (S), so that $x(t)$ and $y(t)$ are eventually of one sign. Let $x(t) \neq 0$, $y(t) \neq 0$ for $t \geqq \tau' \geqq \tau$.

If $x(t) > 0$, $y(t) > 0$ for $t \geqq \tau'$. It is easy to see that $x(t)$ is increasing and $y(t)$ is decreasing, from which we have

$$\lim_{t \to \infty} x(t) = x(\infty) , \quad x(\tau') < x(\infty) \leqq \infty ,$$

$$\lim_{t \to \infty} y(t) = y(\infty) , \quad 0 \leqq y(\infty) < y(\tau') .$$

Integrating (S) from τ' to t, and using H_3 we have

$$G[y(t)] - G[y(\tau')] \geqq A\{F[x(t)] - F[x(\tau')]\}$$

and hence

$$F[x(\infty)] \leqq F[x(\tau')] + \frac{1}{A}\{G[y(\infty)] - G[y(\tau')]\}.$$

From this and H_4 it follows that $x(\infty) < \infty$. Let

$$m = \min_{x(\tau') \leqq x \leqq x(\infty)} f_2(x).$$

Clearly, $m > 0$.

Integrating the second equation of (S) from τ' to t. We obtain

$$\int_{y(\tau')}^{y(t)} \frac{\mathrm{d}y}{g_2(y)} = \int_{\tau'}^{t} b(p) f_2[x(p)] \,\mathrm{d}p \geqq m \int_{\tau'}^{t} b(p) \,\mathrm{d}p$$

so that

$$\int_{\tau'}^{\infty} b(p) \,\mathrm{d}p \leqq \frac{1}{m} \int_{y(\tau')}^{y(\infty)} \frac{\mathrm{d}y}{g_2(y)} < \infty,$$

which contradicts H_2.

If $x(t) > 0$, $y(t) < 0$ for $t \geqq \tau'$, then both $x(t)$ and $y(t)$ are decreasing. So we have

$$0 \leqq x(\infty) < x(\tau'), \quad -\infty \leqq y(\infty) < y(\tau').$$

Integrating (S) from τ' to t and using H_3, we have

$$G[y(t)] - G[y(\tau')] \geqq B\{F[x(t)] - F[x(\tau')]\}.$$

Then

$$G[y(\infty)] \geqq G[y(\tau')] + B\{F[x(\infty)] - F[x(\tau')]\}.$$

From this and H_5 it follows that $y(\infty) > -\infty$. We write

$$M = \max_{y(\infty) \leqq y \leqq y(\tau')} g_1(y).$$

Clearly. $M < 0$.

Integrating the first equation of (S) from τ' to t we get

$$\int_{x(\tau')}^{x(t)} \frac{\mathrm{d}x}{f_1(x)} = \int_{\tau'}^{t} a(p)\, g_1[y(p)] \,\mathrm{d}p \leqq M \int_{\tau'}^{t} a(p) \,\mathrm{d}p.$$

Hence,

$$\int_{\tau'}^{\infty} a(p) \,\mathrm{d}p \leqq \frac{1}{M} \int_{x(\tau')}^{x(\infty)} \frac{\mathrm{d}x}{f_1(x)} < \infty,$$

which contradicts H_2 and H_3.

For the cases $x(t) < 0$, $y(t) < 0$ for $t \geqq \tau'$ and $x(t) < 0$, $y(t) > 0$ for $t \geqq \tau'$, the proofs are similar. Therefore, the sufficiency is proved.

Necessity: Suppose that $F(\infty) < \infty$. Then for given $y_0 > 0$, there exists $x_0 > 0$ such that

$$(1) \qquad F(\infty) - F(x_0) < -\frac{G(y_0)}{B}.$$

Let $(x(t), y(t))$ be a solution of the Cauchy problem (S) with $x(\tau) = x_0$, $y(\tau) = y_0$. By assumption, $(x(t), y(t))$ is oscillatory. Let T be the first zero of $y(t)$ on I. It is easy to see $\tau < T < \infty$ and $x(t) > 0$ for $t \in [\tau, T]$.

Integrating (S) from τ to T and using H_3, we have

$$G[y(T)] - G[y(\tau)] \leqq B\{F[x(T)] - F[x(\tau)]\}.$$

Then

$$-G(y_0) \leqq B[F(\infty) - F(x_0)],$$

which contradicts (1). For the case $F(-\infty) < \infty$, the proof is similar.

