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ORIENTED GRAPHS WITH PRESCRIBED m-CENTER AND m-MEDIAN

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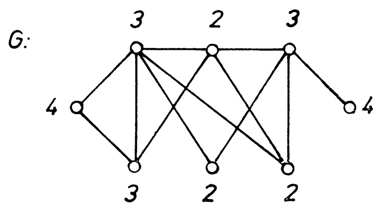
Let D be a strong digraph. For vertices u and v of D, the directed distance $\vec{d}(u, v)$ is the length of a shortest (directed) $u - v$ path in D. The m-distance $md(u, v)$ between u and v is $\max\{\vec{d}(u, v), \vec{d}(v, u)\}$. For subdigraphs F_1 and F_2 of a strong digraph D, the m-distance $md(F_1, F_2)$ between F_1 and F_2 is $\min\{md(u, v) \mid u \in V(F_1), v \in V(F_2)\}$. The m-eccentricity $me(v)$ of a vertex v is $\max\{md(v, u) \mid u \in V(G)\}$. The m-center $mC(D)$ of D is the subdigraph induced by those vertices of minimum m-eccentricity. The m-distance $md(v)$ of v is $\sum_{u \in V(D)} md(v, u)$. The m-median $mM(D)$ is the subdigraph induced by those vertices of minimum m-distance. It is proved that for any two oriented graphs D_1 and D_2 and positive integer k, there exists a strong oriented graph H such that $mC(H) \cong D_1$, $mM(H) \cong D_2$ and $md_H(mC(H), mM(H)) = k$. Also, it is proved that for any three oriented graphs D_1, D_2 and K such that K is isomorphic to an induced subdigraph of both D_1 and D_2, then there exists a strong oriented graph H such that $mC(H) \cong D_1$, $mM(H) \cong D_2$ and $mC(H) \cap mM(H) \cong K$.

The *distance $d(u, v)$ between two vertices u and v* in a connected graph G is the length of a shortest $u - v$ path in G. The *distance between two subgraphs F_1 and F_2* of G is defined by $d(F_1, F_2) = \min\{d(u, v) \mid u \in V(F_1), v \in V(F_2)\}$. The *eccentricity* $e(u)$ of a vertex u is $\max\{d(u, v) \mid v \in V(G)\}$. The *center* $C(G)$ of G is the subgraph induced by those vertices of maximum eccentricity. The eccentricities of the vertices of the graph G of Figure 1 are shown together with the center of G.



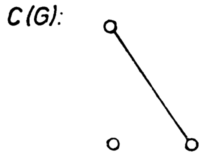


Figure 1

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The *distance of a vertex u* in a connected graph G is defined by $d(u) = \sum_{v \in V(D)} d(u, v)$. The subgraph of G induced by those vertices of minimum distance is called the *median* of G and is denoted by $M(G)$. The vertices of the graph G of Figure 2 are labeled by their distances and the median of G is shown.

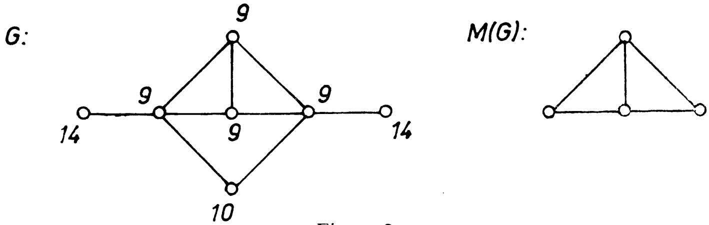

Figure 2

Hendry [1], Holbert [2] and Novotny and Tian [3] studied the relative location of the center and median of a connected graph. Hendry proved that for every two graphs F and G, there exists a connected graph H such that $C(H) \cong F$ and $M(H) \cong G$ where $C(H)$ and $M(H)$ are disjoint. Holbert extended this result by showing that for every two graphs F and G and positive integer k, there exists a connected graph H such that $C(H) \cong F$, $M(H) \cong G$, and $d(C(H), M(H)) = k$. Thus, the center and median can be arbitrarily far apart. On the other hand, these subgraphs can be arbitrarily close as Novotny and Tian showed when they proved for any three graphs F, G and K, where K is isomorphic to an induced subgraph of both F and G, there exists a connected graph H such that $C(H) \cong F$, $M(H) \cong G$ and $C(H) \cap M(H) \cong K$. It is the goal of this paper to present directed analogues of the theorems of Holbert and of Novotny and Tian.