Suppose that $G(-\infty) > -\infty$. Then for given $x_0 > 0$, there exists $y_0 < 0$ such that

$$-G(-\infty) + G(y_0) < AF(x_0). \tag{2}$$

Let $(x(t), y(t))$ be a solution of the Cauchy problem (S) with $x(\tau) = x_0$, $y(\tau) = y_0$, and let T' be the first zero of $x(t)$ on I. Clearly, $\tau < T' < \infty$ and $y(t) < 0$ for $t \in [\tau, T']$.

Integrating (S) from τ to T' and using H_3, we have

$$G[y(T')] - G[y(\tau)] \leqq A\{F[x(T')] - F[x(\tau)]\}.$$

Thus

$$-G(-\infty) + G(y_0) \geqq AF(x_0),$$

which contradicts (2). For the case $G(\infty) > -\infty$, the proof is similar. So the necessity is proved. This completes the proof of Theorem 1.

Remark 1. Letting $a(t) \equiv 1$, $f_1(x) \equiv 1$ and $g_1(y) \equiv y$ in (S), Theorem 1 reduces to Theorem 1 of Liang [7]. The sufficient condition of Theorem 1 extends Theorem 0.1 of Bhatia [1] and Theorem 2 of Utz [10].

2. BOUNDEDNESS

In this section we shall establish a necessary and sufficient condition for all solutions of (S) to be bounded as $t \to \infty$, and also conditions to insure that the trivial solution of (S) is uniformly stable. For this, we first introduce some notation and three lemmas.

Let $(x(t), y(t))$ be a nontrivial oscillatory solution of (S) with the sequences $\{t_{2n}\}$ and $\{t_{2n+1}\}$ of zeros of $x(t)$ and $y(t)$ respectively such that $t_{2n} < t_{2n+1} < t_{2n+2}$ $(n = 0, 1, \ldots)$, $t_n \to \infty$ as $n \to \infty$. Clearly, $x(t_{2n+1})$ and $y(t_{2n})$ are extrema of $x(t)$ and $y(t)$ respectively for $n = 0, 1, \ldots$.

Integrating (S) from t_{2n-1} to t_{2n} and using the integral mean value theorem, we obtain

$$G[y(t_{2n})] = -\frac{b(\tau_{2n-1})}{a(\tau_{2n-1})} F[x(t_{2n-1})] \tag{3}$$

$$(t_{2n-1} < \tau_{2n-1} < t_{2n},\ n = 1, 2, \ldots).$$

Similarly, integrating (S) from t_{2n} to t_{2n+1}, we get

$$G[y(t_{2n})] = -\frac{b(\tau_{2n})}{a(\tau_{2n})} F[x(t_{2n+1})] \tag{4}$$

$$(t_{2n} < \tau_{2n} < t_{2n+1},\ n = 0, 1, \ldots).$$

We denote

$$s_n = \prod_{k=1}^{n} \frac{a(\tau_{2k})\, b(\tau_{2k-1})}{a(\tau_{2k-1})\, b(\tau_{2k})}, \quad s_n' = \prod_{k=1}^{n} \frac{a(\tau_{2k-2})\, b(\tau_{2k-1})}{a(\tau_{2k-1})\, b(\tau_{2k-2})}$$

$$(n = 1, 2, \ldots).$$

The sequences $\{s_n\}$ and $\{s_n'\}$ are called characteristic sequences of the oscillatory solution $(x(t), y(t))$. Clearly, we have

$$s_n' = \frac{b(\tau_{2n})\, a(\tau_0)}{a(\tau_{2n})\, b(\tau_0)} s_n \quad (n = 1, 2, \ldots). \tag{5}$$

From (3) and (4) it follows that the extrema of $x(t)$ and $y(t)$ satisfy

$$F[x(t_{2n+1})] = s_n F[x(t_1)] \quad (n = 1, 2, \ldots) \tag{6}$$

and

$$G[y(t_{2n})] = s_n' G[y(t_0)] \quad (n = 1, 2, \ldots) \tag{7}$$

respectively. So we have the following lemma.

Lemma 1. *Suppose that $(x(t), y(t))$ is an oscillatory solution of* (S). *Then the characteristic sequences $\{s_n\}$ and $\{s_n'\}$ of $(x(t), y(t))$ satisfy* (5); *the sequences $\{x(t_{2n+1})\}$ and $\{y(t_{2n})\}$ satisfy* (3), (4), (6) *and* (7).

Lemma 2. *Suppose that* H_1 *and* H_2 *hold. Then all bounded solutions of* (S) *are oscillatory.*

In fact, the proof of Lemma 2 is similar to that of the sufficiency of Theorem 1, so we omit the proof.

Lemma 3. *Suppose that* H_3 *and the following condition* H_6 *hold.*
H_6: *$b(t)/a(t)$ is a function of bounded variation on I.*
Then the characteristic sequence $\{s_n\}$ of every oscillatory solution of (S) *is convergent and $s_n \to s_\infty > 0$ as $n \to \infty$.*

In fact, the convergence of the infinite product

$$s_\infty = \prod_{k=1}^{\infty} \frac{a(\tau_{2k})\, b(\tau_{2k-1})}{a(\tau_{2k-1})\, b(\tau_{2k})}$$

is equivalent to that of the series

$$\sigma = \sum_{k=1}^{\infty} \left[\frac{b(\tau_{2k-1})}{a(\tau_{2k-1})} - \frac{b(\tau_{2k})}{a(\tau_{2k})}\right] \Big/ \frac{b(\tau_{2k})}{a(\tau_{2k})}.$$

By H_3 and H_6 we get

$$\sum_{k=1}^{\infty}\left|\frac{b(\tau_{2k-1})}{a(\tau_{2k-1})}-\frac{b(\tau_{2k})}{a(\tau_{2k})}\right|\bigg/\frac{b(\tau_{2k})}{a(\tau_{2k})}\leqq\frac{1}{A}\sum_{k=1}^{\infty}\left|\frac{b(\tau_{2k+1})}{a(\tau_{2k+1})}-\frac{b(\tau_{2k})}{a(\tau_{2k})}\right|<\infty,$$

which implies that the series σ is absolutely convergent, and hence s_∞ is convergent. From $s_n > 0$ we get $s_\infty > 0$, so Lemma 3 is proved.

We need the following hypothesis:

H_7: (S) is oscillatory and the characteristic sequence $\{s_n\}$ of every nontrivial solution of (S) is bounded.

The main result of this section may now be stated:

Theorem 2. *Suppose that* H_2 *and* H_3 *are satisfied. Then all solutions of* (S) *are bounded as* $t \to \infty$ *if and only if* H_4, H_5 *and* H_7 *hold.*

Proof. Sufficiency: By H_7 we know that any solution $(x(t), y(t))$ of (S) is oscillatory, $s_n < s < \infty$ $(n = 1, 2, \ldots)$ and hence by Lemma 1, (5)–(7) hold. By (5) we get $s_n' < (B/A)\, s = s' < \infty$. From (6) and (7) we get

$$F[x(t_{2n+1})] < sF[x(t_1)]\,, \quad G[y(t_{2n})] < s'G[y(t_0)]\,.$$

From this and H_4, H_5, it follows that the extrema sequences $\{x(t_{2n+1})\}$ and $\{y(t_{2n})\}$ are bounded. So $(x(t), y(t))$ is bounded. Thus the sufficiency is proved.

Necessity: Let $(x(t), y(t))$ be a solution of (S). By Lemma 2 $(x(t), y(t))$ is oscillatory, and hence by Theorem 1 we know that H_4 and H_5 hold.

From the boundedness of $(x(t), y(t))$, we can assume that $|x(t)| < m$ for $t \in I$. From (6) we have

$$s_n < C/F[x(t_1)] < \infty$$

where $C = \max_{|x| \leqq m} F(x) > 0$ and so H_7 holds. This completes the proof of Theorem 2.

In general, it is difficult to verify H_7, so we introduce some conditions which are easier to apply.

Corollary 1. *Suppose that* H_2, H_3 *and* H_6 *are satifified. Then all solutions of* (S) *are bounded as* $t \to \infty$ *if and only if* H_4 *and* H_5 *hold.*

In fact, using Lemma 3, it is easy to deduce the conclusion from Theorem 2.

Corollary 2. *Suppose that* H_2, H_3 *and the following condition* H_8 *are satisfied.*

$$H_8\colon\ b(t)/a(t) \in C^1\,, \quad \int_\tau^\infty \frac{[b(t)/a(t)]'_-}{b(t)/a(t)}\,dt < \infty\,,$$

$$[b(t)/a(t)]'_- = \max\{0, -[b(t)/a(t)]'\}\,.$$

Then all solutions of (S) *are bounded as* $t \to \infty$ *if and only if* H_4 *and* H_5 *hold.*