For vertices u and v in a strong digraph D, the *directed distance* $\vec{d}(u, v)$ is the length of a shortest (directed) $u - v$ path in D. The *maximum distance* or *m-distance* $md(u, v)$ between u and v is $\max\{\vec{d}(u, v), \vec{d}(v, u)\}$. It is not difficult to show that the m-distance is a metric on the vertex set of a strong digraph. For the digraph D of Figure 3, $\vec{d}(u, v) = 3$ and $\vec{d}(v, u) = 4$, so $md(u, v) = 4$.

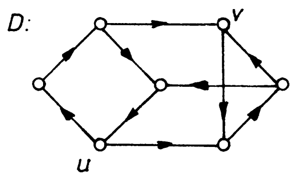

Figure 3

For subdigraphs F_1 and F_2 of a strong digraph D, the *m-distance* between F_1 and F_2 is defined by

$$md_D(F_1, F_2) = \min\{md_D(u, v) \mid u \in V(F_1),\ v \in V(F_2)\}.$$

For the subdigraphs F_1 and F_2 of the digraph D of Figure 4,

$$md_D(F_1, F_2) = md(u, v) = 3 .$$

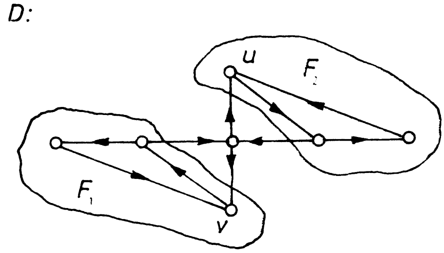

Figure 4

For a given oriented graph D, our first result shows that any subdigraph F of D, whose vertices have the same m-distance in D, can be the m-median of some oriented graph that contains D as an induced subdigraph.

Lemma 1. *Let D be a strong oriented graph and let F be a subdigraph of D with $md_D(u) = md_D(v)$ for all $u, v \in V(F)$. Then there exists an oriented graph H having D as an induced subdigraph such that $mM(H) \cong F$.*

Proof. Suppose $md_D(v) = k$ for all $v \in V(F)$. Let

$$n = \left\lceil \frac{p(F) + k}{2} \right\rceil - p(D) + 1 .$$

We construct an oriented graph H by adding $2n$ new vertices u_i, v_i $(1 \leqq i \leqq n)$ to D and the arcs joining all vertices of F to u_i and from v_i for $1 \leqq i \leqq n$ (see Figure 5).

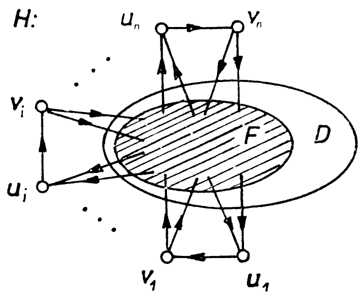

Figure 5

Then,

$$md_H(v) = \sum_{i=1}^{n} (md(v, u_i) + md(v, v_i)) + \sum_{x \in V(D)} md_H(v, x) \leqq$$
$$\leqq 4n + \sum_{x \in V(D)} md_D(v, x) = 4n + md(v) = 4n + k , \quad \text{for} \quad v \in V(F) .$$

For $1 \leqq i \leqq n$, it follows that

$$md(u_i) = \sum_{1 \leqq j \neq i \leqq n} (md(u_i, u_j) + md(u_i, v_j)) + md(u_i, v_i) +$$
$$+ \sum_{x \in V(F)} md_H(u_i, x) + \sum_{x \in V(D) - V(F)} md_H(u_i, x) \geqq$$
$$\geqq 7(n-1) + 2 + 2\,p(F) + 3(p(D) - p(F)) =$$
$$= 7n + 3\,p(D) - p(F) - 5\,.$$