In fact, from H_8 it follows that the characteristic sequence $\{s_n\}$ of any oscillatory solution of (S) satisfies

$$s_n = \exp \sum_{k=1}^{n} \left[\ln \frac{b(\tau_{2k-1})}{a(\tau_{2k-1})} - \ln \frac{b(\tau_{2k})}{a(\tau_{2k})} \right] = \exp \left\{ - \sum_{k=1}^{n} \int_{\tau_{2k-1}}^{\tau_{2k}} \frac{[b(t)/a(t)]'}{b(t)/a(t)} \, dt \right\}$$

$$\leqq \exp \sum_{k=1}^{n} \int_{\tau_{2k-1}}^{\tau_{2k}} \frac{[b(t)/a(t)]'_-}{b(t)/a(t)} \, dt \leqq \exp \int_{\tau}^{\infty} \frac{[b(t)/a(t)]'_-}{b(t)/a(t)} \, dt < \infty .$$

From this and Theorem 2 the conclusion follows.

Remark 2. Letting $a_1(t) \equiv 1, f_1(x) \equiv 1$ and $g_1(y) \equiv y$ in (S). Theorem 2 reduces to Theorem 2 of Liang [7]. The sufficient conditions of Corollary 1 and Corollary 2 also improve and extend some of the theorems due to Bihari [2], Kroopnick [5], Liang [6], Wong and Burton [11] and Wong [12, 13].

Theorem 3. *Suppose that* H_3 *holds and there exist* $\delta_1 > 0$, $\delta_2 > 0$ *such that the solution* $(x(t), y(t))$ *of any Cauchy problem* (S) *with* $x(\tau') = x_0$, $y(\tau') = y_0$, $\tau' \geqq \tau$, $x_0^2 + y_0^2 < \delta_1$ *satisfies*

(i) *it is oscillatory,*

(ii) *the characteristic sequence* $\{s_n\}$ *satisfies* $s_n < \delta_2$.

Then the solution $(0, 0)$ *of* (S) *is uniformly stable.*

Proof. From (i) and Lemma 1 we know that (5)–(7) hold. Integrating (S) from τ' to t_1 gives

$$F[x(t_1)] - F[x(\tau')] = \frac{a(\tau_1')}{b(\tau_1')} \{G[y(t_1)] - G(y(\tau')]\} \quad (\tau' < \tau_1' < t_1)$$

so by H_3 we get

$$F[x(t_1)] \leqq F(x_0) - G(y_0)/A . \tag{8}$$

Integrating (S) from τ' to t_0 gives

$$G[y(t_0)] - G[y(\tau')] = \frac{b(\tau_0')}{a(\tau_0')} \{F[x(t_0)] - F[x(\tau')]\} \quad (\tau' < \tau_0' < t_0)$$

and again by H_3 we get

$$G[y(t_0)] \geqq -BF(x_0) + G(y_0) . \tag{9}$$

Using (8) and (9) in (6) and (7) we obtain

$$F[x(t_{2n+1})] \leqq s_n[F(x_0) - G(y_0)/A] \leqq \delta_2[F(x_0) - G(y_0)/A]$$

$$G[y(t_{2n})] \geqq -s_n'[BF(x_0) - G(y_0)] \geqq -(B\delta_2/A)[BF(x_0) - G(y_0)] .$$

From which it follows that the solution (0, 0) of (S) is uniform stable.

Corollary 3. *Suppose that* H_2, H_3, *and* H_6 *(or* H_8*) hold. Then the solution* $(0, 0)$ *of* (S) *is uniformly stable.*

To see this consider the auxiliary system

$$x' = a(t)\,\varphi_1(x)\,\psi_1(y) , \quad y' = b(t)\,\varphi_2(x)\,\psi_2(y) \tag{S'}$$

where $\varphi_i(x)$, $\psi_i(y)$ $(i = 1, 2)$ are defined by

$$\varphi_i(x) = \begin{cases} f_i(x) & \text{for } |x| < \delta \\ f_i(\delta) & \text{for } x \geqq \delta \\ f_i(-\delta) & \text{for } x \leqq -\delta \end{cases}$$

$$\psi_i(y) = \begin{cases} g_i(y) & \text{for } |y| < \delta \\ g_i(\delta) & \text{for } y \geqq \delta \\ g_i(-\delta) & \text{for } y \leqq -\delta \end{cases}$$

$(i = 1, 2,\ \delta > 0 \text{ const.})$.

It is easy to see that the stability of the solution $(0, 0)$ of (S) is equivalent to that of (S′) and to verify that (S′) satisfies $H_2 - H_5$. So by Theorem 1 (S′) is oscillatory, and hence condition (i) of Theorem 3 is satisfied. Furthermore, from H_3, H_6 (or H_8) and using Lemma 3 (or the proof of Corollary 2) we can show that condition (ii) of Theorem 3 is satisfied.

Remark 3. Corollary 3 improves a result of Wong and Burton [11].

3. SYSTEM (LS)

In this section we shall establish necessary and sufficient conditions for all solutions of (LS) to be periodic, and also show equivalent relations for oscillation, boundedness and periodicity of (LS). We assume that the solution of any Cauchy problem is unique. It is easy to see that the origin $(0, 0)$ is the unique singular point and every orbit of (LS) surrounds the origin.

Theorem 4. *Suppose that $\alpha > 0$ and $\beta > 0$. Then every solution is periodic if and only if H_4 and H_5 hold.*

Proof. Necessity: Since every periodic solution of (LS) is bounded, by Theorem 2 we have that H_4 and H_5 hold.

Sufficiency: We consider the function

$$V(x, y) = \beta F(x) - \alpha G(y),$$

and taking the derivative of $V(x, y)$ along an orbit of (LS) we get

$$V'(x, y) = \frac{\beta f_2(x)}{f_1(x)} [\alpha f_1(x)\, g_1(y)] - \frac{\alpha g_1(y)}{g_2(y)} [\beta f_2(x)\, g_2(y)] \equiv 0 .$$

So from H_2 and H_5 it follows that $V(x. y) = C$ is a closed orbit of (LS) for any $C > 0$. This completes the proof.

Remark 4. The sufficient condition of Theorem 4 extends some results of Bhatia [1] and Utz [9].

Theorem 5. *Suppose that $\alpha > 0$ and $\beta > 0$. Then for all solutions of* (LS), *there* *exists the following equivalent relation*

$$\textit{Oscillation} \Leftrightarrow \textit{Boundedness} \Leftrightarrow \textit{Periodicity}.$$

Proof. Since the characteristic sequence $\{s_n\}$ of oscillatory solution of (LS) satisfies $s_n = 1$ $(n = 1, 2, \ldots)$, the conclusion follows from Theorems 1, 2 and 3.

4. ASYMPTOTIC PERIODICITY

In this section we shall give a necessary and sufficient condition for all solutions of (S) to be asymptotically periodic. This describes the structure of positive sets of solutions of (S). Furthermore, we also show the equivalence of oscillation, boundedness and asymptotic periodicity for solutions of (S) under certain assumptions.

A solution of (S) is said to be *asymptotically periodic* if its orbit approaches a periodic orbit of (LS) in a spiral manner as $t \to \infty$. We introduce the following condition:

H_9: The characteristic sequence $\{s_n\}$ of every oscillatory solution of (S) satisfies: $s_n \to s_\infty > 0$ as $n \to \infty$.

Theorem 6. *Suppose that* H_0 *is satisfied. Then all solutions of* (S) *are asymptotically periodic if and only if* H_4, H_5 *and* H_9 *hold.*

Proof. Sufficiency: By Theorem 4 we know that all orbits of (LS) are closed. By Theorems 1 and 2, every solution $M(t) = (x(t), y(t))$ of (S) is oscillatory and bounded, and so Lemma 1 holds. Letting $n \to \infty$ in (6) and (7) and using H_0, H_9 and (5) we obtain

$$\lim_{n\to\infty} F[x(t_{2n+1})] = s_\infty F[x(t_1)]\,, \tag{10}$$

$$\lim_{n\to\infty} G[y(t_{2n})] = s'_\infty G[y(t_0)]\,, \tag{11}$$

where $s'_\infty = \beta s_\infty\, a(\tau_0)\, \alpha\, b(\tau_0) > 0$.

Letting $n \to \infty$ in (4) and using (10) and (11) give

$$\beta s_\infty F[x(t_1)] = -\alpha s'_\infty G[y(t_0)] = C_1 > 0\,.$$

Now consider the periodic orbit of (LS)

$$V(x, y) = \beta\, F(x) - \alpha\, G(y) = C_1\,. \tag{12}$$

It is clear that $V(x, y) = C_1$ intersects the x-axis at $(x_1, 0)$ and $(x_2, 0)$, and the y-axis at $(0, y_1)$ and $(0, y_2)$. Without loss of generality we can assume that $x_1 > 0$, $x_2 < 0$, $y_1 > 0$ and $y_2 < 0$, and $x(t_{4n-3}) > 0$, $x(t_{4n-1}) < 0$, $y(t_{4n-2}) > 0$ and