Similarly, $md(v_i) \geqq 7n + 3\,p(D) - p(F) - 5$ for $1 \leqq i \leqq n$. If $v \in V(D) - V(F)$, then

$$md_H(v) = \sum_{1 \leqq i \leqq n} (md(v, u_i) + md(v, v_i)) + \sum_{x \in V(D)} md_H(v, x) \geqq$$
$$\geqq 6n + 2(p(D) - 1)\,.$$

Since

$$n = \left\lceil \frac{p(F) + k}{2} \right\rceil - P(D) + 1\,,$$

it follows that

$$7n + 3\,p(D) - p(F) - 5 > 4n + k \quad \text{and} \quad 6n + 2\,p(D) - 2 > 4n + k\,.$$

Therefore, $mM(H) \cong F$. □

In order to apply Lemma 1 to any subdigraph F of D, we prove that under certain conditions the oriented graph D can be imbeded into an oriented graph H such that all vertices of F have the same m-distance in H.

Lemma 2. *Let D be a strong oriented graph and let F be a subdigraph of D with* $\max \{md_D(u, v) \mid u, v \in V(F)\} \leqq 3$. *Then there exists an oriented graph H containing D as an induced subdigraph such that*

(i) *if* $V(H) \neq V(D)$ *then* $\max \{md_H(u, v) \mid u \in V(F), v \in V(H) - V(D)\} = 3$, and

(ii) $md_H(u) = md_H(v)$ for all $u, v \in V(F)$.

Proof. Let $m_\Delta(D) = \max \{md_D(x) \mid x \in V(F)\}$, $m_\delta(D) = \min \{md_D(x) \mid x \in V(F)\}$ and $n = m_\Delta(D) - m_\delta(D)$. If $n = 0$, then $md_D(u) = md_D(v)$ for all $u, v \in V(F)$. Let $H = D$. Then the oriented graph H has the desired property. If $n \geqq 1$, then we denote $S_\Delta(D) = \{x \in V(F) \mid md_D(x) = m_\Delta(D)\}$. Define an oriented graph H_1 by

$$V(H_1) = V(D) \cup \{w_1, x_1, y_1\}$$

and

$$E(H_1) = E(D) \cup \{(w_1, x_1), (x_1, y_1), (y_1, w_1)\} \cup$$
$$\cup \{(w_1, z), (z, y_1) \mid z \in S_\Delta(D)\} \cup$$
$$\cup \{(x_1, z), (z, y_1) \mid z \in V(F) - S_\Delta(D)\}$$

(see Figure 6).

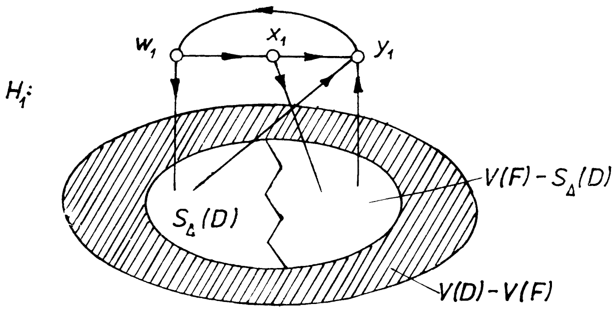

Figure 6

Clearly, D is an induced subdigraph of H_1 and $\max\{md_{H_1}(u, v) \mid u \in V(F), v \in V(H_1) - V(D)\} = 3$. Since $md_D(z_1, z_2) \leqq 3$ for all $z_1, z_2 \in V(F)$, it follows that $md_{H_1}(z, t) = md_D(z, t)$ for all $z \in V(F)$ and $t \in V(D)$. In particular, $md_{H_1}(z_1, z_2) = md_D(z_1, z_2) \leqq 3$ for all $z_1, z_2 \in V(F)$. Therefore, for $z \in S_\Delta(D)$,

$$md_{H_1}(z) = md_D(z) + md_{H_1}(z, w_1) + md_{H_1}(z, x_1) + md_{H_1}(z, y_1) = $$
$$= md_D(z) + 2 + 3 + 2 = md_{H_1}(z) + 7\,.$$