$y(t_{4n}) < 0$. From (10), (11) and (12) it follows that

$$x(t_{4n-3}) \to x_1, \quad x(t_{4n-1}) \to x_2,$$
$$y(t_{4n-2}) \to y_1, \quad y(t_{4n}) \to y_2 \quad \text{as} \quad n \to \infty.$$

We now prove that for arbitrary $\varepsilon > 0$, there exists $\tau^* \geqq \tau$ such that $M(t) \subset A(\varepsilon)$ for all $t \geqq \tau^*$, where

$$A(\varepsilon) = \{(x, y) : C_1 - \varepsilon < V(x, y) < C_1 + \varepsilon\}.$$

In fact, suppose the contrary. Then from the boundedness of $M(t)$ it follows that there exists a sequence $\{t'_j\}$, $t'_j \to \infty$ as $j \to \infty$ such that $M(t'_j) \notin A(\varepsilon)$ for each j and

$$M(t'_j) \to \bar{M} = (\bar{x}, \bar{y}) \quad \text{as} \quad j \to \infty.$$

Clearly, $\bar{M} \notin A(\varepsilon)$. Without loss of generality we can assume that $\bar{x} > 0$ and $\bar{y} > 0$. So we have $t_{4n_j-3} < t'_j < t_{4n_j-2}$ for j large enough. Integrating (S) from t_{4n_j-3} to t'_j we obtain

$$\frac{b(\tau'_{4n_j-3})}{a(\tau'_{4n_j-3})} F[x(t'_j)] - G[y(t'_j)] = \frac{b(\tau'_{4n_j-3})}{a(\tau'_{4n_j-3})} F[x(t'_{4n_j-3})] \tag{13}$$
$$(t_{4n_j-3} < \tau'_{4n_j-3} < t'_j).$$

Letting $j \to \infty$ in (13) and using H_0 and (10), we have

$$\frac{\beta}{\alpha} F(\bar{x}) - G(\bar{y}) = \frac{\beta s_\infty}{\alpha} F[x(t_1)].$$

From this it follows that $V(\bar{x}, \bar{y}) = C_1$. This is impossible. The sufficiency is proved.

Necessity: By assumption we know that any solution of (S) is oscillatory and bounded, and hence by Theorem 2 H_4 and H_5 hold.

As in the proof of sufficiency, we can assume that $x(t_{4n-3}) \to x_1 > 0$, $x(t_{4n-1}) \to$ $\to x_2 < 0$ as $n \to \infty$ and $V(x_1, 0) = V(x_2, 0) = C_1 > 0$. Therefore, it follows that

$$F(x_1) = F(x_2) = C_1/\beta > 0. \tag{14}$$

On the other hand, from (6) we get

$$s_{2k-2} = F[x(t_{4k-3})]/F[x(t_1)] \quad (k = 2, 3, \ldots), \tag{15}$$
$$s_{2k-1} = F[x(t_{4k-1})]/F[x(t_1)] \quad (k = 1, 2, \ldots). \tag{16}$$

Letting $k \to \infty$ in (15) and (16), and using (14) we obtain

$$\lim_{k\to\infty} s_{2k-2} = F(x_1)/F[x(t_1)] = C_1/\{\beta F[x(t_1)]\},$$
$$\lim_{k\to\infty} s_{2k-1} = F(x_2)/F[x(t_1)] = C_1/\{\beta F[x(t_1)]\}.$$

From this it follows that H_9 is valid. This completes the proof of Theorem 6.

Corollary 4. *Suppose that* H_0 *and* H_6 *hold. Then all solutions of* (S) *are asymptotically periodic if and only if* H_4 *and* H_5 *hold.*

In fact, by Lemma 3 H_6 implies H_9. So Corollary 4 follows directly from Theorem 6.

Theorem 7. *Suppose that* H_0 *and* H_9 *are satisfied. Then for all solutions of* (S), *there exists the following equivalent statement*:

$$\text{(17)} \qquad \textit{Oscillation} \Leftrightarrow \textit{Boundedness} \Leftrightarrow \textit{Asymptotic periodicity} .$$

In fact, relation (17) can be deduced from Theorems 1, 2 and 6.

Corollary 5. *Suppose that* H_0 *and* H_6 *hold. Then for all solutions of* (S), *relation* (17) *holds.*

Remark 5. Letting $a(t) \equiv 1, f_1(x) \equiv 1$ and $g_1(y) \equiv y$ in (S). Theorem 6 reduces to Theorem 4 of Liang [7]. The sufficient condition of Corollary 4 improves and extends theorems 5 and 6 of Wong and Burton [11].

5. ALMOST ASYMPTOTIC PERIODICITY

In this section we shall establish a sufficient condition for all solutions of (S) to be almost asymptotically periodic.

A solution $M(t) = (x(t), y(t))$ is said to be *almost asymptotically periodic* if there exists a constant $T > 0$ such that for arbitrary $\varepsilon > 0$, there is a $\tau' \geqq \tau$ so that

$$|M(t + T) - M(t)| < \varepsilon \quad \text{for all} \quad t \geqq \tau' .$$

Theorem 8. *Suppose that* H_0, H_4, H_5 *and* H_9 *are satisfied. Then all solutions of* (S) *are almost asymptotically periodic.*

Proof. As in the proof of Theorem 6, we can assume that $x(t_{4n-3}) \to x_1 > 0$, $x(t_{4n-1}) \to x_2 < 0$, $y(t_{4n-2}) \to y_1 > 0$ and $y(t_{4n}) \to y_2 < 0$ as $n \to \infty$.

It is easy to verify that $M(t + t_{4n-3}) = (x(t + t_{4n-3}), y(t + t_{4n-3}))$ is a solution of the Cauchy problem:

$$\begin{aligned} u' &= \alpha f_1(u)\, g_1(v) + \\ &\quad + [-\alpha + a(t + t_{4n-3})] f_1[x(t + t_{4n-3})]\, g_1(y(t + t_{4n-3})] , \\ v' &= \beta f_2(u)\, g_2(v) + \\ &\quad + [-\beta + b(t + t_{4n-3})] f_2[x(t + t_{4n-3})]\, g_2[y(t + t_{4n-3})] , \\ u(0) &= x(t_{4n-3}) , \\ v(0) &= 0 . \end{aligned}$$

Let $M^*(t) = (x^*(t), y^*(t))$ be a solution of the Cauchy problem (LS) with $x(0) = x_1$, $y(0) = 0$. By Theorem 4 we know $M^*(t)$ is periodic solution (periodic $T > 0$). Using Yoshizawa's method (see [15, § 13]), it is easy to prove that $M(t + t_{4n-3})$ converges to $M^*(t)$ uniformly in $t \in [0, 3T]$ as $n \to \infty$. Thus for each $\varepsilon > 0$ there is an integer $N > 0$ such that $t_{4(n+1)-3} - t_{4n-3} < 2T$ and $|M(t + t_{4n-3}) - M^*(t)| < \varepsilon/2$ for

$n \geqq N$ and $t \in [0, 3T]$. From this it follows that

$$|M(t + t_{4n-3} + T) - M(t + t_{4n-3})| \leqq$$
$$\leqq |M(t + t_{4n-3} + T) - M^*(t + T)| + |M^*(t) - M(t + t_{4n-3})| < \varepsilon$$

for $t \in [0, 2T]$. So we have $|M(t + T) - M(t)| < \varepsilon$ for $t \geqq t_{4N-3}$. This completes the proof of Theorem 8.

Corollary 6. *Suppose that* H_0, H_4, H_5 *and* H_6 *hold. Then all solutions of* (S) *are almost asymptotically periodic.*

Remark 6. Corollary 6 improves and extends Theorem 7 of Wong and Burton [11]. If $a(t) \equiv 1, f_1(x) \equiv 1, g_1(y) \equiv y$ in (S), then Theorem 8 reduces to Theorem 5 of Liang [7]. However, almost asymptotic periodicity is an open question.

6. ASYMPTOTIC STABILITY

In this section we shall establish a necessary and sufficient condition for all solutions of (S) to approach zero as $t \to \infty$. We need the following hypotheses.

H_{10}: (S) is oscillatory and the characteristic sequence $\{s_n\}$ of every solution of (S) satisfies $s_n \to 0$ as $n \to \infty$.

H_{11}: There exists a $\delta > 0$ such that $[b(T)/a(T)] [b(T')/a(T')]^{-1} \leqq \delta$ for any $T' \geqq T \geqq \tau$.