Similarly, $md_{H_1}(z) = md_D(z) + 8$ for $z \in V(F) - S_\Delta(D)$. Define $m_\Delta(H_1) = \max\{md_{H_1}(x) \mid x \in V(F)\}$ and $m_\delta(H_1) = \min\{md_{H_1}(x) \mid x \in V(F)\}$. Then $m_\Delta(H_1) = m_\Delta(D) + 7$ and $m_\delta(H_1) = m_\delta(D) + 8$. Therefore, $m_\Delta(H_1) - m_\delta(H_1) = (m_\Delta(D) + 7) - (m_\delta(D) + 8) = m_\Delta(D) - m_\delta(D) - 1$. Let $S_\Delta(H_1) = \{x \in V(F) \mid md_{H_1}(x) = m_\Delta(H_1)\}$. We define an oriented graph H_2 by

$$V(H_2) = V(H_1) \cup \{w_2, x_2, y_2\}$$

and

$$E(H_2) = E(H_1) \cup \{(w_2, x_2), (x_2, y_2), (y_2, w_2)\} \cup$$
$$\cup \{(w_2, z), (z, y_2) \mid z \in S_\Delta(H_1)\} \cup$$
$$\cup \{(x_2, z), (z, y_2) \mid z \in V(F) - S_\Delta(H_1)\}\,.$$

By a similar argument, it follows that $m_\Delta(H_2) - m_\delta(H_2) = m_\Delta(D) - m_\delta(D) - 2$. We repeat this process $n - 1$ times. Let $H = H_n$. Then $m_\Delta(H) = m_\delta(H)$, namely, $md_H(u) = md_H(v)$ for all $u, v \in V(F)$. In addition, by the construction of H_n, it follows that D is an induced subdigraph of H and $\max\{md_H(u, v) \mid u \in V(F), v \in V(H) - V(D)\} = 3$. □

With the aid of Lemmas 1 and 2, we now are ready to prove that for every pair of oriented graphs D_1 and D_2 there exists an oriented graph H such the m-center and m-median are isomorphic to D_1 and D_2, respectively. Furthermore, the m-distance between D_1 and D_2 in H can be arbitrarily prescribed.

Theorem 3. *Let D_1 and D_2 be oriented graphs. For all integers $k \geqq 2$, there* *exists a strong oriented graph H such that $mC(H) \cong D_1$, $mM(H) \cong D_2$ and $md_H(mC(H), mM(H)) = k$.*

Proof. We first define an oriented graph H_0 by adding two new vertices u and v to D_2 and the arc (u, v) together with the arcs joining all the vertices of D_2 to u and from v. Clearly, H_0 is strong and $md_{H_0}(x, y) \leqq 3$ for all $x, y \in V(D_2)$. By Lemma 2, there exists an oriented graph H_1 containing H_0 as an induced subdigraph such that (i) if $V(H_1) \neq V(H_0)$, then $\max\{md_{H_1}(x, y) \mid x \in V(D_2),\ y \in V(H_1) - V(H_0)\} = 3$, and (ii) $md_{H_1}(x) = md_{H_1}(y)$ for all $x, y \in V(D_2)$. Let $n_1 = \max\{\vec{d}_{H_1}(x, y) \mid x \in V(D_2),\ y \in V(H_1) - V(D_2)\}$ and $n_2 = \max\{\vec{d}_{H_1}(y, x) \mid x \in V(D_2),\ y \in V(H_1) - V(D_2)\}$. Since H_1 is strong, it follows that $n_1, n_2 \geqq 2$. By the construction of H_1, if $H_1 \neq H_0$, then $n_1 = n_2 = 3$. Further, if $H_1 = H_0$, then $n_1 = n_2 = 2$. Therefore, $n_1 = n_2$. Let $t = \max\{3, n_1\}$. We define the oriented graph H_2 by

$$V(H_2) = V(H_1) \cup V(D_1) \cup \{u_i \mid 0 \leqq i \leqq k - 1\} \cup \\ \cup \{v_i \mid 0 \leqq i \leqq k + t\}$$

and

$$E(H_2) = E(H_1) \cup E(D_1) \cup \{(u_0, v_0)\} \cup \{(x, u_0), (v_0, x) \mid x \in V(D_1)\} \cup \\ \cup \{(u_i, u_{i+1}) \mid 1 \leqq i \leqq k - 2\} \cup \{(v_i, v_{i+1}) \mid 1 \leqq i \leqq k + t - 1\} \cup \\ \cup \{(x, u_1), (x, v_1), (v_{k+t}, x) \mid x \in V(D_1)\} \cup \{(u_{k-1}, x) \mid x \in V(D_2)\} \cup \\ \cup \{(x, y) \mid x \in V(D_1),\ y \in V(D_2)\}$$

(see Figure 7).

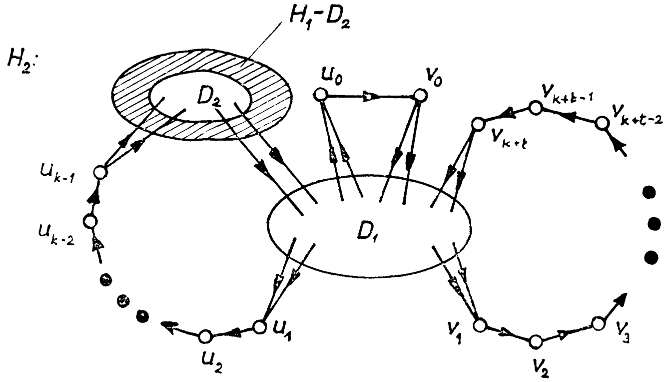

Figure 7

We now show that $mC(H_2) \cong D_1$. Let $x \in V(D_1)$. First observe that

(i) $md_{H_2}(x, y) \leqq 3$ for all $y \in V(D_1)$;
(ii) $md_{H_2}(x, u_0) = md_{H_2}(x, v_0) = 2$;
(iii) $md_{H_2}(x, u_i) \leqq k$ for $1 \leqq i \leqq k - 1$;
(iv) $md_{H_2}(x, v_i) \leqq k + t$ for $1 \leqq i \leqq k + t$; and
(v) $md_{H_2}(x, v_1) = k + t$.

For $y \in V(H_1)$, it follows that

$$\begin{aligned} md_{H_2}(x, y) &= \max\{\vec{d}_{H_2}(x, y), \vec{d}_{H_2}(y, x)\} \leqq \\ &\leqq \max\{\vec{d}_{H_2}(x, z) + \vec{d}_{H_2}(z, y), \vec{d}_{H_2}(y, z) + \vec{d}_{H_2}(z, x)\} \leqq \\ &\leqq \max\{k + \vec{d}_{H_2}(z, y), \vec{d}_{H_2}(y, z) + 1\} \leqq \\ &\leqq \max\{k + \vec{d}_{H_1}(z, y), 1 + \vec{d}_{H_1}(y, z)\}, \text{ where } z \in V(D_2). \end{aligned}$$

Observe that $\vec{d}_{H_1}(z, y) \leqq \max\{\vec{d}_{H_1}(z, y') \mid y' \in V(H_1)\} = \max\{\max\{\vec{d}_{H_1}(z, y') \mid y' \in \in V(D_2)\}, \max\{\vec{d}_{H_1}(z, y') \mid y' \in V(H_1) - V(D_2)\}\} \leqq \max\{3, n_1\} = t$. Similarly, $\vec{d}_{H_1}(y, z) \leqq \max\{3, n_2\} = t$. Therefore,

$$md_{H_2}(x, y) \leqq \max\{k + t, 1 + t\} = k + t \quad \text{for all} \quad y \in V(H_1).$$

Hence, $me_{H_2}(x) = k + t$, for all $x \in V(D_1)$. It is obvious that $me_{H_2}(x) > k + t$, for all $x \in V(H_2) - V(D_1)$. Thus $mC(H_2) \cong D_1$.

Since $k \geqq 2$, it follows that $md_{H_2}(x, y) = md_{H_1}(x, y)$, for all $x \in V(D_2)$, $y \in V(H_1)$. It follows also that

$$md_{H_2}(x, z) = md_{H_2}(y, z), \quad \text{for all} \quad x, y \in V(D_2), \quad z \in V(H_2) - V(H_1).$$