Theorem 9. *Suppose that* H_1, H_2 *and* H_{11} *are valid. Then every solution* $(x(t), y(t))$ *of* (S) *satisfies*

$$\lim_{t \to \infty} x(t) = 0 \quad \textit{and} \quad \lim_{t \to \infty} \frac{a(t)}{b(t)} G[y(t)] = 0 \tag{18}$$

if and only if H_{10} *holds.*

Proof. Sufficiency: By H_{10} and Lemma 1 we know that $(x(t), y(t))$ is oscillatory and (6) holds. From (6) and H_{10} it follows that the first part of (18) is valid. We shall next show that the second part of (18) holds. In fact, integrating (S) from t_{2n-1} to $t \in [t_{2n-1}, t_{2n}]$ we have

$$G[y(t)] = -\frac{b(\tau'_{2n-1})}{a(\tau'_{2n-1})} \{F[x(t)] - F[x(t_{2n-1})]\} .$$

Multiplying the two sides of the above equation by $a(t)/b(t)$ we get

$$\frac{a(t)}{b(t)} G[y(t)] = -\frac{b(\tau'_{2n-1})}{a(\tau'_{2n-1})} \left[\frac{b(t)}{a(t)}\right]^{-1} \{F[x(t)] - F[x(t_{2n-1})\} . \tag{19}$$

Letting $t \to \infty$ in (19) which implies $n \to \infty$, and by H_{11} and $x(t) \to 0$ as $t \to \infty$ we know that the second part of (18) is valid. So the sufficiency is proved.

Necessity: By the assumption, all solutions of (S) are bounded. Therefore, from H_1 and H_2 and by Lemma 3 we know that (S) is oscillatory, and so Lemma 1 holds. Let $\{s_n\}$ be a characteristic sequence of any solution $(x(t), y(t))$ of (S). Using (6) gives

$$s_n = F[x(t_{2n+1})]/F[x(t_1)] \quad (n = 1, 2, \dots),$$

it follows that $s_n \to 0$ as $n \to \infty$. Thus H_{10} is valid. This completes the proof of Theorem 9.

Remark 7. Letting $a(t) \equiv 1$, $f_1(x) \equiv 1$, $g_1(y) \equiv y$ in (S), Theorem 9 reduces to Theorem 4.2 of Liang [8].

References

[1] *Bhatia, N. P.:* Some oscillation theorems for second order differential equations, J. Math. Anal. Appl. *15* (1966), 442—446.

[2] *Bihari, I.:* Research of the boundedness and stability of the solutions of nonlinear differential equations, Acta Math. Hungar. *8* (1957), 261—278.

[3] *Hatvani, L.:* On the asymptotic behavior of solutions of $(p(t)\,x')' + q(t)\,f(x) = 0$, Publ. Math. Debrecen *19* (1972), 225—237.

[4] *Kwong, Man Kam* and *Wong, J. S. W.:* Oscillation of Emden-Fowler Systems, Diff. Integ. Eqs. *1* (1988), 133—141.

[5] *Kroopnik, A.:* Oscillation properties of $(m(t)\,x')' + a(t)\,b(x) = 0$, J. Math. Anal. Appl. *63* (1978), 141—144.

[6] *Liang, Z.:* Asymptotic character of the solutions of a class of second order nonlinear differential equations, Shuxue Jinzhan *9* (1966), 251—264. MR 38#4767.

[7] *Liang, Z.:* Asymptotically periodic solutions of a class of second order nonlinear differential equations, Proc. Amer. Math. Soc. *99* (1987), 693—699.

[8] *Liang, Z.:* Necessary and sufficient conditions of the stability for a class of second order nonlinear oscillations, Funkcial. Ekvac. *30* (1987), 45—55.

[9] *Petty, C. M.* and *Johnson, W. E.:* Properties of solutions of $u'' + c(t)\,f(u)\,h(u') = 0$ with explicit initial conditions, SIAM J. Math. Anal. *4* (1973), 269—282.

[10] *Utz, W. R.:* Properties of solutions of $u'' + g(t)\,u^{2n-1} = 0$, Monatsh. Math. *66* (1962), 55—60.

[11] *Wong, J. S. W.* and *Burton, T. A.:* Some properties of solutions of $u''\ 1\ a(t)\,f(u)\,g(u') = 0$ (II), Monatsh. Math. *69* (1965), 368—374.

[12] *Wong, J. S. W.:* Some properties of solutions of $u'' + a(t)\,f(u)\,g(u') = 0$ (III), SIAM J. Appl. Math. *14* (1966), 209—214.

[13] *Wong, J. S. W.:* Boundedness theorems of solutions of $u'' + a(t)\,f(u)\,g(u') = 0$ (IV), Enseign. Math. *13* (1967), 157—165.

[14] *Wong, J. S. W.:* On the generalized Emden-Fowler equations, SIAM Rev. *17* (1975), 339—360.

[15] *Yoshizawa, T.:* Stability theory by Liapunov's second method, Math. Soc. Japan, Tokyo, 1966.

Authors' address: Department of Mathematics, University of Alberta Edmonton, Canada T6G 2G1.