Therefore,

$$\begin{aligned} md_{H_2}(x) &= md_{H_1}(x) + \sum_{z \in V(H_2) - V(H_1)} md_{H_2}(x, z) = \\ &= md_{H_1}(y) + \sum_{z \in V(H_2) - V(H_1)} md_{H_2}(y, z) = md_{H_2}(y), \end{aligned}$$

for all $x, y \in V(D_2)$. Hence, by Lemma 1, there exists an oriented graph H containing H_2 as an induced subdigraph such that $mM(H) \cong D_2$. Further, by the construction of H in the proof of Lemma 1, it follows that $md_H(x, y) = 2$ for all $x \in V(D_2)$, $y \in V(H) - V(H_2)$. Therefore $mC(H) = mC(H_2) \cong D_1$. It is obvious that $md_H(mC(H), mM(H)) = k$. □

We now prove the other extreme case where the m-center and m-median of an oriented graph can be overlap on any common induced subdigraph.

Theorem 4. *Let D_1, D_2 be oriented graphs. Let K be a nonempty oriented graph isomorphic to an induced subdigraph of both D_1 and D_2. Then there exists an oriented graph H such that $mC(H) \cong D_1$, $mM(H) \cong D_2$ and $mC(H) \cap mM(H) \cong K$.*

Proof. Suppose $V(D_1) = \{u_1, u_2, \ldots, u_{p_1}\}$ and $V(D_2) = \{v_1, v_2, \ldots, v_{p_2}\}$. Without loss of generality, we assume that $p(K) = k$, $\langle\{u_1, u_2, \ldots, u_k\}\rangle \cong \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}\rangle \cong \cong K$, and that $u_j \to v_{i_j}$ $(j = 1, 2, \ldots, k)$ is an isomorphism between $\langle\{u_1, u_2, \ldots, u_k\}\rangle$ and $\langle\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}\rangle$. We first construct an oriented graph H_0 by identifying u_j and v_{i_j}, and labeling the resulting vertex again by u_j for $1 \leqq j \leqq k$. We now define an oriented graph H_1 by

$$V(H_1) = V(H_0) \cup \{u, v\} \cup \{w_i, w'_i \mid 1 \leqq i \leqq 6\}$$

and

$$E(H_1) = E(H_0) \cup \{(u, v)\} \cup \{(w_i, w_{i+1}), (w'_i, w'_{i+1}) \mid 1 \leqq i \leqq 5\} \cup$$

$\cup \{(x, u), (v, x) \mid x \in V(H_0)\} \cup$
$\cup \{(u_i, w_1), (w_6, u_i), (u_i, w_1'), (w_6, u_i) \mid 1 \leqq i \leqq p_1\}$
(see Figure 8).

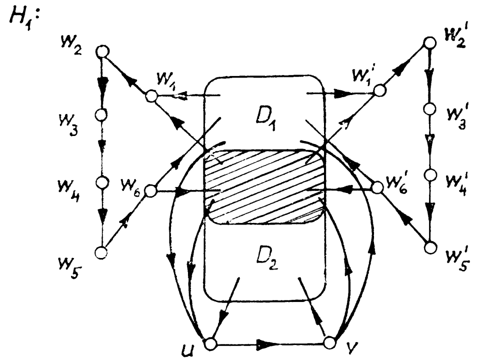

Figure 8

It is clear that m-rad $H_1 = 6$ and $mC(H_1) \cong D_1$. By Lemma 2, there exists an oriented graph H_2 containing H_1 as an subdigraph such that (i) if $V(H_2) \neq V(H_1)$, then $\max \{md_{H_2}(x, y) \mid x \in V(D_2),\ y \in V(H_2) - V(H_1)\} = 3$ and (ii) $md_{H_2}(x) = md_{H_2}(y)$ for all $x, y \in V(D_2)$. Thus $md_{H_2}(x, y) \leqq 6$ for $x \in V(D_1),\ y \in V(H_2) - V(H_1)$, from which it is easy to see that m-rad $H_2 = m$-rad $H_1 = 6$ and $mC(H_2) = mC(H_1) \cong D_1$. By Lemma 1, there exists an oriented graph H containing H_2 as an induced subdigraph such that $mM(H) \cong D_2$. The construction of H in the proof of Lemma 1 implies that $md_H(x, y) = 2$ for $x \in V(D_2),\ y \in V(H) - V(H_2)$. Therefore $mC(H) = mC(H_2) \cong D_1$. □

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